Inverse Wavelet Reconstruction for Resolving the Gibbs Phenomenon

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Abstract—The Gibbs phenomenon refers to the lack of uniform convergence which occurs in many orthogonal basis approximations to piecewise smooth functions. This lack of uniform convergence manifests itself in spurious oscillations near the points of discontinuity and a low order of convergence away from the discontinuities. Here we describe a numerical procedure for overcoming the Gibbs phenomenon called the inverse wavelet reconstruction method. The method takes the Fourier coefficients of an oscillatory partial sum and uses them to construct the wavelet coefficients of a non-oscillatory wavelet series.

Index Terms—Gibbs phenomenon, Inverse wavelet reconstruction, Inverse polynomial reconstruction.

Fourier and orthogonal polynomial series are known for their highly accurate expansions for smooth functions. In fact it is known that the more derivatives a function has, the faster the approximation will converge. However, when a function possesses jump-discontinuities the approximation will fail to converge uniformly. In addition, spurious oscillations will cause a loss of accuracy throughout the entire domain. This lack of uniform convergence is known as the Gibbs phenomenon. Methods for post-processing approximations which suffer from the Gibbs phenomenon include the Gegenbauer reconstruction method of Gottlieb and Shu [7,9], the method of Pade approximants due to Driscoll and Fornberg [2], the method of spectral mollifiers due to Gottlieb and Tadmor [8] and Tadmor and Tanner [18, 19], the inverse polynomial reconstruction method of Shizgal and Jung [13,14,15,16,17], and the Freund polynomial reconstruction method of Gelb and Tanner [6]. These reconstruction methods can be combined with an effective method for edge-detection developed by Gelb and Tadmor [3,4,5,6], to yield an exponentially accurate reconstruction of the original function. In this paper we describe a numerical method for overcoming the Gibbs phenomenon following the work of Shizgal and Jung, called the Inverse Wavelet Reconstruction method.

We begin with a brief review of the essential definitions of wavelets which we will need. Recall that a wavelet is a function $\psi \in L^2(\mathbb{R})$ satisfying:

$$\int_{-\infty}^{\infty} \psi(x) \, dx = 0$$

(1)

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and

$$\int_{-\infty}^{\infty} \left| \frac{\hat{\psi}(\xi)}{\xi} \right| \, d\xi < \infty,$$

(2)

where $\hat{\psi}$ here is the Fourier transform of $\psi$. The function $\psi$ is also known as an analyzing wavelet or a mother wavelet since any function $f \in L^2(\mathbb{R})$ can be expressed as a continuous sum of translations and dilations involving $\psi$ according to the continuous wavelet transform. The continuous wavelet transform is given by

$$(W_\psi, f)(b, a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(t) \psi\left(\frac{t-b}{a}\right) \, dt$$

and the inverse continuous wavelet transform is given by

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [(W_\psi f)(b,a)] \frac{\psi_{b,a}(t)}{a^2} \, db \, dt$$

(3)

where

$$\psi_{b,a}(t) = |a|^{-1/2} \psi\left(\frac{t-b}{a}\right)$$

and

$$C_\psi = \int_{-\infty}^{\infty} \left| \frac{\hat{\psi}(\xi)}{\xi} \right| \, d\xi.$$ 

A discrete wavelet series is given for fixed constants $a_0, b_0$ by

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{j,k} \psi_{j,k}(x)$$

(4)

where the discrete wavelet series coefficients are given by

$$c_{j,k} = (W_\psi f)\left(\frac{b_0 k}{a_0}, \frac{1}{a_0}\right)$$

and

$$\psi_{j,k}(x) = a_0^{j/2} \psi\left(a_0^j x - b_0 k\right).$$

In general the wavelet functions $\psi_{j,k}$ do not constitute an orthonormal basis. Instead they constitute what is known as a frame. The family of functions $\psi_{j,k}$ is a frame if

$$A \|f\|^2 \leq \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |(f, \psi_{j,k})| \leq B \|f\|^2$$

for positive constants $A$ and $B$. When $A = B$ the functions $\psi_{j,k}$ are known as a tight frame. For more information see Daubechies [1]. The inverse wavelet reconstruction method makes use of wavelet bases and frames.
The function also has a Gegenbauer-

\[ f(x) = \sum_{m=0}^{\infty} \hat{f}^\lambda(m) C^\lambda_m(x) \]

where

\[ \hat{f}^\lambda(m) = \frac{1}{h^\lambda} \int_{-1}^{1} f(x) C^\lambda_m(x) (1-x^2)^\lambda \frac{1}{2} dx \]

and

\[ h^\lambda = \pi^{1/2} \Gamma(m + 2\lambda) \Gamma(\lambda + \frac{1}{2}) \Gamma(\lambda) / m! \Gamma(2\lambda) \]

The Fourier coefficients \( \hat{f}^\lambda(n) \) can be expressed in terms of Gegenbauer-\( \lambda \) coefficients as follows.

\[ \hat{f}^\lambda(n) = \frac{1}{2} \int_{-1}^{1} f(x) e^{-i\pi nx} dx \]

\[ = \frac{1}{2} \int_{-1}^{1} \sum_{m=0}^{\infty} \hat{f}^\lambda(m) C^\lambda_m(x) e^{-i\pi nx} dx \]

\[ = \sum_{m=0}^{\infty} \hat{f}^\lambda(m) \left( \frac{1}{2} \int_{-1}^{1} C^\lambda_m(x) e^{-i\pi nx} dx \right) \]

\[ = \sum_{m=0}^{\infty} \hat{f}^\lambda(m) C^\lambda_{m,n} \]

The connection coefficients \( C^\lambda_{m,n} \) are given by

\[ C^\lambda_{m,n} = \frac{1}{2} \int_{-1}^{1} C^\lambda_m(x) e^{-i\pi nx} dx \]

and can be computed numerically. An explicit formula for \( C^\lambda_{m,n} \) was derived by this author, a form of which appeared in Greene [10], and is given by:

\[ c^\lambda_{m,n} = \begin{cases} \frac{[m/2]}{2^n} \sqrt{2n} J_{m-2j+1/2}(\pi n) \\ \lambda - \frac{1}{2} j (\lambda - m - j (\frac{3}{2}) m - 2j), \ n \neq 0 \end{cases} \]

\[ \frac{1}{j!^{1/2} n^{1/2}} m, \ m \text{ even.} \]

The inverse polynomial reconstruction procedure is obtained by truncating the infinite series above and solving the system of equations

\[ \hat{f}(n) = \sum_{m=0}^{2N} \hat{f}^\lambda(m) C^\lambda_{m,n}, \ n = -N...N. \]

One then computes the reconstruction approximation

\[ S^\lambda_{2N} f(x) = \sum_{m=0}^{2N} \hat{f}^\lambda(m) C^\lambda_m(x). \]

The authors show that this final polynomial approximation does not depend on the choice of \( \lambda \).

B. Inverse Wavelet Reconstruction

We explore here an analogous approach which seeks to reconstruct the original function in terms of wavelets. We begin with a Fourier partial sum as before and assume that the function \( f(x) \) can also be expressed as a discrete wavelet series:

\[ f(x) = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{m,l} \psi_{m,l}(x) \]

where

\[ c_{m,l} = \int_{-\infty}^{\infty} f(x) \psi_{m,l}(x) dx \]

and

\[ \psi_{m,l}(x) = a_0^{-m/2} \psi(a_0^m x - b_0 l). \]

We now derive a formula expressing the Fourier coefficients in terms of wavelet coefficients.

\[ \hat{f}(n) = \frac{1}{2} \int_{-1}^{1} f(x) e^{-i\pi nx} dx \]

\[ = \frac{1}{2} \int_{-1}^{1} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{m,l} \psi_{m,l}(x) e^{-i\pi nx} dx \]

\[ = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{m,l} \left( \frac{1}{2} \int_{-1}^{1} \psi_{m,l}(x) e^{-i\pi nx} dx \right) \]

\[ = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{m,l} \tilde{\psi}_{m,l}(n) \]

The inverse wavelet reconstruction method is obtained by truncating the doubly infinite sum described above and solving for the wavelet coefficients. We suggest solving the following system of equations.

\[ \hat{f}(n) = \sum_{m=-M}^{M-1} \sum_{l=-L}^{L-1} c_{m,l} \tilde{\psi}_{m,l}(n) \]

where

\[ n = -N...N - 1, \text{ and } N = 2ML \]

for the wavelet coefficients \( c_{m,l} \). One then computes the wavelet reconstruction approximation

\[ S_{M,L} f(x) = \sum_{m=-M}^{M-1} \sum_{l=-L}^{L-1} c_{m,l} \psi_{m,l}(x). \]
The terms $\hat{\psi}_{m,l}(n)$ are the $n$th Fourier coefficients of $\psi_{m,l}(x)$. Since we are solving a system of $2N$ equations in $(2M)(2L) = 4ML$ unknown wavelet coefficients, in order for the system to be invertible we must have $2N = 4ML$ or $N = 2ML$. It is this convention which we will use for our numerical implementation below.

Numerical experiments indicate that the invertibility condition can be relaxed and fewer wavelet coefficients can be computed if one solves the overdetermined system in a least squares sense using Matlab’s backslash operator. This is also illustrated below.

II. NUMERICAL RESULTS

Numerical experiments show that the inverse wavelet reconstruction approach does yield rapidly converging uniform approximations for a variety of wavelet families. We illustrate our method with several wavelet families. We use the Mexican hat wavelets [1] p. 75

$$\psi(x) = \frac{2}{\sqrt{3\pi}} (1 - x^2) e^{-x^2/2},$$

the Morlet wavelets [1] p. 76

$$\psi(x) = \cos (\pi x) e^{-x^2/2},$$


$$\tilde{\psi} (\xi) = \sqrt{\frac{3}{2\pi}} e^{i\xi/2} \frac{\sin^4(\xi/4)}{\xi^2} \left( 1 + 2 \cos^2(\xi/2) \right) \left( 1 + 2 \cos^2(\xi/4) \right)^{1/2} \frac{1}{\xi^2}.$$  

We consider the test functions $f_1(x) = \tan^{-1}(x)$, $f_2(x) = \sin (\cos (4x))$ and $f_3(x) = 1$. The functions $f_1$, $f_2$, and $f_3$ both experience the Gibbs phenomenon at $x = \pm 1$. The function $f_3$ is naturally oscillatory and its inverse polynomial reconstruction was studied in [16]. The periodic extension of the function $f_2$ is continuous at $x = \pm 1$, however its first derivative is discontinuous resulting in a slowly converging Fourier series.

We implement the inverse wavelet reconstruction method by computing Fourier coefficients for the test functions in question using trapezoidal rule quadrature. This makes our Fourier series a pseudospectral series. For the method to be accurate we compute the coefficients $\hat{\psi}_{m,l}(n)$ using the same trapezoidal rule quadrature. Results reported previously [11] were less accurate because they were implemented using Matlab’s built in quadl function.

Rather than display semilogarithmic error plots for each choice of wavelet and each choice of the parameters $a_0$ and $b_0$, which would be cumbersome, we choose instead to display in the tables below the base 10 logarithm of the maximum absolute error of the reconstruction. This gives the order of the approximation in the uniform norm. To reflect the fact that the accuracy of the approximation throughout the interval is typically better than the maximum error, we also display the logarithm of the median absolute error.

No attempt has been made to optimize the parameters $a_0$ and $b_0$. There may be wavelet families as well as choices for $a_0$ and $b_0$ which significantly outperform the choices displayed here. Our goal has simply been to illustrate that the method works and yields highly accurate results.

In the tables below we consider 16, 36, 64, 100, and 144 Fourier coefficients and solve for an equal number of wavelet coefficients. The sequence is generated by setting $M = L$ so that the number of coefficients is given by $2N = 4M^2$ for $m = 2, 3, 4, 5,$ and 6. It turns out that one need not be bound by this invertibility condition. The method is apparently stable when the number of Fourier coefficients is greater than the number of wavelet coefficients, when one solves the system in a least squares sense using Matlab’s backslash operator. This is illustrated below where 20, 40, 80 and 160 Fourier coefficients are taken and only 16, 36, 64, and 100 wavelet coefficients are computed.

A comparison of the maximum absolute error for several choices of wavelets illustrating apparent uniform convergence for $f_1(x)$.

A comparison of the median absolute error for several choices of wavelets illustrating the typical error throughout the interval for $f_1(x)$.

A comparison of the maximum absolute error for $f_2(x)$.

A comparison of the median absolute error for $f_2(x)$.
A 40 term pseudospectral Fourier series for \( f_1(x) = \tan^{-1} x \).

A 40 term pseudospectral Fourier series for \( f_2(x) = \sin (\cos 4x) \).

A 40 term pseudospectral Fourier series for \( f_3(x) = \cos (1.4\pi (x + 1)) \).

Absolute error for inverse Mexican Hat reconstruction of \( f_1(x) \), \( a_0 = 2, b_0 = 0.25 \) from 16, 36, 64, 100 Fourier coefficients using the same number of wavelet coefficients (the invertible case).

Absolute error for inverse Mexican Hat reconstruction of \( f_1(x) \), \( a_0 = 2, b_0 = 0.25 \) from 20, 40, 80, 160 Fourier coefficients using 16, 36, 64, 144 wavelet coefficients respectively, solved in a least squares sense.

It is apparent that as the number of Fourier coefficients increases the error of the reconstruction initially decreases rapidly, but soon levels off at around \( 10^{-14} \) regardless of how many Fourier coefficients are used. This is due to the fact that the linear system is ill-conditioned and machine epsilon here is approximately \( 2.2 \times 10^{-16} \). All the wavelets shown perform comparably well, although some do slightly better than others. The results for the inverse wavelet reconstruction method in all cases appear comparable to the results reported for the inverse polynomial reconstruction method.

III. **Conclusions and Future Work**

Numerical results indicate that the inverse wavelet reconstruction method yields an accurate and uniformly converging reconstruction approximation for a variety of wavelets. Work in progress includes reconstruction involving the use of scaling.
functions, reconstruction from series other than Fourier series, such as orthogonal polynomials or wavelets, a comparison with other methods, and a proof of convergence.

REFERENCES


