Fuzzy Relation Equations on the van Hiele Levels of Geometric Reasoning

Michael Gr. Voskoglou

Abstract - Fuzzy Relation Equations, which are associated with the composition of binary fuzzy relations, is a dynamic tool of fuzzy mathematics that has been used by many researchers in several real word applications. In the paper at hands Fuzzy Relation Equations are applied on the Van-Hiele levels of geometric reasoning to study the student progress on learning Euclidean Geometry and examples are presented illustrating this application. The Van-Hiele theory suggests that students can progress in Geometry through five states of increasing structural complexity.

Key-Words - *Fuzzy Set (FS), Membership Function (MF), Fuzzy Binary Relation (FBR), Fuzzy Relation Equations (FRE), Fuzzy Assessment Methods, Van-Hiele (VH) Levels of Geometric Reasoning.*

I. INTRODUCTION

Euclidean Geometry, although it is not on the focus of the current mathematical research being overlapped by the modern geometries that use more dynamic tools for their development (like algebra, calculus, topological theories, etc.), it has doubtlessly a great pedagogical value. It cultivates the student reasoning, fantasy and cognitive skills connecting directly mathematics to the real world.

However, researchers and educators agree that students face significant difficulties in learning the Euclidean Geometry, which fluctuate from the understanding of space to the development of geometric reasoning and the ability of constructing the proofs and solutions of several geometric propositions and problems. Therefore for students there exists an uncertainty in general about the good understanding of several geometric procedures, methods and concepts. From the teacher's point of view there also exists a degree of fuzziness about the student acquisition of various geometric topics. All the above remarks are good reasons for one to attempt applying principles of **Fuzzy Logic (FL)** for the assessment of student geometric reasoning skills.

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Perdikaris [1] presented a fuzzy framework for the **van Hiele (vH) level theory** of geometric reasoning by introducing a finite absorbing Markov chain [2] on the fuzzy linguistic labels (characterizations) of no, low, intermediate, high and complete acquisition respectively of each vH level. He also used the fuzzy possibilities of student profiles and the generalization in possibility theory of the classical Shannon's entropy for measuring a system's uncertainty to compare the intelligence of student groups in the vH theory [3]. However, his method is problematic, since it assigns non-zero membership degrees to student profiles in which the student performance in a vH level is assumed to be worse than that in the next level, a thing that is not possible to happen.

This problem was resolved by Voskoglou in [4], where he developed a similar model for the process of learning a subject matter in the classroom, by assigning non zero membership degrees only to **well defined** student profiles. These are profiles of the form (x, y, z), where x is a linguistic characterization the same or better than y, which is the same or better than z. But even the improved Voskoglou's model remains complex needing laborious calculations in its final step of application. In a more recent paper [5] Voskoglou presented a simplified version of this model, which is much easier to be applied in practice, together with two alternative fuzzy methods for the assessment of the acquisition of the vH levels in Geometry by students. Those methods involve the application of the **Centre of Gravity (COG) defuzzification technique** and the use of **Triangular Fuzzy Numbers (TFNs)** respectively.

In the present work **Fuzzy Relation Equations (FRE)** are applied on the vH levels, in an effort to obtain more comprehensive information about student geometric reasoning skills. The rest of the paper is formulated as follows: In Section II a brief account is given of the vH level theory, while Section III contains the background from FRE theory which is indispensable for the understanding of the paper. The assessment model using FRE is presented in Section IV and applications are presented in Section V illustrating its usefulness in practice. The paper closes with the conclusion stated in Section VI.

II. THE VH LEVEL THEORY OF GEOMETRIC REASONING

The vH theory of geometric reasoning [6, 7] suggests that students can progress through five levels of increasing structural complexity. A higher level contains all knowledge of any lower level and some additional knowledge which is not explicit at the lower levels. Therefore, each level appears as a meta-theory of the previous one [8]. The five vH levels include:

- L_1 **(Visualization)***:* Students perceive the geometric figures as entities according to their appearance, without explicit regard to their properties.
- L_2 **(Analysis)***:* Students establish the properties of geometric figures by means of an informal analysis of their component parts and begin to recognize them by their properties.
- L_3 **(Abstraction)***:* Students become able to relate the properties of figures, to distinguish between the necessity and sufficiency of a set of properties in determining a concept and to form abstract definitions.
- L_4 **(Deduction):** Students reason formally within the context of a geometric system and they gasp the significance of deduction as means of developing geometric theory.
- L_5 **(Rigor):** Students understand the foundations of geometry and can compare geometric systems based on different axioms.

Obviously the level L_5 is very difficult, if not impossible, to appear in secondary classrooms, while level L_4 also appears very rarely.

Although van Hiele [7] claimed that the above levels are discrete – which means that the transition from a level to the next one does not happen gradually but all at once – alternative researches by Burnes & Shaughnessy [9], Fuys et al. [10], Wilson [11], Guttierrez et al. [12] and by Perdikaris [13] suggest that the vH levels are *continuous* characterized by transitions between the adjacent levels.

III. FUZZY RELATION EQUATIONS

First recall that a **Fuzzy Set (FS)** A on the universe U, introduced by Zadeh in 1965 [14], is determined by a map *m*: U $\rightarrow$ [0, 1], called the *membership function (MF)* of A. The book [15] is proposed as a general reference for readers being not familiar with the basic principles of FS theory.

Definition 1: Let X, Y be two crisp sets. Then a *fuzzy binary relation (FBR)* R(X, Y) is a FS on the Cartesian product X x Y of the form:

$$R(X, Y) = \{(r, m_R(r)) : r = (x, y) \in X \times Y\},$$

where $m_R : X \times Y \rightarrow [0, 1]$ is the corresponding MF.

When X = $\{x_1, \ldots\ldots\ldots, x_n\}$ and Y = $\{y_1, \ldots\ldots, y_m\}$, then a FBR R(X, Y) can be represented by a n x m matrix of the form:

$$R = \begin{array}{c} \\ x_1 \\ . \\ . \\ . \\ x_n \end{array} \begin{array}{c} y_1 \ldots\ldots\ldots\ldots \; y_m \\ \begin{pmatrix} r_{11} & . & . & . & r_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ r_{m1} & . & . & . & r_{mn} \end{pmatrix} \end{array} = [r_{ij}]$$

In the above matrix $r_{ij} = m_R(x_i, y_j)$, with i = 1,…, n and j =1,…, m. The matrix R is called the **membership matrix** of the FBR R(X, Y).

The basic ideas of fuzzy relations, which were introduced by Zadeh [16] and were further investigated by other researchers, are extensively covered in the book [17].

Definition 2: Consider two FBRs P(X, Y) and Q(Y, Z) with a common set Y. Then, the standard composition of these relations, which is denoted by
P(X, Y) $\circ$ Q(Y, Z) produces a FBR R(X, Z) with MF m_R defined by:

$$m_R(x_i, z_j) = \max_{y \in Y} \min\,[m_P(x_i, y), m_Q(y, z_j)] \quad (1),$$

for all i=1,…,n and all j=1,…,m. This composition is often referred as *max-min composition.*

Compositions of FBRs are conveniently performed in terms of their membership matrices. In fact, if P = $[p_{ik}]$ and Q=$[q_{kj}]$ are the membership matrices of the relations P(X, Y) and Q(Y, Z) respectively, then by relation (1) we get that the membership matrix of R(X, Y) = P(X, Y) $\circ$ Q(Y, Z) is the matrix R = $[r_{ij}]$, with

$$r_{ij} = \max_k \min(p_{ik}, q_{kj}) \qquad (2)$$

Example 1: If $P = \begin{array}{c} \\ x_1 \\ x_2 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 0.2 & 0.4 & 0.8 \\ 0.1 & 0.5 & 1 \\ 0.4 & 0.7 & 0.3 \end{pmatrix} \end{array}$ and

$Q = \begin{array}{c} \\ y_1 \\ y_2 \\ y_2 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 0.2 & 0.7 & 0 & 0.4 \\ 0.8 & 0.1 & 0.5 & 0.6 \\ 1 & 0.3 & 0.2 & 0.9 \end{pmatrix} \end{array}$ are the membership matrices of

P(X, Y) and Q(Y, Z) respectively, then by equation (2) the membership matrix of R(X, Z) is the matrix

$$R = P \circ Q = \begin{array}{c} \\ x_1 \\ x_2 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 0.8 & 0.3 & 0.4 & 0.8 \\ 1 & 0.3 & 0.5 & 0.9 \\ 0.7 & 0.4 & 0.5 & 0.6 \end{pmatrix} \end{array}.$$

Observe that the same elements of P and Q are used in the calculation of R as would be used in the regular multiplication of matrices, but the product and sum operations are here replaced with the min and max operations respectively.

Definition 3: Consider the FBRs P(X, Y), Q(Y, Z) and R(X, Z), defined on the sets, $X = \{x_i : i \in N_n\}$, $Y = \{y_j : j \in N_m\}$, $Z = \{z_k : k \in N_s\}$, where $N_t = \{1,2,\dots,t\}$, for t = n, m, k, and let $P=[p_{ij}]$, $Q=[q_{jk}]$ and $R=[r_{ik}]$ be the membership matrices of P(X, Y), Q(Y, Z) and R(X, Z) respectively. Assume that the above three relations constrain each other in such a way that $P \circ Q = R$, where $\circ$ denotes the max-min composition. Therefore, for each i in N_n and each k in N_s we have that

$$\max_{j \in J} [\min (p_{ij}, q_{jk})] = r_{ik} \quad (3).$$

Therefore the matrix equation $P \circ Q = R$ encompasses n x s simultaneous equations of the form (3). When two of the components in each of the equations (3) are given and one is unknown, these equations are referred as FRE.

The notion of FRE was first proposed by Sanchez [18] and was further investigated by other researchers [19, 20, 21].

IV. FRE ON THE VH LEVELS OF GEOMETRIC REASONING

Let us consider the crisp sets X = {M}, Y = {A, B, C, D, F} and $Z = \{L_1, L_2, L_2, L_4\}$, where M denotes the "*average student*" of a class, A = Excellent, B = Very Good, C = Good, D = Fair and F = Failed are *linguistic labels (grades)* used for the assessment of the student performance and L_1, L_2, L_2, L_4 are the first four of the vH levels of geometric reasoning. The last level L_5 has been omitted, since its appearance in the classroom is almost impossible.

Further, let n be the total number of students of a certain class and let n_i be the numbers of students who obtained the grade i assessing their performance, $i \in Y$. Then one can represent the average student of the class as a FS on Y in the form

$$M = \{(i, \frac{n_i}{n}): i \in Y\}.$$

The FS M induces a FBR P(X, Y) with membership matrix

$$P = [\frac{n_A}{n} \ \frac{n_B}{n} \ \frac{n_C}{n} \ \frac{n_D}{n} \ \frac{n_F}{n}].$$

In an analogous way the average student of the class can be represented as a FS on Z of the form

$$M = \{(j, m(j)): j \in Z\},$$

where $m: Z \rightarrow [0, 1]$ is the corresponding MF. In this case the FS M induces a FBR (X, Z) with membership matrix

$$R = [m(L_1) \ \ m(L_2) \ \ m(L_3) \ \ m(L_4)].$$

We consider also the fuzzy binary relation Q(Y, Z) with membership matrix the 5X4 matrix $Q = [q_{ij}]$, where $q_{ij} = m(i, j)$ with $i \in Y$ and $j \in Z$ and the FRE encompassed by the matrix equation $P \circ Q = R$. When the matrix Q is fixed and the row-matrix P is known, then the equation $P \circ Q = R$ has always a unique solution with respect to R, which enables the representation of the average student of a class as a FS on the set of the first four vH levels. This is useful for the instructor for designing his/her future teaching plans. On the contrary, when the matrices Q and R are known, then the equation $P \circ Q = R$ could have no solution or could have more than one solution with respect to P, which makes the corresponding situation more complicated.

All the above will be illustrated in the next section with suitable examples.

V. EXAMPLES

Example 2: The following study was performed in collaboration with the teacher of mathematics on a class of 60 students aged 15 years of a secondary school in the city of Patras, Greece. After the end of the teaching of similar polygons a written test was given to students, the results of which are depicted in Table 1.

Table 1. Student Performance

Grade	No. of Students
A	30
B	10
C	7
D	5
F	8
Total	60

In this case the average student M of the class can be represented as a FS on Y = {A, B, C, D, F} by

$$M = \{(A, \frac{30}{60}), (B, \frac{10}{60}), (C, \frac{7}{60}), (D, \frac{5}{60}), (F, \frac{8}{60})\}$$

$$\approx \{(A, 0.5), (B, 0.17), (C, 0.12), (D, 0.08), (F, 0.13)\}.$$

Therefore M induces a FBR P(X, Y), where X={M}, with membership matrix

$$P = [0.5 \ 0.17 \ 0.12 \ 0.08 \ 0.13].$$

Also, using statistical data of the last five years on student geometric reasoning skills we fixed the membership matrix Q of the FBR Q(Y, Z), where $Z = \{L_1, L_2, L_2, L_4\}$, in the form:

$$Q = \begin{array}{c} \\ A \\ B \\ C \\ D \\ F \end{array} \begin{array}{c} \begin{array}{cccc} L_1 & L_2 & L_3 & L_4 \end{array} \\ \begin{pmatrix} 0.8 & 0.6 & 0.2 & 0 \\ 0.2 & 0.2 & 0.2 & 0.1 \\ 0 & 0.1 & 0.3 & 0.1 \\ 0 & 0.1 & 0.1 & 0.2 \\ 0 & 0 & 0.2 & 0.6 \end{pmatrix} \end{array}$$

Next, using the max-min composition of FBRs one finds that the membership matrix of R(X, Z) = P(X, Y) o Q (Y, Z) is equal to

$$R = P \text{ o } Q = [0.5 \ \ 0.5 \ \ 0.2 \ 0.13].$$

Therefore the average student of the class can be expressed as a fuzzy set on Z by

$$M = \{(L_1, 0.5), (L_2, 0.5), (L_3, 0.2), (L_4, 0.13)\}.$$

The conclusions obtained from the above expression of M are the following:

- Half of the students of the class became able to recognize the figures of similar polygons (L_1) and to establish their properties (L_2).
- On the contrary, only the 20% of the students became able to distinguish between the necessity and sufficiency of a set of properties for forming abstract definitions of the corresponding concepts (L_3).
- Finally, only the 13% of the students were able to reason formally and tackled satisfactorily the proofs of the corresponding results.

It becomes evident that the above solutions were very useful for the teacher for reorganizing his future teaching plans in order to achieve better results on the student understanding of the similar polygons.

Let us now consider the case where the membership matrices Q and R are known and we want to determine the matrix P representing the average student of the class as a fuzzy set on Y. This is a complicated case because we may have more than one solution or no solution at all. The following two examples illustrate this situation:

Example 3: Consider the membership matrices Q and R of the previous example and set $P = [p_1\ p_2\ p_3\ p_4\ p_5]$.

Then the matrix equation P o Q = R encompasses the following equations:

$\max \{\min (p_1, 0.8), \min (p_2, 0.2), \min (p_3, 0), \min (p_4, 0), \min(p_5, 0)\}= 0.5$

$\max \{\min (p_1, 0.6), \min (p_2, 0.2), \min (p_3, 0.1), \min (p_4, 0.1), \min(p_5, 0)\}= 0.5$

$\max \{\min (p_1, 0.2), \min (p_2, 0.2), \min (p_3, 0.3), \min (p_4, 0.1), \min (p_5, 0.2)\}= 0.2$

$\max \{\min (p_1, 0), \min (p_2, 0.1), \min (p_3, 0.1), \min (p_4, 0.2), \min(p_5, 0.6)\}= 0.13$

The first of the above equations is true if, and only if, $p_1 = 0.5$, which satisfies the second and third equations as well. Also, the fourth equation is true if, and only if, $p_3 = 0.13$ or $p_4 = 0.13$ or $p_5 = 0.13$. Therefore, any combination of values of p_1, p_2, p_3, p_4, p_5 in [0, 1] such that $p_1 = 0.5$ and $p_3 = 0.13$ or $p_4 = 0.13$ or $p_5 = 0.13$ is a solution of P o Q = R.

Let S(Q, R) = {P: P o Q = R } be the set of all solutions of P o Q = R. Then one can define a partial ordering on S(Q, R) by

$$P \leq P' \Leftrightarrow p_i \leq p'_i, \ \forall\, i = 1, 2, 3, 4, 5.$$

It is well known that whenever S(Q, R) is a non empty set, it always contains a unique maximum solution and it may contain several minimal solutions [18]. It is further known that S(Q, R) is fully characterized by the maximum and minimal solutions in the sense that all its other elements are between the maximal and each of the minimal solutions [18]. A method of determining the maximal and minimal solutions of P o Q = R with respect to P is developed in [21].

Example 4: Let $Q = [q_{ij}]$, i = 1, 2, 3, 4, 5 and j = 1, 2, 3, 4 be as in Example 2 and let R = [1 0.5 0.2 0.13]. Then the first equation encompassed by the matrix equation P o Q = R is $\max \{\min (p_1, 0.8), \min (p_2, 0.2), \min (p_3, 0), \min (p_4, 0), \min(p_5, 0)\}= 1$.

In this case it is easy to observe that the above equation has no solution with respect to p_1, p_2, p_3, p_4, p_5, therefore P o Q = R has no solution with respect to P.

In general, writing $R = \{r_1\ r_2\ r_3\ r_4\}$, it becomes evident that we have no solution if $\max_j q_{ij} < r_j$, for some j = 1, 2, 3, 4.

VI. CONCLUSION

In the present work we have considered a high-school class of *n* students learning Euclidean Geometry and we have applied a FRE model on the vH levels of student geometric reasoning defined with the help of three FBRs with membership matrices P, Q and R respectively satisfying the equation P o Q = R.

The matrix $P = [\frac{n_A}{n}\ \frac{n_B}{n}\ \frac{n_C}{n}\ \frac{n_D}{n}\ \frac{n_F}{n}]$, where n_i denotes the number of students whose progress has been assessed by the grade i = A, B, B, C, D, F, is fixed representing the "average student" of the class. On the other hand, $Q = [q_{ij}]$ is the 5 x 4 matrix in which q_{ij} denotes the membership degree of (i, j) in the FBR Q, where j = L_1, L_2, L_3, L_4 represent the first four vH levels (the fifth and last vH level has been omitted, since it is almost impossible to appear in a secondary student class). Also $R= [m(L_1)\ m(L_2)\ m(L_3)\ m(L_4)]$ is the matrix of the membership degrees of the first four vH levels in the FBR Q.

When the matrix Q is known (from statistical data), then the equation P o Q = R has a unique solution with respect to R that gives valuable information to the teacher about the student progress in learning the corresponding geometric topic. On the contrary, when the matrix R is known, then the above equation has more than one or no solution at all with respect to Q, which makes the situation to be more complicated.

Through the examples, illustrating the applicability of our model in real situations, it becomes evident that the FRE theory is a very useful tool that could be used not only for student, but also for the assessment of other human or machine (e.g. computers [22]) activities.

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