Application of finite Markov chains to decision - making

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***Abstract*—** During the last 50-60 years the rapid technological progress together with the radical changes happened to the local and international economies and other relevant reasons led to a continuously increasing complexity of the problems of our everyday life connected to decision making, i.e. to the process of choosing a solution between two or more alternatives, aiming to achieve the best possible results. Thus, from the beginning of the 1950's a progressive development started of a systematic methodology for the decision making process, which is based on Probability Theory, Statistics, Economics, Psychology, etc. and it is termed as Statistical Decision Theory. In the present paper we develop a finite Markov chain model for the mathematical description of the decision making process. An example is also presented to illustrate our results.

Keywords— Decision-Making (DM), Finite Markov Chains (FMC), Absorbing Markov Chains (AMC), Fundamental Matrix of an AMC.

I. INTRODUCTION

Decision Making (DM) is the process of choosing a solution between two or more alternatives, aiming to achieve the best possible results for a given problem. Obviously the above process has sense if, and only if, there exist more than one *feasible solutions* together with a suitable criterion that helps the decision maker (d-m) to choose the best among these solutions. It is recalled that a solution is characterized as feasible, if it satisfies all the restrictions imposed onto the real system by the statement of the problem as well as all natural restrictions imposed onto the problem by the real system. For example, if x denotes the quantity of stock of a product, it must be $x \geq 0$. The choice of the suitable criterion, especially when the results of DM are affected by random events, depends upon the desired goals of the d-m; e.g. optimistic or conservative criterion etc).

The rapid technological progress , the impressive development of the transport means, the globalization of our modern society, the enormous changes happened to the local and international economies and other relevant reasons led during the last 50-60 years to a continuously increasing complexity of the problems of our everyday life. As a result the DM process became in many cases a very difficult task, so that it is impossible to be based on the d-m's experience, intuition and skills only, as it usually used to happen in the past. Thus, from the beginning of the 1950's a progressive development started of a systematic methodology for the DM process, which is based on Probability Theory, Statistics, Economics, Psychology, etc. and it is termed as *Statistical Decision Theory* [1].

According to the nowadays existing standards the DM process involves the following steps:

- $\mathbf{d_1}$**:** *Analysis* of the decision-problem, i.e. understanding, simplifying and reformulating the problem in a way permitting the application of the principles of SDT on it.
- $\mathbf{d_2}$**:** Collection from the real system and interpretation of all the necessary information related to the problem.
- $\mathbf{d_3}$**:** Determination of all the alternative feasible solutions.
- $\mathbf{d_4}$**:** Choice of the best solution in terms of the suitable (according to the decision maker's goals and targets) criterion.

One could add one more step to the DM process, the *verification of the chosen decision* according to the results obtained by applying it in practice. However, this step is extended to areas, which due to their depth and importance for the administrative rationalism have become autonomous. Therefore, it is usually examined separately from the other steps of the DM process.

Note that the first three steps of the DM process are *continuous* in the sense that the completion of each one of them usually needs some time, during which the decision maker's reasoning is characterized by transitions between hierarchically neighbouring steps. In other words *the DM process is not a linear process*. Accordingly its flow-diagram is represented in Figure 1 below:

$$\mathbf{d_1} \leftrightarrow \mathbf{d_2} \leftrightarrow \mathbf{d_3} \rightarrow \mathbf{d_4}$$

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Fig. 1: The flow-diagram of the DM process

In the present paper we shall develop a Markov chain model for the mathematical description of the DM process. The model is presented in Section II, whereas in Section III an example is given illustrating the model's use in practice. Finally, Section IV is devoted to our conclusions and to a short discussion on our plans for further research on the subject.

II. THE MARKOV CHAIN MODEL

Roughly speaking, a *Markov chain* (MC) is a special type of a stochastic process that moves in a sequence of steps (phases) through a set of states and whose main characteristic, known as the *Markov property*, is that it has memory of only one step. This means that the probability of entering a certain state at a certain step of the process depends only on the state occupied in the previous step. However, during the process of mathematical modelling there usually exists a need to simplify the real system in a way that enables the formulation of it to a form ready for mathematical treatment (*assumed real system* [2]). This makes a number of authors to express the Markov property in a wider context by accepting that the probability of entering a certain state at a certain step of the process, although it is not necessarily independent from older steps, it depends *mainly* on the state occupied in the previous step [3, Chapter 12].

When a MC has a finite number of states, it is called a *finite Markov chain (FMC).* For general facts on FMC's we refer to the book [4] of Kemeny and Snell.

The theory of MCs offers in general ideal conditions for the study and mathematical modelling of a certain kind of real situations depending on random variables [5-9]. Most of the corresponding problems can be solved by distinguishing between two types of MCs, the *Ergodic* [4, Chapter V] and the *Absorbing* ones [4, Chapter III].

For obtaining a mathematical formulation of the DM process we introduce here a finite MC on its steps by adopting the notion of the Markov property in its wider context. This means that the states of our chain are the steps d_i, i = 1, 2, 3, 4, of the DM process introduced in the previous section. It is logical to accept that d_1 is always the starting state. Further, we observe that, when the chain reaches the state d_4 (end of the DM process) it is impossible to leave it. This means that d_4 is the unique *absorbing state* of the chain. Therefore, since it is possible from any state to reach the absorbing state d_4, not necessarily in one step (see Figure 1), our MC is an Absorbing Markov chain (AMC).

Let us now denote by p_{ij} the *transition probability* from state d_i to d_j , i,j = 1,2,3,4 . Then the *transition matrix* of the MC is

$$A = \begin{array}{c} \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & d_3 & d_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ p_{21} & 0 & p_{23} & 0 \\ 0 & p_{32} & 0 & p_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

We have that $p_{21}+p_{23} = p_{32}+p_{34} = 1$ (1), since the right member of equation (1) calculates the probability of a certain event.

Let us also denote by $\varphi_0, \varphi_1, \varphi_2, ...$ the several steps of the chain and let $P_i = [p_1^{(i)}\ p_2^{(i)}\ p_3^{(i)}\ p_4^{(i)}]$ be the row - matrix giving the probabilities for the chain to be in each one of its states at the step φ_i , i = 0,1,2,.... . Then, since d_1 is always the starting state, we have that $P_0 = [1\ 0\ 0\ 0]$. Further it is well known that $P_{i+1} = P_i A$. Therefore we have:

$P_1 = P_0A = [0\ 1\ 0\ 0]$

$P_2 = P_1 A = [p_{21}\ 0\ p_{23}\ 0]$

$P_3 = P_2 A = [0\ \ p_{21}+ p_{23}p_{32}\ \ 0\ \ p_{23}p_{34}]$ (2).

$P_4 = P_3A = [p_{21}^2+p_{21}p_{23}p_{32}\ \ 0\ \ p_{21}p_{23}+ p_{23}^2 p_{32}\ \ p_{23}p_{34}]$ and so on.

In general an inductive argument shows that $P_n = P_0A^n$, n = 1, 2, 3,... . This formula enables one to make *short-run forecasts* for the evolution of the situation modelled by the corresponding MC.

We now bring the transition matrix A to its *standard form* A* by listing the absorbing state first and then we make a partition of A* to sub-matrices as follows:

$$A^* = \begin{array}{c} \\ d_4 \\ d_1 \\ d_2 \\ d_3 \end{array} \begin{array}{c} \begin{array}{cccc} d_4 & d_1 & d_2 & d_3 \end{array} \\ \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & p_{21} & 0 & p_{23} \\ p_{34} & 0 & p_{32} & 0 \end{array}\right] \end{array}.$$

Denote by $Q = \begin{bmatrix} 0 & 1 & 0 \\ p_{21} & 0 & p_{23} \\ 0 & p_{32} & 0 \end{bmatrix}$ the transition matrix of the non absorbing states and denote by I_3 the 3x3 unitary matrix. It is well known that I_3-Q is always an invertible matrix (e.g. see [9], section 2).

The *fundamental matrix* N of the chain is defined to be the inverse matrix of I_3-Q. Therefore, applying standard techniques from the Linear Algebra we have:

$$N = (I_3 - Q)^{-1} = \frac{1}{D(I_3 - Q)} \text{ adj } (I_3 - Q).$$

In the above formula D(I_3-Q) denotes the determinant of I_3-Q and adj (I_3-Q) denotes the *adjoin matrix* of. I_3-Q. It is recalled that the elements of adj (I_3-Q) are the algebraic complements of the elements of the transpose matrix of I_3-Q.

By a straightforward calculation and using equation (1) one obtains that

$$N = \frac{1}{p_{23}p_{34}} \begin{bmatrix} 1-p_{32}p_{23} & 1 & p_{23} \\ p_{21} & 1 & p_{23} \\ p_{21}p_{32} & p_{32} & p_{23} \end{bmatrix} = [n_{ij}] \quad , i,j = 1,2,3$$

(3)

It is well known [4, Chapter III] that *the ij-th entry of N gives the mean number of times in state d_j before the absorption, when the chain is started in state d_i.*

Therefore, since in our case d_1 is always the starting state, the mean number of steps taken before absorption is given by:

$$t = \sum_{i=1}^{3} n_{1i} = \frac{2+p_{23}p_{34}}{p_{23}p_{34}} \quad (4).$$

Obviously, the bigger is the value of t, the more the difficulties that a d-m faces during the DM process. In other words t provides an indication for the difficulty of the DM process. Another indication for the difficulty of the DM process could be the time spent by the d-m to complete the process, etc.

III. EXAMPLE

The following example illustrates the use of the MC model constructed in Section II in practice:

In the province of Akhaia of Greece the manager of a local company A, which produces, bottles and trades wine, has employed a specialist to help him in deciding about the proper place for building a new factory. The deal is to pay the specialist for six working hours (w. h.) whenever an analysis of the DM problem is required, for 54 w. h. whenever collection and interpretation of the necessary information is needed, for 28 w. h. whenever the determination of all feasible solutions is attempted and for 9 w. h. for the final choice of the best decision. The manager wants to determine the probability for the DM process to be terminated in four steps and to estimate the mean number of steps needed before taking the decision as well as the expected number of w. h. to be paid to the specialist for his services.

We shall analyze the DM process for the above DM problem according to the lines of the above presented MC model:

d_1: Analysis of the DM problem

The analysis of the DM problem, performed by the specialist, showed that the profitability of the decision to be taken depends upon the types (qualities) of the oil produced by the existing in the area where A acts competitive companies.

d_2: Collection and interpretation of the necessary information

The relevant investigation has shown that there is only one competitive company in the area, say B, which produces three different types of wine, say W_1, W_2 and W_3.

d_3: Determination of the feasible solutions

The general situation of the area (communications, traffic, the already existing factories and storehouses of the companies A and B etc), combined with the funds available by the company A for the construction of the new factory, suggest that there are four favourable places , say P_1 , P_2 , P_3 and P_4 for the possible construction of the new factory. However, the need of some new information (data of the market's research) became necessary for the specialist at this point in order to be able to proceed to the choice of the best solution.

$d_3 \rightarrow d_2$: Going back from d_3 to d_2

The market's research has shown that the expected net profits of the company A with respect to the favourable places for the construction of the new factory and the types of the wine to be produced by the company B are those shown in Table 1 below:

Table1: Net profits of the company A

$$\begin{array}{c} \\ W_1 \\ W_2 \\ W_3 \end{array} \begin{array}{c} \begin{matrix} P_1 & P_2 & P_3 & P_4 \end{matrix} \\ \begin{bmatrix} 3 & 8 & 5 & 4 \\ 4 & 2 & 6 & 5 \\ 2 & 1 & 1 & -1 \end{bmatrix} \end{array}$$

$d_2 \rightarrow d_3$: New transition from d_2 to d_3

From Table 1 it becomes evident that the feasible solution P_4 is worse than P_3 and therefore P_4 is rejected.

d_4: Choice of the best solution

The manager of the company does not want to risk for earning low profits by constructing the new factory, which means that the specialist must adopt a conservative criterion for the choice of the best place for building it. In such cases the most frequently used criterion is *the maximin of payoffs* (profits), due to *Wald*. The Wald's criterion, based on the *Murphy's* law assuming that the worst possible fact to be happen will finally happen, suggests to maximize the minimal possible for each case profits. In other words, since the minimal expected profit from the choice of P_1 is 2 monetary units and the minimal profit from the choice of P_2 or of P_3 is 1 monetary unit (see Table 1), according to the Wald's criterion the place P_1 must be chosen for building the new factory.

Data evaluation

From the above analysis of the DM process it becomes evident that $p_{21} = 0$ and $p_{23} = 1$. We also claim that $p_{32} = p_{34} = 0.5$. In fact, when the MC reaches the state d_3 for first time, the probability of returning to d_2 at the next step is 1, since the collection and interpretation of new information is necessary. Further, the second time that the MC reaches d_3 the probability of returning to d_2 at the next step is 0, since no more information is needed for the choice of the best solution. Therefore the transition probability p_{32} is equal to the mean value $\frac{0+1}{2}$. Therefore, $p_{34} = 1- p_{32} = 0.5$.

Replacing the above values of the transition probabilities to the third of relations (2) we find that $P_3 = [0\ 0.5\ 0\ 0.5]$, i.e. $p_4^{(3)} = 0.5$. This means that the probability for the DM process to be terminated in 4 steps is 50%. This could happen, if there was no feasible solution worse than one of the others and therefore we didn't reject any of them, as we did above for P_4.

Further, from relation (3) we obtain that

$$N = \frac{1}{0.5}\begin{bmatrix} 0.5 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0.5 & 1 \end{bmatrix}$$

Therefore $n_{11} = 1$ and $n_{12} = n_{13} = 2$. Thus, the mean number of steps for the DM process before taking the decision is t = 5 steps, while the expected number of w. h. to be paid to the specialist is equal to 1*6 + 2*54 + 2*28 + 1*9 = 179 w. h.

V. DISCUSSION AND CONCLUSIONS

DM is a very important process for individuals, groups, enterprises etc., aiming to achieve the best possible results by choosing a solution between two or more alternatives. In the present paper a mathematical model was developed for the DM process by introducing an AMC on its steps. This enables one to make short-run forecast for the evolution of the DM process. Further, with the help of the fundamental matrix of the AMC we calculated the mean number of steps needed for the completion of the DM process. An example concerning a local wine company was also presented illustrating our results.

The theory of MCs, being a smart combination of Probability and Linear Algebra, offers in general ideal conditions for the study and mathematical modelling of a certain kind of processes depending on random variables. In earlier works we have developed similar to the above MC models to describe several other situations in the areas of Management, Education and Artificial Intelligence; e.g. see the books [10, 11] and the relevant references given in them. Therefore it looks interesting and useful to search in future for more real examples, where one could apply analogous MC models for the mathematical description and evaluation of the corresponding situations.

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