

A Harmonical Model for Approximating the Identity in Min-Plus Convolution

Ming Li¹ and Wei Zhao²

Abstract— Min-plus convolution is an algebra system that has applications to computer networks. Mathematically, the identity of min-plus convolution plays a key role in theory. On the other hand, the mathematical representation of the identity, which is computable with digital computers, is essential for further developing min-plus convolution (e.g., de-convolution) in practice. However, the identical element in min-plus convolution is defined as infinity over infinite interval, making digital computation of the identity difficult because digital computers only provide finite range of numbers for numerical computations. Consequently, the issue of numerical approximation of the identical element is worth discussing. This paper proposes a harmonic model for finite approximation of the identical element in the min-plus convolution.

Keywords— Min-plus convolution, Fourier analysis, approximation, generalized functions.

I. INTRODUCTION

MIN-plus convolution has gained applications to, such as graph [1], discrete systems [2], quality of service in computer communication networks [3-6, 11-16]. This paper discusses a computation model of the identity of min-plus convolution that plays a key role in network calculus due to the importance of the identity in an algebra system from a view of mathematics. Such a discussion is rarely seen, to the best of our knowledge.

Let \mathbb{R}^+ be the set of positive real numbers. For real functions $f(x)$, $g(x)$ ($x \in \mathbb{R}^+$), the operation of min-plus convolution is given by

$$f(x) \otimes g(x) = \inf_{0 \leq u \leq x} \{f(u) + g(x-u)\} \quad (1-a)$$

where \otimes stands for the operation of min-plus convolution. In practical computations, the above infimum can be replaced by minimum. That is, for $0 \leq u \leq x$,

$$f(x) \otimes g(x) = \min\{f(u) + g(x-u)\} \quad (1-b)$$

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¹Ming Li (corresponding author) is with the School of Information Science & Technology, East China Normal University, No. 500, Dong-Chuan Road, Shanghai 200241, PR. China. (Tel.: +86-21-5434 5193; fax: +86-21-5434 5119; e-mails: ming_lihk@yahoo.com, mli@ee.ecnu.edu.cn).

²Wei Zhao is with Rensselaer Polytechnic Institute, 110 Eighth Street, 1C05 Science, Troy, NY 12180-3590, USA. (e-mail: zhaow3@rpi.edu).

The existence of the identity in min-plus convolution is given by [5]. In order to define the identical element for \otimes , the enlargement of \mathbb{R}^+ is needed. In doing so, we consider $\mathbb{R}_+ = \mathbb{R}^+ \cup \{\infty\}$ [1, 2]. Then, the following

$$I(x) = \begin{cases} \infty, & x > 0 \\ 0, & x \leq 0 \end{cases} \quad (2)$$

is defined as the identical element for \otimes because

$$f(x) \otimes I(x) = I(x) \otimes f(x) = f(x).$$

From a view of functions, $I(x)$ is a function of x , which represents infinity over infinite interval. From a view of computations, however, numerical computations of ∞ cannot be performed by digital computers as number range provided by digital computers for numerical computations is finite. Hence, comes a problem how to numerically approximate I .

As known, the existence of the identity in min-plus convolution implies that the de-convolution of min-plus convolution exists though literature regarding de-convolution of min-plus convolution is rarely seen. However, de-convolution may come into practical use only if the representation of $I(x)$ by using ordinary functions is available. Due to ∞ , finite approximation has to be taken into account. This paper presents a harmonic model for finite approximation of $I(x)$ based on the Fourier analysis in generalized functions.

The paper is organized as follows. A harmonic model of I is derived in Section 2. Section 3 illustrates the numeric simulation of the present model. Section 4 concludes the paper.

II. HARMONIC MODEL OF $I(x)$

Let $\delta(x)$ be the notation of the Dirac- δ function. Then [6, Chap. III],

$$\begin{cases} \delta(x-x_0) = \begin{cases} 0, & x \neq x_0 \\ \infty, & x = x_0 \end{cases} \\ \int_{-\infty}^{\infty} \delta(x-x_0) dx = 1. \end{cases} \quad (3)$$

The above integral does not exist in ordinary functions [7, Chap. 10] but $\delta(x)$ has its root in the theory of generalized functions [8, 9]. Eq. (4) is an expression of $\delta(x)$ (other equivalent expressions of $\delta(x)$ can be found in references, e.g., [7-10]).

$$\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \cos(kx). \quad (4)$$

By taking into account a generalized periodic function given

by

$$\sum_{n=0}^{\infty} \delta(x - nT) \quad (T > 0),$$

the discrete I is given by

$$I(k) = \sum_{n=0}^{\infty} \delta(k - nT). \tag{5}$$

In the continuous case,

$$I(x) = \lim_{T \rightarrow 0} \sum_{n=0}^{\infty} \delta(x - nT). \tag{6}$$

Following [9, p. 63] [10, p. 67-68], the discrete I can be expressed as

$$I(k) = \frac{2}{T} + \frac{4}{T} \sum_{n=1}^{\infty} \cos\left(\frac{2n\pi k}{T}\right). \tag{7}$$

In the limit case,

$$I(x) = \lim_{T \rightarrow 0} \left[\frac{2}{T} + \frac{4}{T} \sum_{n=1}^{\infty} \cos\left(\frac{2n\pi x}{T}\right) \right]. \tag{8}$$

III. NUMERIC SIMULATION

In practice, finite items of (7) are used for approximation. The finite approximation of (7) is given by

$$I(k) \approx \frac{2}{T} + \frac{4}{T} \sum_{n=1}^N \cos\left(\frac{2n\pi k}{T}\right). \tag{9}$$

Errors caused due to finite approximation is expressed by

$$e(N) = \frac{4}{T} \sum_{n=N+1}^{\infty} \cos\left(\frac{2n\pi k}{T}\right). \tag{10}$$

A. Error Caused by Finite Approximation

Let $N = 10$, $T = 9$ and $k = 0, 1, \dots, 300$. Then, the computation result with (9) is shown in Fig. 1. The errors are reflected in two aspects. One is that there are some components between each two conjoint x s, which obscure the computation result. This error is called error I. The other is that the magnitude of computation result is not 'infinite'. We call it error II.

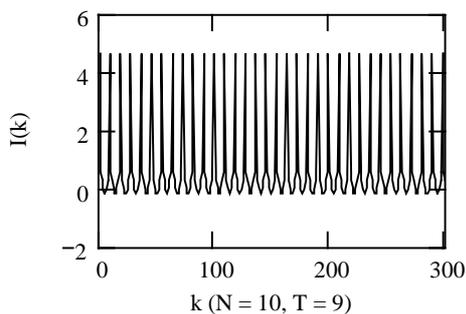
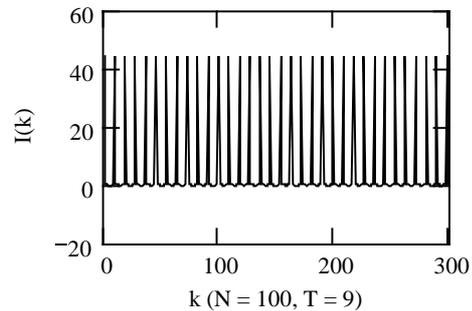


Fig. 1. Finite approximation with error I and error II.

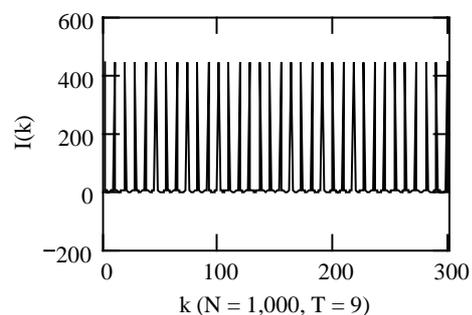
B. Convergence

The convergence implies that both errors should decrease as $N \rightarrow \infty$. The convergence is illustrated by Fig. 2 that shows the results when $N = 100, 1,000, 10,000$, and $100,000$, respectively. From those figures, by eye, we see that two types of errors

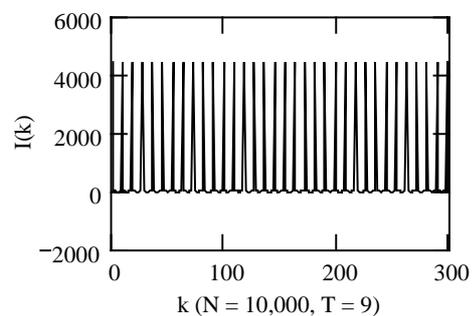
decrease as N increases. We use Fig. 3 to demonstrate the convergence.



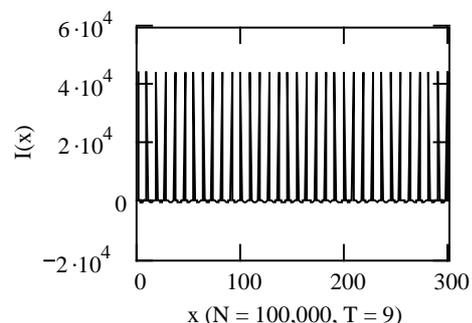
(a)



(b)

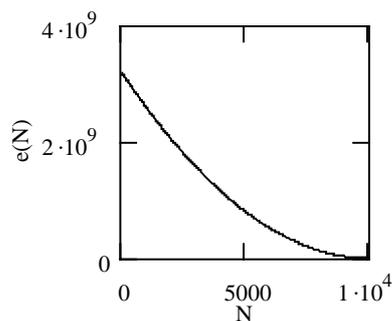


(c)



(d)

Fig. 2. Observation of convergence.

Fig. 3. Error $e(N)$.

IV. CONCLUSION

We have derived a harmonically numeric model on finite approximation of the identical element in min-plus convolution. The numerical simulation and convergence have been demonstrated.

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Ming Li was born in 1955 in Wuxi, China. He completed his undergraduate program in electronics engineering at Tsinghua University. He received the M.S. degree in ship structural mechanics from China Ship Scientific Research Center (CSSRC) and Ph.D. degree in computer science from City University of Hong Kong, respectively. From 1990 to 1995, he was a researcher in CSSRC. From 1995-1999, he was with the Automation Department, Wuxi University of Light Industry. From 2002 to 2004, he was with the School of Computing, National University of Singapore.

In 2004, he joined East China Normal University (ECNU) as a professor in both electronics engineering and computer science. He is currently a Division Head for Communications & Information Systems at ECNU. His research areas relate to applied statistics, computer science, measurement & control. He has published over 70 papers in international journals and international conferences in those areas. Dr. Li is a member of IEEE.



Wei Zhao is a professor of computer science and the dean for the School of Science at Rensselaer Polytechnic Institute. His research interests include distributed computing, real-time systems, computer networks, and cyberspace security. Zhao received a PhD in computer and information sciences from the University of Massachusetts, Amherst. He is a Fellow of the IEEE. He has published over 280 papers in international journals, international conferences, and book chapters.