Finding simple roots by seventh- and eighth-order derivative-free methods

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Abstract- Nonlinear equation solving by without memory iterative methods is taken into account in the present research. Recently, Khattri and Argyros in [S.K. Khattri, I.K. Argyros, Sixth order derivative free family of iterative methods, Appl. Math. Comput. 217 (2011), 5500-5507], proposed a sixth-order family of derivative-free methods including four function evaluations per full cycle to reach the index of efficiency 1.565. In this work, we develop new derivative-free without memory methods, based on the above-mentioned work, in which the convergence rates reach the seventh- and eighth-order respectively. And subsequently, the index of efficiency will be increased to 1.626 and 1.682. This shows that our proposed methods are more economic than their work in terms of convergence rate and the efficiency index. Moreover, the numerical examples are considered to support the theoretical results and put on show that the contributions in this paper hit the targets.

Keywords- Nonlinear equations, simple root, iterative methods, derivative-free, efficiency index, order of convergence, Steffensens's method.

I. PREREQUASITIES

There are many iterative methods to find the simple roots of single valued nonlinear equations [5, 6, 9, 10, 11, 13, 18]. It could be mentioned that all of them have some strong points and drawbacks. For example, the best known iterative method, which was given by Newton-Raphson includes one derivative evaluation. In application-oriented problems and/or when the evaluation of derivatives of the nonlinear functions is not possible and/or takes up a high computational time, the use of such derivative-involved methods is limited; or they are very hard to implement. To remedy this, first Steffensen gave an iterative method, $x_{n+1} = x_n - [f(x_n)^2]/[f(x_n) - f(x_n - f(x_n))]$, with two function evaluations and the same convergence rate as Newton-Raphson's method.

As a matter of fact, he replaced the first-order derivative evaluation in the Newton-Raphson's iteration by backward finite difference (FD) approximation (or forward FD) of order one. Even though the problem of derivative evaluation was solved by Steffensen, there exist another problem in the attained scheme, that is to say, the convergence rate is only two.

In addition, by using the definition of efficiency index, which was given in the book of Traub [3], the classical efficiency index of Steffensen's or Newton-Raphson's schemes are $2^{1/2} \approx 1.4142$. To overcome on this barrier, many researchers have been trying to increase the convergence rate and efficiency index of the known methods; see for example [1, 7, 8, 15]. In fact, the procedure to construct better schemes is to compose some without memory schemes to each other; e.g. Newton-Raphson with Newton-Raphson or Steffensen with Newton-Raphson, etc., and then approximate the new-appeared first derivatives of the function by a means of already known data.

To provide third or fourth order of convergence, one should consider a two-step cycle and to give methods of orders more or equal than 5 and also less or equal than eight, one should take into account of three-step cycles. Such iterative methods are also known as multi-point high-order iterations [3].

In 2011, Khattri and Argyros presented a sixth-order family of multi-point derivative-free methods including three steps; four parameters ($\kappa \in \mathbb{R} - \{0\}$, α, β, and $\eta \in \mathbb{R}$); and four function evaluations as comes next [1]

$$\begin{cases} y_n = x_n - \dfrac{\kappa f(x_n)^2}{f(x_n) - f(x_n - \kappa f(x_n))}, \\ z_n = y_n - \dfrac{\kappa f(x_n) f(y_n)}{f(x_n) - f(x_n - \kappa f(x_n))}\left[1 + \dfrac{f(y_n)}{f(x_n)} + \dfrac{f(y_n)}{f(x_n - f(x_n))} + \alpha\left(\dfrac{f(y_n)}{f(x_n)}\right)^2 + \beta\left(\dfrac{f(y_n)}{f(x_n - f(x_n))}\right)^2\right], \\ x_{n+1} = z_n - \dfrac{\kappa f(x_n) f(z_n)}{f(x_n) - f(x_n - \kappa f(x_n))}\left[1 + \dfrac{f(y_n)}{f(x_n)} + \dfrac{f(y_n)}{f(x_n - f(x_n))} + \alpha\left(\dfrac{f(y_n)}{f(x_n)}\right)^2 + \beta\left(\dfrac{f(y_n)}{f(x_n - f(x_n))}\right)^2 + \eta\dfrac{f(z_n)}{f(y_n)}\right]. \end{cases} \tag{1}$$

In this work, we develop (1) by providing seventh- and eighth-order derivative-free classes of methods including the same number of function evaluations as (1), but with better efficiency indices. Due to this, the rest of this research unfolds the contents as follows. Section II furnishes new techniques of two-parameter without memory iterative methods with seventh-order convergence in details. This section is follows by Section III where a new class of derivative-free methods will be constructed. The numerical comparisons are made in Section IV to reveal the efficacy of the proposed contributions in this paper. Finally, Section V is provided to give a conclusion.

II. New Seventh-Order Techniques

Let us consider the one variable function $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be sufficiently smooth and for simplicity assume that $w_n = x_n - f(x_n)$, while $\kappa = 1$ is also taken into account in (1). Note that the existing of κ is only needed if we try to provide with memory iterations, which is not in our scope at the moment. However, we use this free parameter in the next section.

According to [1], the iteration (1) reaches to the second order at the end of the first step; fourth-order at the end of the second one; and sixth-order of convergence at the end of the third step by using four function evaluations per full cycle. In fact, Khattri and Argyros have considered a weight function in the third step of (1) to reach the convergence order six.

In order to develop (1) and reaches a family of iterations with better convergence rate and efficiency index; we propose the weight function in the last step in what follows

$$W(x_n, w_n, y_n, z_n) = 1 + [2 - f[x_n, w_n]]\frac{f(y_n)}{f(w_n)} + \left[\frac{1}{1 - f[x_n, w_n]}\right]\left(\frac{f(y_n)}{f(x_n)}\right)^2 + \frac{f(z_n)}{f(y_n)} + \theta\frac{f(z_n)}{f(x_n)} + \tau\frac{f(z_n)}{f(w_n)}, \qquad (2)$$

three-step without memory iterative family of bi-parametric methods for solving one variable nonlinear equations wherein $\theta, \tau \in \mathbb{R}$ and $f[x_n, w_n]$ is the divided difference which can be described as $f[x_n, w_n] = \frac{f(x_n) - f(w_n)}{f(x_n)}$. Thus, using (2) and for simplicity $\alpha = \beta = 0$, we attain the following

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, \\ z_n = y_n - \dfrac{f(y_n)}{f[x_n, w_n]}\left[1 + \dfrac{f(y_n)}{f(x_n)} + \dfrac{f(y_n)}{f(w_n)}\right], \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f[x_n, w_n]}\left[1 + [2 - f[x_n, w_n]]\dfrac{f(y_n)}{f(w_n)} + \left[\dfrac{1}{1 - f[x_n, w_n]}\right]\left(\dfrac{f(y_n)}{f(x_n)}\right)^2 + \dfrac{f(z_n)}{f(y_n)} + \theta\dfrac{f(z_n)}{f(x_n)} + \tau\dfrac{f(z_n)}{f(w_n)}\right]. \end{cases} \qquad (3)$$

Theorem 1 indicates that (3) possesses the seventh-order of convergence.

Theorem 1. *Let us consider α as the simple zero of the nonlinear equation $f(x) = 0$ in the domain D. And assume that $f(x)$ is sufficiently smooth in the neighborhood of the root, i.e. D. Then, the derivative-free iterative family which is defined by (3) is of order seven and satisfies in the following error equation*

$$e_{n+1} = -\frac{1}{c_1^6}(-1 + c_1)c_2^2((5 + (-5 + c_1)c_1)c_2^2 + (-1 + c_1)c_1c_3)((-1 + c_1)c_1c_3(4 + c_1(-2 + \theta) - \theta - \tau) + c_2^2(16 - 5\theta - 5\tau + c_1(-24 + 10\theta + c_1(10 + c_1(-1 + \theta) - 6\theta - \tau) + 5\tau)))e_n^7 + O(e_n^8). \qquad (4)$$

wherein $c_k = f^{(k)}(\alpha)/k!, \forall k = 1,2,3, \ldots$, $e_n = x_n - \alpha$ and θ, τ are two real valued free parameters.

Proof. Using Taylor's series and symbolic computation; we can determine the asymptotic error constant of the three-step without memory family of methods (3). Now we expand $f(x_n)$ about the simple zero α. Hence, we have

$$f(x_n) = c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + O(e_n^8). \qquad (5)$$

Taking into account of (5), we also have

$$\frac{f(x_n)}{f[x_n, w_n]} = e_n^1 + \frac{(-1 + c_1)c_2e_n^2}{c_1} + \frac{1}{c_1^2}((2 + (-2 + c_1)c_1)c_2^2 - (-2 + c_1)(-1 + c_1)c_1c_3)e_n^3 + \frac{1}{c_1^3}((-4 + c_1(5 + (-3 + c_1)c_1))c_2^3 + c_1(7 + c_1(-10 + (7 - 2c_1)c_1))c_2c_3 + (-1 + c_1)c_1^2(3 + (-3 + c_1)c_1)c_4)e_n^4 + O(e_n^5).$$

By considering (5) and the first step of (3), we obtain

$$y_n = \alpha + \left(-1 + \frac{1}{c_1}\right)c_2e_n^2 + \frac{(-(2 + (-2 + c_1)c_1)c_2^2 + c_1(-1 + c_1)(-2 + c_1)c_3)}{c_1^2}e_n^3 + \cdots + O(e_n^8). \qquad (6)$$

We should expand $f(y_n)$ around the root and using (6). Accordingly, we have

$$f(y_n) = (c_2 - c_1c_2)e_n^2 + \left(\left(2 - \frac{2}{c_1} - c_1\right)c_2^2 + (-2+c_1)(-1+c_1)c_3\right)e_n^3 + \cdots + O(e_n^8). \tag{7}$$

Writing the Taylor's series expansion at the end of the second step by using (7), we get that

$$z_n = \alpha - \frac{(-1+c_1)c_2\big((5+(-5+c_1)c_1)c_2^2 + (-1+c_1)c_1c_3\big)}{c_1^3}e_n^4 + \frac{1}{c_1^4}(\big(-36+c_1\big(80-3c_1(22+(-8+c_1)c_1)\big)\big)c_2^4$$
$$+(-1+c_1)c_1\left(-32+c_1\big(46+c_1(-22+3c_1)\big)\right)c_2^2c_3 + (-2+c_1)(-1+c_1)^2c_1^2c_3^2 + (-2+c_1)(-1+c_1)^2c_1^2c_2c_4)e_n^5$$
$$+\cdots+O(e_n^8). \tag{8}$$

Additionally, we have

$$f(z_n) = \frac{(-1+c_1)c_2\big((5+(-5+c_1)c_1)c_2^2 + (-1+c_1)c_1c_3\big)}{c_1^2}e_n^4 + \frac{1}{c_1^3}(\big(-36+c_1\big(80-3c_1(22+(-8+c_1)c_1)\big)\big)c_2^4$$
$$+(-1+c_1)c_1\left(-32+c_1\big(46+c_1(-22+3c_1)\big)\right)c_2^2c_3 + (-2+c_1)(-1+c_1)^2c_1^2c_3^2 + (-2+c_1)(-1+c_1)^2c_1^2c_2c_4)e_n^5$$
$$+\cdots+O(e_n^8). \tag{9}$$

At this time it is required to find the Taylor's series expansion of the proposed weight function (2) about the simple root. Therefore, we have

$$W(x_n,w_n,y_n,z_n) = 1 + \left(-1+\frac{2}{c_1}\right)c_2e_n^1 + \frac{(-c_2^2+(3+(-3+c_1)c_1)c_3)}{c_1}e_n^2 + \frac{1}{c_1^3}(-2+c_1)c_1^2(2+(-2+c_1)c_1)c_4$$
$$-c_1c_2c_3(-4+\theta+c_1(10+c_1(-4+\theta)-2\theta-\tau)+\tau) + c_2^3(-16+5\theta+5\tau$$
$$+c_1\big(24-10\theta-5\tau+c_1(-10+c_1+6\theta-c_1\theta+\tau)\big))e_n^3 + \cdots + O(e_n^8). \tag{10}$$

Consequesntly, using (8), (9) and (10) results in

$$e_{n+1} = x_{n+1} - \alpha = -\frac{1}{c_1^6}(-1+c_1)c_2^2((5+(-5+c_1)c_1)c_2^2 + (-1+c_1)c_1c_3)((-1+c_1)c_1c_3(4+c_1(-2$$
$$+\theta)-\theta-\tau) + c_2^2(16-5\theta-5\tau+c_1(-24+10\theta+c_1(10+c1(-1+\theta)-6\theta-\tau)+5\tau)))e_n^7 + O(e_n^8), \tag{11}$$

which shows that our bi-parametric family of derivative-free without memory methods reaches the convergence rate seven by consuming only four function evaluations per full cycle. This completes the proof. ■

Taking into account (3), and the number of evaluations per full cycle; we obtain that the efficiency index of (3) is $7^{1/4} \approx 1.626$, which is bigger than that of (1), i.e. 1.565 and lots of other available derivative-free methods in literature. This shows that our proposed scheme is more economic than (1) in terms of the convergence rate and efficiency index for finding simple root. That is to say, with the same number of evaluations with (1), we have provided a more robust scheme. To have a more simplified version of (3), if we consider $\theta = \tau = 0$, then we attain the following three-step without memory method

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n,w_n]}, \\ z_n = y_n - \dfrac{f(y_n)}{f[x_n,w_n]}\left[1 + \dfrac{f(y_n)}{f(x_n)} + \dfrac{f(y_n)}{f(w_n)}\right], \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f[x_n,w_n]}\left[1 + [2 - f[x_n,w_n]]\dfrac{f(y_n)}{f(w_n)} + \left[\dfrac{1}{1-f[x_n,w_n]}\right]\left(\dfrac{f(y_n)}{f(x_n)}\right)^2 + \dfrac{f(z_n)}{f(y_n)}\right], \end{cases} \tag{12}$$

where its error equation can be given as comes next

$$e_{n+1}=\frac{(-2+c_1)(-1+c_1)c_2^2\big((5+(-5+c_1)c_1)c_2^2+(-1+c_1)c_1c_3\big)\big((8+(-8+c_1)c_1)c_2^2+2(-1+c_1)c_1c_3\big)}{c_1^6}e_n^7 +O(e_n^8). \tag{13}$$

III. Novel Optimal Eighth-Order Methods

The only problem with the family of iterations given in the previous section is that, its computational efficiency is less than that of optimal three-step without memory iterations in the sense of Kung and Traub [2]. Therefore, it would be interesting from practical and analytical point of view to increase the order of convergence from seven to eight without forcing more function evaluations per full computing step. Toward this new goal, we make use of weight function approach.

We consider the following very general three-step four-point without memory iteration including one free non-zero parameter. As we mentioned before, this parameter is so much useful in building with memory high order iterations.

$$\begin{cases} y_n=x_n-\dfrac{f(x_n)}{f[x_n,w_n]}, w_n=x_n-\beta f(x_n),\\ z_n=y_n-\dfrac{f(y_n)}{f[x_n,w_n]}\left[1+\dfrac{f(y_n)}{f(x_n)}+\dfrac{f(y_n)}{f(w_n)}\right],\\ x_{n+1}=z_n-\dfrac{f(z_n)}{f[x_n,w_n]}\left[G\left(\dfrac{f(y_n)}{f(w_n)}\right)+H\left(\dfrac{f(z_n)}{f(y_n)}\right)+K\left(\dfrac{f(z_n)}{f(w_n)}\right)\right]. \end{cases} \tag{14}$$

Clearly, iteration class (14) requires four function evaluations and is free from any derivative evaluation in seeking for the simple roots of nonlinear scalar equations. Thus, now the weight functions in (14) should be chosen such that the class (14) hits the eighth-order of convergence and be optimal in the sense of Kung and Traub. This is done in Theorem 2.

Theorem 2. *Let us consider α as the simple zero of the nonlinear equation $f(x)=0$ in the domain D. And assume that $f(x)$ is sufficiently smooth in the neighborhood of the root, i.e. D. Then, the derivative-free iterative class, which is defined by (14) is of order eight when*

$$\begin{cases} G(0)=1, G'(0)=2-\beta f[x_n,w_n], G''(0)=2-2\beta f[x_n,w_n],\\ G^{(3)}(0)=-24+6\beta f[x_n,w_n]\big(6+\beta f[x_n,w_n](-4+\beta f[x_n,w_n])\big),\\ \text{and } |G^{(4)}(0)|\le\infty,\\ H(0)=0, H'(0)=1, \text{and } |H''(0)|\le\infty\\ K(0)=0, K'(0)=4-2\beta f[x_n,w_n], \text{and } |K''(0)|\le\infty, \end{cases} \tag{15}$$

Proof. Using $c_k=f^{(k)}(\alpha)/k!, \forall k=1,2,3,\ldots$, $e_n=x_n-\alpha$, Taylor's series and symbolic computation; we can have the similar relation as the proof of the Theorem 1. Thus, here we only include the following error equations

$$\frac{f(z_n)}{f[x_n,w_n]}=-\frac{1}{c_1^3}c_2(-1+c_1\beta)(c_1c_3(-1+c_1\beta) +c_2^2(5+c_1\beta(-5+c_1\beta)))e_n^4+\frac{1}{c_1^4}(c_1^2c_3^2(-2 +c_1\beta)(-1+c_1\beta)^2+c_1^2c_2c_4(-2+c_1\beta)(-1+c_1\beta)^2 +c_1c_2^2c_3(-1+c_1\beta)(-34+c_1\beta(49+c_1\beta(-23 +3c_1\beta)))+c_2^4(-46+c_1\beta(105-4c_1\beta(22 +c_1\beta(-8+c_1\beta)))))e_n^5+\cdots+O(e_n^9). \tag{16}$$

and

$$z_n-\frac{f(z_n)}{f[x_n,w_n]}\left[G\left(\frac{f(y_n)}{f(w_n)}\right)+H\left(\frac{f(z_n)}{f(y_n)}\right)+K\left(\frac{f(z_n)}{f(w_n)}\right)\right] -\alpha=\frac{1}{c_1^3}c_2(-1+c_1\beta)(c_1c_3(-1+c_1\beta)+c_2^2(5 +c_1\beta(-5+c_1\beta)))(-1+G(0)+H(0)+K(0))e_n^4 -\frac{1}{c_1^4}(c_1^2c_3^2(-2+c_1\beta)(-1+c_1\beta)^2(-1+G(0) +H(0)+K(0))+c_1^2c_2c_4(-2+c_1\beta)(-1+c_1\beta)^2(-1 +G(0)+H(0)+K(0))+c_1c_2^2c_3(-1+c_1\beta)(32-34G(0) -34H(0)-34K(0)+c_1\beta(-46+49G(0)+49H(0) +49K(0)+c_1\beta(22-23G(0)-23H(0)-23K(0) +3c_1\beta(-1+G(0)+H(0)+K(0)))-G'(0))+G'(0)) -c_2^4(-36+46G(0)+46H(0)+46K(0)-5G'(0) +c_1\beta(c_1\beta((22+c_1\beta(-8+c_1\beta))(-3+4G(0) +4H(0)+4K(0))+(-6+c_1\beta)G'(0))-5(21G(0) +21H(0)+21K(0)-2(8+G'(0))))))e_n^5 +\cdots+O(e_n^9). \tag{17}$$

This clearly reveals that the weight functions in (14) must be chosen as (15) to make the order optimal. Thus, using (15) in (14), we can have the following general error equations for the optimal three-step derivative-free class (14)

$$e_{n+1}=\frac{1}{24c_1^7}c_2(-1+c_1\beta)(c_1c_3(-1+c_1\beta)+c_2^2(5 +c_1\beta(-5+c_1\beta)))(-24c_1^2c_2c_4(-1+c_1\beta)^2 +12c_1^2c_3^2(-1+c_1\beta)^2(-2+H''(0))+24c_1c_2^2c_3(-1 +c_1\beta)(-19+5H''(0)+c_1\beta(19-3c_1\beta+(-5 +c_1\beta)H''(0)))+c_2^4(12(5+c_1\beta(-5+c_1\beta))(-2(9 +c_1\beta(-9+c_1\beta))+(5+c_1\beta(-5+c_1\beta))H''(0)) +G^{(4)}(0)))e_n^8+O(e_n^9). \tag{18}$$

This concludes the proof and shows that our class of iteration reaches the optimal convergence order eight using four function evaluations. ■

Although available high-order derivative-involved methods in literature are quite powerful, see e.g [4, 12, 14, 16, 17], there are many obstacles to using them on real-world applications. When someone first implements one of those schemes, the most common observations are that some derivative information is not available or hardly to calculate. Once the method is working, then attention is usually turned to the computational derivative evaluation cost, which can be prohibitive for medium to large-scale optimization problems. But the contribution presented in this research article overcomes on this drawback. Now, by considering the condition given in (15) for the weight functions in (14), we can produce any eighth-order method free from derivative. Some of such optimal schemes are listed below. For example, we have

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, w_n = x_n - f(x_n), \\ z_n = y_n - \dfrac{f(y_n)}{f[x_n, w_n]}\left[1 + \dfrac{f(y_n)}{f(x_n)} + \dfrac{f(y_n)}{f(w_n)}\right], \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f[x_n, w_n]}[A_1], \end{cases} \tag{19}$$

where

$$A_1 = 1 + (2 - f[x_n, w_n])\frac{f(y_n)}{f(w_n)} + (1 - f[x_n, w_n])\left(\frac{f(y_n)}{f(w_n)}\right)^2 + \Big(-4 + f[x_n, w_n]\big(6 + f[x_n, w_n](-4 + f[x_n, w_n])\big)\Big)\left(\frac{f(y_n)}{f(w_n)}\right)^3 + \frac{f(z_n)}{f(y_n)} + (4 - 2f[x_n, w_n])\frac{f(z_n)}{f(w_n)}.$$

Another efficient derivative-free method can be defined as

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, w_n = x_n - f(x_n), \\ z_n = y_n - \dfrac{f(y_n)}{f[x_n, w_n]}\left[1 + \dfrac{f(y_n)}{f(x_n)} + \dfrac{f(y_n)}{f(w_n)}\right], \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f[x_n, w_n]}[A_2], \end{cases} \tag{20}$$

with

$$A_2 = 1 + (2 - f[x_n, w_n])\frac{f(y_n)}{f(w_n)} + (1 - f[x_n, w_n])\left(\frac{f(y_n)}{f(w_n)}\right)^2 + \Big(-4 + f[x_n, w_n]\big(6 + f[x_n, w_n](-4 + f[x_n, w_n])\big)\Big)\left(\frac{f(y_n)}{f(w_n)}\right)^3 + \frac{f(z_n)}{f(y_n)} + \left(\frac{f(z_n)}{f(y_n)}\right)^2 + (4 - 2f[x_n, w_n])\frac{f(z_n)}{f(w_n)}.$$

We can also have the following iteration free from derivative from the class (14)-(15)

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, w_n = x_n - f(x_n), \\ z_n = y_n - \dfrac{f(y_n)}{f[x_n, w_n]}\left[1 + \dfrac{f(y_n)}{f(x_n)} + \dfrac{f(y_n)}{f(w_n)}\right], \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f[x_n, w_n]}[A_1], \end{cases} \tag{21}$$

$$A_3 = 1 + (2 - f[x_n, w_n])\frac{f(y_n)}{f(w_n)} + (1 - f[x_n, w_n])\left(\frac{f(y_n)}{f(w_n)}\right)^2 + \Big(-4 + f[x_n, w_n]\big(6 + f[x_n, w_n](-4 + f[x_n, w_n])\big)\Big)\left(\frac{f(y_n)}{f(w_n)}\right)^3 + \frac{f(z_n)}{f(y_n)} + \left(\frac{f(z_n)}{f(y_n)}\right)^2 + (4 - 2f[x_n, w_n])\frac{f(z_n)}{f(w_n)} + \left(\frac{f(z_n)}{f(w_n)}\right)^2.$$

As we mentioned above, the free non-zero parameter β can be chosen to reach better accuracy. In this regard, we consider $\beta = \frac{1}{100}$, to produce better root solver without derivative computation per full cycle

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, w_n = x_n - \dfrac{1}{100}f(x_n), \\ z_n = y_n - \dfrac{f(y_n)}{f[x_n, w_n]}\left[1 + \dfrac{f(y_n)}{f(x_n)} + \dfrac{f(y_n)}{f(w_n)}\right], \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f[x_n, w_n]}[A_4], \end{cases} \tag{22}$$

with

$$A_4 = 1 + \left(2 - \frac{1}{100}f[x_n, w_n]\right)\frac{f(y_n)}{f(w_n)} + (1 - \frac{1}{100}f[x_n, w_n])\left(\frac{f(y_n)}{f(w_n)}\right)^2 + (-4 + \frac{1}{100}f[x_n, w_n](6 + \frac{1}{100}f[x_n, w_n](-4 + \frac{1}{100}f[x_n, w_n])))\left(\frac{f(y_n)}{f(w_n)}\right)^3 + \frac{f(z_n)}{f(y_n)} + \left(\frac{f(z_n)}{f(y_n)}\right)^2 + \left(4 - \frac{1}{50}f[x_n, w_n]\right)\frac{f(z_n)}{f(w_n)}.$$

In terms of computational point of view, each member from the class (14)-(15) requires four function evaluations to reach the convergence order four. Therefore, it is optimal in the sense of Kung and Traub (1974). The optimal efficiency index for our contributed class is 1.682.

IV. Numerical Reports

In order to demonstrate the accuracy of a method, it is necessary to study the numerical results of the presented scheme and the schemes available in literature. Khattri and Argyros in [1] showed that the performance of (1) is better than the sixth-order existing methods in literature. Therefore herein, we only compare (12) with (1)- $\alpha = \beta = \eta = 0, \kappa = 1$ and Steffensen's method under a fair situation. The test functions, their simple roots and the starting approximations are listed in Table 1.

Experimental results for our contributed methods from the three-step derivative-free class (14)-(15) can give better feedbacks, i.e. provide better accuracy than those illustrated in Table 2, by choosing very small positive values for β. In fact, by choosing very small positive value for β, the error equation will be narrowed, as in (22). Also note that, if we approximate by an iteration through the data of the first step per cycle, then with memory iterations from our class will be attained which herein we do not drag the topic into these kind of iterative processes.

Table 1. Test nonlinear function, their roots and the starting points

Test Functions	Roots	Starting points
$f_1(x) = x^5 - x^2 + 7x - 41$	1.9878112719284984566488037279366485686…	1.97
$f_2(x) = x^5 - \sin x$	0.9610369414967730615237286599110949112…	0.98
$f_3(x) = (\cos x)^5 - \sin x$	0.517947975980637494267992204184369598…	0.45
$f_4(x) = \sqrt{\cos(x^2)} - \ln(x\sqrt{x})$	1.217890626801970654238566992291492270…	1
$f_5(x) = \tan(\sin(x^2)) \times \sin(x) - x^3 + 17$	2.581711667829765694742300553293398099…	2.8
$f_6(x) = \cos(x) + \ln(x) \times \sqrt{x^3 + 7} - 10$	3.845238953520693454366605119828417436…	5
$f_7(x) = x^3 \tan^{-1}(x) - 1$	1.068947758536760226054678150950059868…	1.3
$f_8(x) = \tan^{-1}(x) + \sin(\tan^{-1}(x)) + 1$	-0.56063741038397049258041123222860129…	-0.9
$f_9(x) = \sin(x^2 + x - 3) + x^5 - x + 1$	0.969472186368112610071446279236264209…	1
$f_{10}(x) = \sin(x^2 + x - 3) + x^5 - x + 1$	-1.130365453500527510641311728942152770…	-1.1
$f_{11}(x) = \sin(x^2 + x - 3)$	-0.82925270904004231372285114567154234504…	-1.5
$f_{12}(x) = \sin(x^2 + x - 3)$	1.302775637731994646559610633735247973125	1.5

All computations were performed in MATLAB 7.6 using variable precision arithmetic (VPA) to increase the number of significant digits. Herein, we accept an approximate solution rather than the exact root, depending on the precision of the computer. Thus, we have considered the following stopping criterion $|f(x_n)| \leq 10^{-1000}$.

The results of comparisons are given in Table 2 in terms of the number significant digits for each test function after the specified number of iteration, that is, e.g. 0.1e-36 shows that the absolute value of the given nonlinear function (f_1) after three iterations is zero up to 36 decimal places.

The computer specifications are: Intel(R) Core(TM) 2 Quad CPU, Q9550 @ 2.83GHz with 2.00GB of RAM. As we can see from Table 2, our contributed methods perform better in comparison with relation (1), i.e., it can be even comparable with all of the quoted sixth-order methods in [1]. The computational order of convergence, namely COC which is defined as follows

$$COC \approx \frac{\ln\left|\frac{e_{n+1}}{e_n}\right|}{\ln\left|\frac{e_n}{e_{n-1}}\right|}, \tag{14}$$

where $e_n = x_n - \alpha$, is very close to 7 (to at least the fourth decimal place) for the our method (12), while it is around 6 for (1) and 8 for (20) and (22). This also manifests that the illustrative practical results coincide well with the theoretical results given in Theorem 1.

We here remark that, it is widely known that quadratically iterative methods such as Steffensen's iterative scheme double the number of correct digits in the convergence phase for the simple roots. As a matter of fact, if an iterative method converges with order p, then after each iteration the number of correct significant digits in the approximation increases by a factor of approximately p. That is why the high-order methods converge faster. Accordingly, our method of order seven increases the number of correct significant digits by a factor of approximately seven per full iteration. Subsequesntly, this rate is around eight for (20) and (22). We also should remark that the accuracy of the iterative root solvers completely relies on the distance between the starting points and the sought zero, i.e. a bad starting approximation may leads to divergency.

Table 2. Comparison of some derivative-free method

f		Steffensen	(1)	(12)	(20)	(22)
f_1	IT	9	3	3	3	3
	TNE	18	12	12	12	12
	$\|f\|$	0.1e-36	0.1e-21	0.1e-35	0.5e-75	0.1e-937
f_2	IT	8	3	3	3	3
	TNE	16	12	12	12	12
	$\|f\|$	0.3e-222	0.4e-252	0.1e-362	0.1e-559	0.1e-555
f_3	IT	8	3	3	3	3
	TNE	16	12	12	12	12
	$\|f\|$	0.1e-401	0.1e-228	0.7e-403	0.1e-691	0.2e-703
f_4	IT	8	3	3	3	3
	TNE	16	12	12	12	12
	$\|f\|$	0.4e-13	0.2e-30	0.3e-36	0.3e-62	0.6e-27
f_5	IT	8	3	3	3	3
	TNE	16	12	12	12	12

	$\lvert f \rvert$	0.3e-8	0.1e-11	0.2e-19	0.1e-35	0.1e-299
f_6	IT	8	3	3	3	3
	TNE	16	12	12	12	12
	$\lvert f \rvert$	0.1e-43	0.1e-71	0.2e-87	0.1e-127	0.8e-298
f_7	IT	9	3	3	3	3
	TNE	18	12	12	12	12
	$\lvert f \rvert$	0.1e-73	0.3e-41	0.4e-61	0.8e-94	0.1e-225
f_8	IT	8	3	3	3	3
	TNE	16	12	12	12	12
	$\lvert f \rvert$	0.3e-327	0.7e-227	0.4e-372	0.1e-482	0.6e-273
f_9	IT	8	3	3	3	3
	TNE	16	12	12	12	12
	$\lvert f \rvert$	0.9e-112	0.3e-177	0.2e-198	0.2e-296	0.4e-436
f_{10}	IT	9	3	3	3	3
	TNE	18	12	12	12	12
	$\lvert f \rvert$	0.1e-232	0.1e-117	0.2e-194	0.6e-304	0.1e-506
f_{11}	IT	8	3	3	3	3
	TNE	16	12	12	12	12
	$\lvert f \rvert$	0.7e-193	0.2e-103	0.8e-195	0.6e-222	0.3e-141
f_{12}	IT	8	3	3	3	3
	TNE	16	12	12	12	12
	$\lvert f \rvert$	0.4e-227	0.4e-72	0.7e-104	0.5e-219	0.4e-270

V. Conclusion

It is widely known that many problems in different scientific fields of studies are reduced to solve single valued nonlinear equations. On the other hand, the construction of iterative without memory methods for approximating the solution of nonlinear equations or systems is an interesting task in numerical analysis. During the last years, numerous papers, devoted to the mentioned iterative methods, have appeared in several journals.

The existence of an extensive literature on these iterative methods reveals that this topic is a dynamic branch of the numerical and nonlinear studies with interesting and promising applications (the study of dynamical models of chemical reactors, radioactive transfer, preliminary orbit determination, etc).

For these reasons, we have constructed a bi-parametric family of derivative-free without memory methods in which there are four function evaluations per full cycle. Taking into consideration of the efficiency index of multi-point iterations, we have attained that our proposed family possess 1.626 as its index of efficiency, which is bigger than that of the newly published work (1). This idea was developed by giving an optimal eighth-order class of three-step derivative-free techniques.

The convergence rate of the presented contributions were established theoretically and its performance was tested through numerical examples. Our contributions are very promising, when the calculation of derivatives of the function takes up a great deal of time or impossible.

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