A Stability Test for Control Systems with Delays Based on the Nyquist Criterion

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***Abstract*—** The aim of this contribution is to revise and extend results about stability and stabilization of a retarded quasipolynomial and systems obtained using the Mikhaylov criterion in our papers earlier. Not only retarded linear time-invariant time-delay systems (LTI-TDS) are considered in this paper; neutral as well as distributed-delay systems are the matter of the research. A LTI-TDS system of retarded type is said to be asymptotically stable if all its poles rest in the open left half plane. Asymptotic stability of neutral systems described by its spectrum is not sufficient to express the notion of stability at whole since neutral LTI-TDS are sensitive to infinitesimal delay changes. This yields the concept of so called strong stability involving this fact. Moreover, stability can not be studied using the characteristic quasipolynomial when distributed delays in either input-output or internal relation appear in a model. The contribution transforms the formulation of the Mikhaylov criterion (the argument principle) into the language of the Nyquist criterion for the open loop of a control system. The classical simple feedback loop is considered. Illustrative examples are presented to clarify the results.

***Keywords*—**Stabilization, stability, time delay systems, Nyquist criterion, argument principle, distributed delays.

I. Introduction

ASYMPTOTIC stability, spectrum analysis and stabilization of linear time-invariant time-delay systems (LTI-TDSs) have been challenging tasks in control theory during last decades. Due to their infinite dimensional nature, these theoretical problems are nontrivial even for simple-modeled systems. A vast bulk of various significant results was obtained and reported; see for instance [1] – [7], without any attempt to be exhaustive.

In state-space LTI-TDSs are expressed by a set of functional differential equations (FDEs) [8], whereas the input-output description can be represented by the Laplace transfer function as a fraction of so-called quasipolynomials in one complex variable. Delay in the feedback can significantly deteriorate the quality of control performance, namely stability and periodicity. Although the asymptotic stability of LTI-TDSs is defined in the space of state variables and it can be easier to deal with in this space, we investigate our results on the basis of transfer functions since some elegant control algorithms stem form the input-output description. It is essential to discern retarded and neutral LTI-TDSs as well as lumped and distributed delays. For lumped delays, the denominator quasipolynomial decides about the control system asymptotic stability because of the fact that its zeros are system poles with the same meaning as for polynomials; however, the spectrum is infinite due to a quasipolynomial transcendental form. Dealing with distributed delays (either in state or input variables) is a rather more involved since some roots of transfer function numerator and denominator coincide and thus the system poles do not agree with denominator zeros. Moreover, stability of neutral systems can not be sufficiently studied only in terms of asymptotic stability because of the fact that neutral TDSs can be destabilized by even infinitesimally small changes in delays. This led to the concept of so called strong stability [9] which is closely related to notion of formal stability [10].

This paper extends and corrects results obtained for the stability of a retarded quasipolynomial with two delays in [11] and those for stabilization of the control feedback with a first order LTI-TDS in [12]. Since the crucial theorem in the former one is not fully correct, its revisited version is presented and proved in this contribution. The findings in papers mentioned above were obtained via the argument principle (or via the Mikhaylov stability criterion) for retarded LTI-TDSs [13]. Applying the argument principle for the control feedback along with the knowledge the open loop frequency response results in the use of the well known Nyquist criterion. The notorious precept about the number of open loop unstable poles, however, is not easy to utilize in the case of LTI-TDSs due to their infinite spectrum [14]-[15]. In addition, parlous and complex cases of neutral and distributed delays are discussed and comprehend in this research. Hence, we simply derive the generalized Nyquist criterion for a wide class of LTI-TDSs.

Theoretical results obtained herein are supported by simulations in Matlab-Simulink to clarify and prove the statements.

The paper is organized as follows: A possible LTI-TDS

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model, some basic preliminaries about asymptotic, formal and strong stability and the argument principle are introduced in Chapter II. Chapter III contains a revision of previous results about root locus (stability) of some retarded quasipolynomials. In Chapter IV, divided into several subsections, generalized Nyquist criteria and related lemmas for a simple control feedback, for retarded, neutral and distributed-delay LTI-TDSs are introduced. Chapter V. contains two simulation examples elucidating and supporting the presented results. Conclusions and references finalize the paper.

II. Stability Preliminaries

A. LTI-TDSs Model

A state-space description of a LTI-TDS can be provided by the set of FDEs

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} &= \sum_{i=1}^{N_H}\mathbf{H}_i \frac{\mathrm{d}\mathbf{x}(t-\eta_i)}{\mathrm{d}t} \\ &+ \mathbf{A}_0\mathbf{x}(t) + \sum_{i=1}^{N_A}\mathbf{A}_i\mathbf{x}(t-\eta_i) \\ &+ \mathbf{B}_0\mathbf{u}(t) + \sum_{i=1}^{N_B}\mathbf{B}_i\mathbf{u}(t-\eta_i) \\ &+ \int_0^L \mathbf{A}(\tau)\mathbf{x}(t-\tau)\mathrm{d}\tau + \int_0^L \mathbf{B}(\tau)\mathbf{u}(t-\tau)\mathrm{d}\tau \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is a vector of state variables, $\mathbf{u} \in \mathbb{R}^m$ stands for a vector of inputs, $\mathbf{y} \in \mathbb{R}^l$ represents a vector of outputs, $\mathbf{A}_i$, $\mathbf{A}(\tau)$, $\mathbf{B}_i$, $\mathbf{B}(\tau)$, $\mathbf{C}$, $\mathbf{H}_i$ are real matrices of compatible dimensions, $0 \le \eta_i \le L$ stand for *lumped* (point-wise) delays and convolution integrals express *distributed* delays. If $\mathbf{H}_i \neq \mathbf{0}$ for any $i = 1,2,...N_H$, model (1) is called *neutral*; on the other hand, if $\mathbf{H}_i = \mathbf{0}$ for every $i = 1,2,...N_H$, so called *retarded* LTI-TDS is obtained.

Integrals in (1) can be exactly reformulated into sums of lumped delays using the Laplace transform, see e.g. [16], [17] or approximately via a standard numerical approximation methods. The exact transform correspondence is as follows

$$\begin{aligned}\mathcal{L}\left\{\int_0^L \mathbf{A}(\tau)\mathbf{x}(t-\tau)\mathrm{d}\tau\right\} &= \mathbf{X}(s)\int_0^L \mathbf{A}(\tau)\exp(-s\tau)\mathrm{d}\tau \\ \mathcal{L}\left\{\int_0^L \mathbf{B}(\tau)\mathbf{u}(t-\tau)\mathrm{d}\tau\right\} &= \mathbf{U}(s)\int_0^L \mathbf{B}(\tau)\exp(-s\tau)\mathrm{d}\tau\end{aligned} \tag{2}$$

where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform operation. Subsequent utilization of the reverse Laplace transform yields the state equation in the form

$$\begin{aligned}\frac{\mathrm{d}\mathbf{z}(t)}{\mathrm{d}t} &= \sum_{i=1}^{N_H}\widetilde{\mathbf{H}}_i \frac{\mathrm{d}\mathbf{z}(t-\eta_i)}{\mathrm{d}t} + \widetilde{\mathbf{A}}_0\mathbf{z}(t) + \sum_{i=1}^{N_A+1}\widetilde{\mathbf{A}}_i\mathbf{z}(t-\eta_i) \\ &+ \widetilde{\mathbf{B}}_0\mathbf{z}(t) + \sum_{i=1}^{N_B+1}\widetilde{\mathbf{B}}_i\mathbf{z}(t-\eta_i) \\ \mathbf{z}(t) &= \left[\mathbf{x}(t) \quad \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}\right]^{\mathrm{T}}\end{aligned} \tag{3}$$

where $\eta_{N_A+1} = \eta_{N_B+1} = L$.

Considering model (1) and zero initial conditions, the following input-output description of a general multi-input multi-output (MIMO) system in the form of the transfer matrix using the Laplace transform is obtained

$$\begin{aligned}\mathbf{Y}(s) &= \mathbf{G}(s)\mathbf{U}(s) = \frac{\mathbf{C}\mathrm{adj}[s\mathbf{I}-\mathbf{A}(s)]\mathbf{B}(s)}{\det[s\mathbf{I}-\mathbf{A}(s)]}\mathbf{U}(s) \\ \mathbf{A}(s) &= s\sum_{i=1}^{N_H}\mathbf{H}_i\exp(-s\eta_i) + \mathbf{A}_0 + \sum_{i=1}^{N_A}\mathbf{A}_i\exp(-s\eta_i) \\ &+ \int_0^L \widetilde{\mathbf{A}}(\tau)\exp(-s\tau)\mathrm{d}\tau \\ \mathbf{B}(s) &= \mathbf{B}_0 + \sum_{i=1}^{N_B}\mathbf{B}_i\exp(-s\eta_i) + \int_0^L \widetilde{\mathbf{B}}(\tau)\exp(-s\tau)\mathrm{d}\tau\end{aligned} \tag{4}$$

The main advantage of the TDS system description in the form of the transfer function lies in its practical usability when system analysis and control design. All transfer functions in $\mathbf{G}(s)$ (or a transfer function in SISO case) have identical denominator in the form

$$\begin{aligned}m(s) &= \mathrm{num}\det[s\mathbf{I}-\mathbf{A}(s)] \\ &= \mathrm{num}M(s) = s^n + \sum_{i=0}^{n}\sum_{j=1}^{h_i} m_{ij}s^i\exp(-s\eta_{ij}),\eta_{ij} \ge 0\end{aligned} \tag{5}$$

where prefix num means the numerator of the determinant, and $\sum_{j=1}^{h_n} m_{nj}\exp(-\eta_{nj}s) \neq \text{constant}$ holds for a neutral system; otherwise, the system is retarded. The expression on the right-hand side of (5) represents a so called quasipolynomial [18]. Indeed, $M(s)$ is a ratio of quasipolynomials (i.e. a meromorphic function) in general due to distributed state (internal) delays, and all roots of the denominator of $M(s)$ are those of the numerator in this case. As a consequence, a transfer function (in a SISO case) can be expressed as a meromorphic function as well.

For instance, consider a system of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\int_0^1 x(t-\tau)\mathrm{d}\tau + u(t), y(t) = x(t) \tag{6}$$

has the transfer function

$$\frac{Y(s)}{U(s)}=\frac{1}{s+\frac{1-\exp(-s)}{s}}=\frac{1}{M(s)}$$
$$=\frac{s}{s^2+1-\exp(-s)}=\frac{s}{m(s)} \quad (7)$$

Clearly, both the numerator and denominator of (7) have the same zero s = 0, whereas the rest of the denominator spectrum lies in the open left half plane. Thus, all system poles are located in $\mathbb{C}_0^-$.

B. LTI-TDSs Stability

Definition 1 (*LTI-TDS asymptotic stability*). LTI-TDS is asymptotically stable if all poles are located in the open left half plane, $\mathbb{C}_0^-$, i.e. there is no s satisfying

$$M(s)=0, \operatorname{Re} s \geq 0 \quad (8)$$

■

In the case of neutral systems, one has to be more careful when deciding about the stability since there may be infinite braches of poles tending to the imaginary axis. Strictly negative roots of the characteristic (quasi)polynomial (or meromorphic function), thus, do not guarantee a satisfactory stable behavior of a system from the asymptotic (and robust) point of view. Let us introduce an associated difference equation and two stability notions for neutral LTI-TDS which are close to each other in the meaning.

Definition 2. Given a SISO neutral LTI-TDS (1), an *associated difference equation* is defined as

$$\mathbf{x}(t)-\sum_{i=1}^{N_H}\mathbf{H}_i\mathbf{x}(t-\eta_i)=0 \quad (9)$$

■

Definition 3. A neutral TDS is said to be *formally stable* if

$$\operatorname{rank}\left[I-\sum_{i=1}^{N_H}\mathbf{H}_i\exp(-s\eta_i)\right]=n, \forall s: s\geq 0 \quad (10)$$

■

see e.g. in [20], [21]. It also means that the a neutral LTI-TDS has only a finite number of poles in the (closed) right-half complex plane ($\mathbb{C}^+$) [10]. Clearly from (9) and (10), a system is formally stable if characteristic equation

$$m_D(s)=\det\left[I-\sum_{i=1}^{N_H}\mathbf{H}_i\exp(-s\eta_i)\right]=0 \quad (11)$$

expressing the spectrum of the difference equation has all its solutions in $\mathbb{C}_0^-$.

The feature of a neutral TDS that rightmost solution of (11) is not continuous in its delays, see e.g. [22], gives rise to another (yet a germane) stability notion.

Definition 4. The difference equation (9) is *strongly stable* if it remains exponentially stable when subjected to small variations in delays (i.e. a TDS remains formally stable). ■

Theorem 1. (a) A neutral LTI-TDS is strongly stable if and only if

$$\gamma_0 := \max\left\{r_\Omega\left(\sum_{i=1}^{N_H}\mathbf{H}_i\exp(s\theta_i)\right): \theta_i\in[0,2\pi), 1\leq k\leq m\right\}<1 \quad (12)$$

where $r_\Omega(\cdot)$ denotes the spectral radius.

(b) Alternatively, necessary and sufficient strong stability condition in the Laplace transform can be formulated as

$$\sum_{j=1}^{h_i}\left|m_{nj}\right|<1 \quad (13)$$

see e.g. [9], [23] where m_{nj} are coefficients for the highest s-power in (5). ■

A sufficient condition for this type of stability is e.g.

$$\sum_{i=1}^{N_H}\left\|\mathbf{H}_i\right\|<1 \quad (14)$$

where $\|\cdot\|$ denotes a matrix norm. A strongly stable system is robust against infinitesimal changes in delays of a neutral LTI-TDS which can destroy the asymptotic stability of the difference equation.

Clearly, strong stability implies formal stability; contrariwise, a formally stable LTI-TDS can be destabilized in the formal sense by an infinitesimal change in delays.

C. Retarded Quasipolynomial Stability

Let us recall some basic results about the spectrum and argument (increment) principle for retarded quasipolynomials, respectively, for retarded LTI-TDSs (with characteristic quasipolynomial of retarded type).

Definition 5. Retarded quasipolynomial of the general form (5) is said to be *asymptotically stable* if it has no root in the closed right half s-plane ($\mathbb{C}_0^-$),, i.e. if there is no s such that

$$m(s)=0, \operatorname{Re}\{s\}\geq 0 \quad (15)$$

■

Definition 5 is a direct analogy to Definition 1.

Proposition 1 (Number of unstable roots) [19]. Consider a quasipolynomial (5) of retarded type. Then the number N_U of poles of $m(s)$ located in the closed right half s-plane (i.e. unstable ones) is

$$N_U=\frac{n}{2}-\frac{1}{\pi}\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg} m(s) \quad (16)$$

■

The direct implication of Proposition 1 is the following theorem [12].

Theorem 2 (Argument increment principle for retarded quasipolynomials). Consider a retarded quasipolynomial $m(s)$. If $m(0)>0$ and $m(s)\neq 0$ for any imaginary $s=\mathrm{j}\omega$, $\omega\in\mathbb{R}$, function $m(s)$ has no zero in $\mathbb{C}^+$ if and only if the argument of $m(s)$ reaches the increment

$$\Delta_{s=\mathrm{j}\omega,\omega\in[0,\infty)}\arg m(s)=\frac{n\pi}{2} \tag{17}$$

■

D. Neutral Quasipolynomial Stability

Analysis of neutral LTI-TDS via the argument increment is a rather more complicated due to the absence of a limit of $\Delta\arg m(s)$; however, it holds true the following [23].

Theorem 3 (Argument increment principle for neutral quasipolynomials). Consider quasipolynomial $m(s)$ of neutral type satisfying $m(0)>0$, $m(s)\neq 0$ for any imaginary $s=\mathrm{j}\omega$, $\omega\in\mathbb{R}$, and (13). Then $m(s)$ is strongly and asymptotically stable if and only if

$$\frac{n\pi}{2}-\Phi\leq \Delta_{s=\mathrm{j}\omega,\omega\in[0,\infty)}\arg m(s)\leq\frac{n\pi}{2}+\Phi \tag{18}$$

where

$$\Phi=\arcsin\left(\sum_{j=1}^{h_n}\left|m_{nj}\right|\right) \tag{19}$$

■

Nevertheless, if the quasipolynomial is formally stable, i.e. it has only a finite number of zeros located in $\mathbb{C}^+$, the number of such unstable zeros is given by formula (16). Condition (13) ensures i.a. that the argument change Φ in (19) is finite (see proof of Theorem 1 in [23]), more precisely, $\Phi\in(0,\pi/2)$. If (13) does not hold true, the quasipolynomial is not strongly stable, yet it can be formally stable. Thus, (13) is a sufficient condition for formal stability of the neutral quasipolynomial and it implies that (16) can be utilized for the relation between the "main" part of the argument change (divisible by $\pi/2$ and ignoring Φ) and the number of unstable roots.

For example, consider a neutral quasipolynomial

$$m(s)=(1-0.5\exp(-s)+0.55\exp(-2s))s+1 \tag{20}$$

which is not strongly stable due to Theorem 1b. However, it has no unstable zero and the "main" part of the overall phase shift is $\pi/2$, see the Mikhaylov curve in Fig. 1, hence it is asymptotically and formally stable.

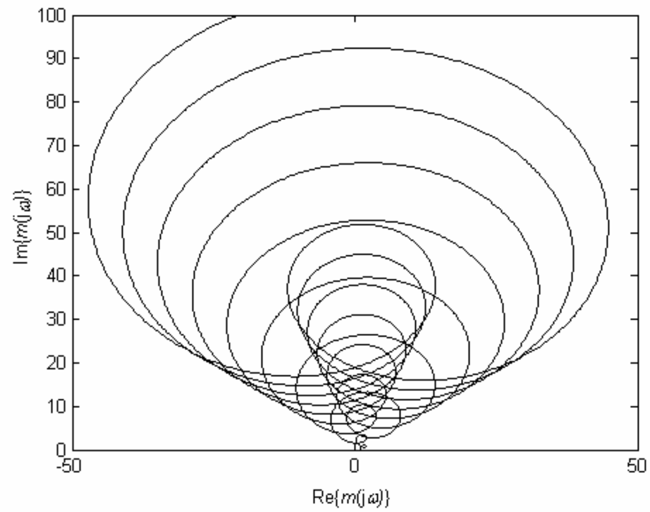

Fig. 1 Mikhaylov plot of neutral quasipolynomial (20)

III. Retarded Quasipolynomial of Degree One - Revision

The following results have been derived for simple quasipolynomials with n = 1 and h_0 = 1 and h_0 = 2, respectively.

Theorem 4 [12]. Consider the quasipolynomial

$$m(s)=s+a\exp(-\vartheta s)+kq \tag{21}$$

where $a\neq 0\in\mathbb{R}$; $k,\vartheta>0\in\mathbb{R}$ are fixed, whereas q is selectable. Then, if

$$a\vartheta\leq 1 \tag{22}$$

the quasipolynomial (21) is asymptotically stable if and only if

$$q>\frac{-a}{k} \tag{23}$$

In the contrary, if

$$a\vartheta>1 \tag{24}$$

the quasipolynomial (20) is asymptotically stable if and only if

$$q>\frac{-a\cos(\vartheta\omega_0)}{k} \tag{25}$$

where the *crossover frequency* ω_0 is the minimum nonzero element of the set

$$\Omega_0:=\{\omega:\omega>0,\mathrm{Im}\{m(\mathrm{j}\omega)\}=0\} \tag{26}$$

■

Definition 6. Consider quasipolynomial

$$m(s)=s+a\exp(-\vartheta s)+kq\exp(-\tau s) \tag{27}$$

with $a \neq 0 \in \mathbb{R}$; $k, \vartheta, \tau > 0 \in \mathbb{R}$. Here, the set of *crossover frequencies* is defined as

$$\Omega_1 := \{\omega : \omega > 0, \mathrm{Im}\{m(\mathrm{j}\omega)\} = \mathrm{Re}\{m(\mathrm{j}\omega)\} = 0\} \tag{28}$$

The *critical frequency* ω_C is defined as

$$\omega_C := \min\left\{\omega : \omega \in \Omega_1, \underset{s=\omega\mathrm{j}, \omega \in [0, \omega_C)}{\Delta \arg m(s)} = 0, \underset{s=\omega\mathrm{j}, \omega \in [\omega_C, \infty)}{\Delta \arg m(s)} = \frac{\pi}{2}\right\} \tag{29}$$

for a particular *critical gain* q_C given by

$$q_C = \frac{\omega_C - a\sin(\vartheta\omega_C)}{k\sin(\tau\omega_C)} \tag{30}$$

■

Remark 1 [11]. Elements $\omega_1 \in \Omega_1$ are calculated as all solutions of the transcendental equation

$$\omega_1 \cos(\tau\omega_1) = a(\sin((\vartheta - \tau)\omega_1)) \tag{31}$$

■

The following theorem constitutes the revisited result presented as Theorem 1 in [11].

Theorem 5. Consider the following five possibilities:

a) If $\sin(\tau\omega_C) = 0$ and $\cos(\tau\omega_C) > 0$, $\cos(\tau\omega_C) < 0$, then quasipolynomial (27) is stable if and only if

$$\max\left(\frac{-a\cos(\vartheta\omega_C)}{k\cos(\tau\omega_C)}, \frac{-a}{k}\right) < q \tag{32}$$

$$\frac{-a}{k} < q < \frac{-a\cos(\vartheta\omega_C)}{k\cos(\tau\omega_C)} \tag{33}$$

respectively.

b) If $\cos(\tau\omega_C) = 0$ and $\sin(\tau\omega_C) > 0$, $\sin(\tau\omega_C) < 0$, then quasipolynomial (27) is stable if and only if

$$\frac{-a}{k} < q < \frac{\omega_C - a\sin(\vartheta\omega_C)}{k\sin(\tau\omega_C)} \tag{34}$$

$$\max\left(\frac{\omega_C - a\sin(\vartheta\omega_C)}{k\sin(\tau\omega_C)}, \frac{-a}{k}\right) < q \tag{35}$$

c) If $\sin(\tau\omega_C) > 0$ and $\cos(\tau\omega_C) < 0$, $\sin(\tau\omega_C) < 0$ and $\cos(\tau\omega_C) > 0$, then quasipolynomial (27) is stable if and only if (33) or (34), (32) or (35), hold, respectively.

d) If $\sin(\tau\omega_C) > 0$ and $\cos(\tau\omega_C) > 0$, then if

$$|\cos(\tau\omega_C)| > |\sin(\tau\omega_C)| \tag{36}$$

then quasipolynomial (27) is stable if and only if (32) holds; otherwise, the quasipolynomial is stable if and only if (33) holds.

e) If $\sin(\tau\omega_C) < 0$ and $\cos(\tau\omega_C) < 0$, then if (36) holds, quasipolynomial (27) is stable if and only if (33) is satisfied. Otherwise, if condition (36) does not hold true, the quasipolynomial is stable if and only if (35) holds.

Recall that ω_C is the critical frequency. ■

Proof. (Necessity.) For all the cases in the theorem, the Mikhaylov curve of stable quasipolynomial (27) starts on the positive real axis, and thus the necessary stability condition

$$\frac{-a}{k} < q \tag{37}$$

included in (32)-(35) holds, as proved in Lemma 2 in [11]. Lemma 3 in [11] states the condition

$$a\vartheta + kq\tau \leq 1 \tag{38}$$

guaranties that the initial change of the Mikhaylov curve in the imaginary axis is positive. i.e. the curve tends to move to the first quadrant for $\omega = 0$; however, according to Observation 1 in [11], it immediately moves to the fourth quadrant. Otherwise, if

$$a\vartheta + kq\tau > 1 \tag{39}$$

is satisfied, the curve passes through the fourth quadrant already for an infinitesimally small ω. The critical (marginal) case is characterized by ω_C and q_C where the curve crosses the origin of the complex plane and a small change of q would cause the quasipolynomial stability, i.e. the overall phase change would be $\pi/2$, see Remark 1 in [11]. The limit stable case thus obviously means that $\mathrm{Re}\{m(\mathrm{j}\omega_C)\} > 0$ and $\mathrm{Im}\{m(\mathrm{j}\omega_C)\} > 0$ must hold simultaneously; here we can use relations

$$\mathrm{Re}\{m(\mathrm{j}\omega)\} = a\cos(\vartheta\omega) + kq\cos(\tau\omega) \tag{40}$$

$$\mathrm{Im}\{m(\mathrm{j}\omega)\} = \omega - a\sin(\vartheta\omega) - kq\sin(\tau\omega) \tag{41}$$

Consider case a) in the theorem and take $\cos(\tau\omega_C) > 0$. Since $\sin(\tau\omega_C) = 0$, we can not deal with (41), whereas (40) gives (32) immediately. Analogously, a case when $\cos(\tau\omega_C) < 0$ results in the right-hand side of (33).

If conditions b) hold, inequalities (34) and (35) are obtained from (40) in the similar way as in the previous paragraph.

In the case c), condition $\mathrm{Re}\{m(\mathrm{j}\omega_C)\} > 0$ using (40) yields results (32) and (33) which are as the same as conditions (34) and (34), respectively, obtained from $\mathrm{Im}\{m(\mathrm{j}\omega_C)\} > 0$ with

(41).

The most involved cases in the theorem are d) and e) since conditions $\operatorname{Re}\{m(\mathrm{j}\omega_C)\}>0$ and $\operatorname{Im}\{m(\mathrm{j}\omega_C)\}>0$ collide here – one gives the upper limit for q whereas the second yields the lower one. To decide which of them is valid, one has to test the sensitivity of the Mikhaylov plot in the vicinity of $q = q_C$. If the infinitesimal change of the curve in the real axis is higher than that in the imaginary one, condition $\operatorname{Re}\{m(\mathrm{j}\omega_C)\}>0$ establishes the behavior of the curve near the origin (i.e. it has the higher priority). Contrariwise, if the plot shifts in the imaginary axis faster than in the real one, the stability is given by condition $\operatorname{Im}\{m(\mathrm{j}\omega_C)\}>0$ because it influences the Mikhaylov plot near the critical point more decidedly.

Hence, if

$$\begin{gathered}\left|\left[\frac{\mathrm{d}}{\mathrm{d}q}\operatorname{Re}\{m(\mathrm{j}\omega)\}\right]_{\substack{\omega=\omega_C\\ q=q_C}}\right| > \left|\left[\frac{\mathrm{d}}{\mathrm{d}q}\operatorname{Im}\{m(\mathrm{j}\omega)\}\right]_{\substack{\omega=\omega_C\\ q=q_C}}\right| \\ |k\cos(\tau\omega_C)| > |-k\sin(\tau\omega_C)| \\ |\cos(\tau\omega_C)| > |\sin(\tau\omega_C)|\end{gathered} \tag{42}$$

then (40) decides about the behavior of the Mikhaylov plot near the origin, which results in (32) for $\cos(\tau\omega_C)>0$ and in (33) for $\cos(\tau\omega_C)<0$, respectively.

Otherwise, if (42) does not hold true, the imaginary part (41) of the quasipolynomial (27) dominates in the critical point, which gives (34) for $\sin(\tau\omega_C)>0$ and (35) for $\sin(\tau\omega_C)<0$.

(Sufficiency.) Bound (37) included in (37)-(40) guarantees that the Mikhaylov curve initiates on the positive real axis, see Lemma 2 in [11]. Lemma 3 in [11] verifies that the curve reaches infinity in the imaginary axis for $\omega \to \infty$, and Lemma 4 states that it is bounded in the real axis. Moreover, if (38) holds the Mikhaylov curve tends to move to the first quadrant and, consequently, to the fourth quadrant for $\omega = 0$; otherwise, it moves to the fourth quadrant for $\omega = \Delta$ when (39) is satisfied. For the quasipolynomial stability, expressed by the overall phase shift $\pi/2$, it is now sufficient to show that the curve does not encircle the origin of the complex plane in the clockwise direction.

Let the critical stability case be expressed by ω_C and q_C and consider case a) first. Since $\sin(\tau\omega_C)=0$, condition $\operatorname{Im}\{m(\mathrm{j}\omega_C)\}>0$ could not be guaranteed from (41) and $\operatorname{Im}\{m(\mathrm{j}\omega_C)\}=0$ remains for any q. However, inequalities (32) and (33) yield $\operatorname{Re}\{m(\mathrm{j}\omega_C)\}>0$ from (40) using $\cos(\tau\omega_C)>0$ and $\cos(\tau\omega_C)<0$, respectively, for a particular $q > q_C$ and $q < q_C$, respectively. Thus, it means that the real axis is crossed in the positive semi-axis first on the critical frequency and thus, with respect to Remark 1 in [11], the origin is encircled in the anti-clockwise direction with the overall phase shift $\pi/2$.

Second, assume the case b). Similarly as in the previous paragraph, $\cos(\tau\omega_C)=0$ gives $\operatorname{Re}\{m(\mathrm{j}\omega_C)\}=0$ for any q. Inequalities (34) and (35) together with $\sin(\tau\omega_C)>0$ and $\sin(\tau\omega_C)<0$, respectively, result in $\operatorname{Im}\{m(\mathrm{j}\omega_C)\}>0$, from (40). Thus, the overall phase shift is $\pi/2$ again.

In c), pairs of conditions (33) and (34), (32) and (35), agree with $\operatorname{Re}\{m(\mathrm{j}\omega_C)\}>0$ and $\operatorname{Im}\{m(\mathrm{j}\omega_C)\}>0$ simultaneously for $\sin(\tau\omega_C)>0$ and $\cos(\tau\omega_C)<0$, $\sin(\tau\omega_C)<0$ and $\cos(\tau\omega_C)>0$, respectively, which implies the desired phase shift for the stability.

Condition (36) in d) and e) expresses the fact that the absolute value of a derivative of the Mikhaylov curve in the critical point is higher in the real than in the imaginary one. Thus, condition $\operatorname{Re}\{m(\mathrm{j}\omega_C)\}>0$ is stricter than $\operatorname{Im}\{m(\mathrm{j}\omega_C)\}>0$ when decision about the behavior of the plot in the vicinity of the origin for ω_C. Inequalities (32) and (33) correspond to $\operatorname{Re}\{m(\mathrm{j}\omega_C)\}>0$ for $\cos(\tau\omega_C)>0$ and $\cos(\tau\omega_C)<0$, respectively, which means that the critical point is not encircled.

In the contrary, if (36) does not hold true, i.e. $\operatorname{Im}\{m(\mathrm{j}\omega_C)\}>0$ decides about the critical behavior, inequalities (34) and (35) correspond to $\operatorname{Im}\{m(\mathrm{j}\omega_C)\}>0$ for $\sin(\tau\omega_C)>0$ and $\sin(\tau\omega_C)<0$, respectively, which guarantees the stability again. □

Corollary 1. Definition 6 and Theorem 5 suggest situations when the quasipolynomial stabilization by the suitable choice of q is not possible. These are two unpleasant possibilities:

1) If ω_C does not exist. Thus, although Ω_0 is non-empty set, it may not contain $\omega_0 = \omega_C$.

2) If q could not satisfy (33) or (34), i.e. if

$$\frac{\omega_C - a\sin(\vartheta\omega_C)}{k\sin(\tau\omega_C)} \le \frac{-a}{k} \tag{43}$$

or

$$\frac{-a\cos(\vartheta\omega_C)}{k\cos(\tau\omega_C)} \le \frac{-a}{k} \tag{44}$$

depending on the particular case from Theorem 5. ■

Remark 3. Assume that a particular value of q does satisfy Theorem 5, then the Mikhaylov plot begins either on the positive real semi-axis but the overall phase change differs from $\pi/2$ or it starts on the negative one for $\omega = 0$. In the former case, the overall phase shift is

$$\underset{s=\omega\mathrm{j},\,\omega\in[0,\infty)}{\Delta\arg} m(s) = -\frac{3\pi}{2} - 2k\pi,\ k \in \mathbb{N} \tag{45}$$

The latter case yields

$$\Delta \arg_{s=\omega j,\omega\in[0,\infty)} m(s) = -\frac{\pi}{2} - 2k\pi,\ k \in \mathbb{N} \tag{46}$$

■

Remark 3 is a direct sequel of Proposition 1.

IV. Generalized Nyquist Criterion for LTI-TDS

In this chapter the Nyquist criterion for retarded and neutral LTI-TDS with both lumped and distributed delays based on the argument principle is presented. As usual, the Nyquist criterion gives information about the closed-loop stability based on the knowledge of the overall phase shift (argument increment) of the open-loop transfer function $G_O(s)$ around the critical point -1.

Consider a simple control system as in Fig. 1 and express the plant and controller transfer functions, respectively, as

$$G(s) = b(s)/a(s),\ G_R(s) = q(s)/p(s) \tag{47}$$

where $a(s), b(s), q(s), p(s)$ are retarded quasipolynomials and $G(s)$ is strictly proper and $G_R(s)$ is proper (the properness is defined as for delay-free systems using the highest s-power). Then the corresponding closed loop reference-to-output (i.e. complementary sensitivity) transfer function reads

$$G_{WY}(s) = \frac{Y(s)}{W(s)} = \frac{G_R(s)G(s)}{1+G(s)G_R(s)} = \frac{G_0(s)}{1+G_0(s)} = \frac{\dfrac{q(s)b(s)}{p(s)a(s)}}{\dfrac{p(s)a(s)+q(s)b(s)}{p(s)a(s)}} \tag{48}$$

where the characteristic quasipolynomial $m(s)$ is

$$m(s) = p(s)a(s) + q(s)b(s) \tag{49}$$

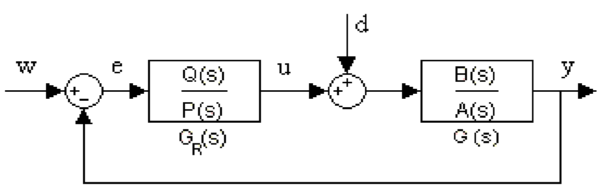

Fig. 2 Simple control feedback loop

Recall that in the case of input-output or internal distributed delays, zeros of (49) do not agree with poles of (48) there are some common (unstable) roots of $a(s), b(s)$ and/or those of $q(s), p(s)$.

A. Retarded LTI-TDS with lumped delays

For retarded LTI-TDSs without distributed delays we can formulate and prove the following theorem.

Theorem 6 (The Nyquist criterion for retarded LTI-TDSs with lumped delays). Let the plant and the controller have transfer functions as in (47) without distributed delays and the control system be in a simple form as in Fig. 1. Let retarded quasipolynomials $a(s)$ and $p(s)$ have no root on the imaginary axis, i.e. $a(s) \neq 0, p(s) \neq 0$ for any imaginary $s = j\omega,\ \omega \in \mathbb{R}$.

Then, if

$$\Delta \arg_{s=\omega j,\omega\in[0,\infty)} p(s)a(s) = l\pi/2 \tag{50}$$

the closed-loop system is asymptotically stable if

$$\Delta \arg_{s=\omega j,\omega\in[0,\infty)} (1+G_O(s)) = (n-l)\frac{\pi}{2} \tag{51}$$

where n is the highest s-power in the closed-loop characteristic quasipolynomial $m(s)$ as in (49) which equals the sum of the highest s-powers of $a(s)$ and $p(s)$. ■

Proof. The highest s-power n of $m(s) = p(s)a(s) + q(s)b(s)$ equals that of $p(s)a(s)$ due to the properness. If

$$\Delta \arg_{s=j\omega,\omega\in[0,\infty)} m(s) = n\pi/2 \tag{52}$$

then the closed-loop system is asymptotically stable according to Theorem 2 (i.e. its characteristic quasipolynomial has all zeros in $\mathbb{C}_0^-$), and, simultaneously, since retarded quasipolynomials are analytic functions, it holds that

$$\Delta \arg_{s=j\omega,\omega\in[0,\infty)} m(s)/(a(s)p(s)) = n\pi/2 - l\pi/2 \tag{53}$$

Moreover, from (47) and (48) it is obvious that

$$\Delta \arg_{s=j\omega,\omega\in[0,\infty)} m(s)/(a(s)p(s)) = \Delta \arg_{s=j\omega,\omega\in[0,\infty)} (1+G_0(s)) \tag{54}$$

and the proof is finished. □

Thus, to test the closed-loop asymptotic stability, one can figure the Nyquist plot of $G_O(s)$ and count its overall number of encirclements around the critical point -1, or more precisely, the overall phase shift of the curve around the point.

Now, the natural question is, whether the notorious precept about the number of unstable poles of $G_O(s)$ (as for delay-free systems) can be used. The answer is the following modification of Theorem 6.

Theorem 7 (The Nyquist criterion for retarded LTI-TDSs with lumped delays – an alternative formulation). Let the plant and the controller have transfer functions as in (47) with lumped delays only, and the control system be in a simple form as in Fig. 1. Let retarded quasipolynomials $a(s)$ and

$p(s)$ have no root on the imaginary axis, i.e. $a(s) \neq 0, p(s) \neq 0$ for any imaginary $s = \mathrm{j}\omega$, $\omega \in \mathbb{R}$.

Then, the closed-loop system is asymptotically stable if

$$\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta \arg} \left(1 + G_O(s)\right) = n_U \pi \tag{55}$$

where n_U is the number of poles of $G_O(s)$ with positive real parts (i.e. unstable poles). ■

Proof. Assume results from Theorem 6 and Proposition 1. If there in no pure complex conjugate pair of poles of $G_O(s)$ (i.e. roots of $a(s)p(s)$), all its unstable poles have positive real parts, the number of which is given by (16). If notations (50) and (55) are taken into account, one can write

$$n_U = \frac{(n-l)}{2} \Rightarrow l = n - 2n_U \tag{56}$$

Substitution (56) into (51) yields (55), finally. □

B. Neutral LTI-TDS with lumped delays

If the plant or the controller is of a neutral type, the Nyquist criterion satisfying both the asymptotic and strong stability can be easily formulated in the light of Theorem 3 and the knowledge of relation between strong and formal stability and the number of unstable quasipolynomial zeros, described in subchapter II.D.

Theorem 8 (The Nyquist criterion for neutral LTI-TDSs with lumped delays). Let the plant and the controller have transfer functions as in (47) with lumped delays only and let the control system be of the scheme as in Fig. 1. Let neutral quasipolynomials $a(s)$ and $p(s)$ have no root on the imaginary axis, i.e. $a(s) \neq 0, p(s) \neq 0$ for any imaginary $s = \mathrm{j}\omega$, $\omega \in \mathbb{R}$, and define the denominator of $G_O(s)$ as

$$m_{ap}(s) = p(s)a(s) = s^n + \sum_{i=0}^{n} \sum_{j=1}^{h_{ap,i}} m_{ap,ij} s^i \exp(-s\eta_{ij}) \tag{57}$$

for which (13) holds true.

Then, if

$$\underset{s=\omega\mathrm{j},\omega\in[0,\infty)}{\Delta \arg} m_{ap}(s) \in \left(l\pi/2 - \Phi_{ap}, l\pi/2 + \Phi_{ap}\right) \tag{58}$$

where

$$\Phi_{ap} = \arcsin\left(\sum_{j=1}^{h_{ap,n}} \left|m_{ap,nj}\right|\right) \tag{59}$$

then the closed-loop system is asymptotically stable if (51) holds true. Note that n is the highest s-power in the closed-loop characteristic quasipolynomial $m(s)$ as in (49), which equals the highest s-power of the $G_O(s)$ denominator $m_{ap}(s)$. ■

Proof. Follow the proof of Theorem 6. If

$$\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta \arg} m(s) \in \left(n\pi/2 - \Phi, n\pi/2 + \Phi\right),\ \Phi = \arcsin\left(\sum_{j=1}^{h_n} \left|m_{nj}\right|\right) \tag{60}$$

then the closed-loop system is asymptotically and strongly stable according to Theorem 3. Since $\deg m(s) = \deg m_{ap}(s) = n$, $\Phi = \Phi_{ap}$, and (59) ensures the strong stability of both $m(s)$, $m_{ap}(s)$. Because of the fact that neutral quasipolynomials are analytic functions, using (47) and (48) it holds that

$$\begin{aligned}\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta \arg} m(s)/m_{ap}(s) &= n\pi/2 \pm \Phi - l\pi/2 \mp \Phi_{ap} \\ &= (n-l)\frac{\pi}{2} \\ &= \underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta \arg} \left(1 + G_0(s)\right)\end{aligned} \tag{61}$$

□

Remark 4. If ones want to study asymptotic stability solely, condition (61) can be used as well without considering (13); however, for strong stability (13) is a necessary initial conditions. ■

As was mentioned, since strong stability condition (13) ensures that the number of unstable zeros of a retarded quasipolynomial is finite, the relation between the main part of the overall argument shift (that divisible by $\pi/2$) and the number of unstable zeros is given by (16). If we use this fact on (61) and $m_{ap}(s)$, one can easily prove that (55) from Theorem 7 holds also for neutral systems with lumped delays.

C. LTI-TDS with distributed delays

In the case of input-output distributed delays, there is a polynomial factor in $a(s)$, the (unstable) zeros of which are those of $b(s)$. Viceversa, if some distributed delays are included in system dynamics, an unstable factor (or factors) appears in $b(s)$ where all its zeros are also included in $a(s)$.

Let us study the stability of the characteristic meromorphic function defined in (5) first. Hence

$$M(s) = \det[s\mathbf{I} - \mathbf{A}(s)] = \frac{M_n(s)}{M_d(s)} \tag{62}$$

where $M_n(s)$ is a (retarded or neutral) quasipolynomial of degree n_M and $M_d(s)$ is a polynomial of a degree d_M with n_{uM} zeros in $\mathbb{C}^+$ which are those of $M_n(s)$. Then the following theorem can be formulated.

Theorem 9 (Argument increment principle for a meromorphic function with distributed delays). Consider the meromorphic function $M(s)$ as in (62) where $M_n(s) \neq 0, M_d(s) \neq 0$ for any imaginary $s = \mathrm{j}\omega$, $\omega \in \mathbb{R}$. Then

a) If $M_n(s)$ is a retarded quasipolynomial, $M(s)$ has no zero in $\mathbb{C}^+$ if and only if

$$\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg} M(s)=\frac{(n_M-d_M)\pi}{2} \tag{63}$$

b) If $M_n(s)$ is a neutral quasipolynomial satisfying (13), $M(s)$ has no zero in $\mathbb{C}^+$ and it is strongly stable if and only if

$$\frac{(n_M-d_M)\pi}{2}-\Phi_M \le \underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg} M(s) \le \frac{(n_M-d_M)\pi}{2}+\Phi_M \tag{64}$$

where

$$\begin{aligned}&\Phi=\arcsin\left(\sum_{j=1}^{h_{n,M_u}}\left|M_{u,nj}\right|\right)\\ &M_u(s)=s^{n_M}+\sum_{i=0}^{n_M}\sum_{j=1}^{h_i}M_{u,ij}s^i\exp(-s\eta_{ij})\end{aligned} \tag{65}$$

■

Proof. Let us make a proof of the case a). The second part of the proof can be done analogously using the fact that $M_n(s)$ is strongly stable and (16) can be taken into account.

Assume two cases. First, let (quasi)polynomials $M_n(s)$, $M_d(s)$ have all their zeros located in $\mathbb{C}_0^-$. Since both functions are analytic, from Theorem 2 it holds that

$$\underset{\omega\in[0,\infty)}{\Delta\arg} M(\mathrm{j}\omega)=\underset{\omega\in[0,\infty)}{\Delta\arg} M_n(\mathrm{j}\omega)-\underset{\omega\in[0,\infty)}{\Delta\arg} M_d(\mathrm{j}\omega)=(n_M-d_M)\frac{\pi}{2} \tag{66}$$

Second, let all n_{uM} zeros of $M_d(s)$ in are those of $M_n(s)$ and there is no other one in $M_n(s)$. From (16) we have

$$\begin{aligned}&\underset{\omega\in[0,\infty)}{\Delta\arg} M_n(\mathrm{j}\omega)=(n_M-2n_{uM})\frac{\pi}{2}\\ &\underset{\omega\in[0,\infty)}{\Delta\arg} M_d(\mathrm{j}\omega)=(d_M-2n_{uM})\frac{\pi}{2}\end{aligned} \tag{67}$$

which gives (66) and (63) again.

The inverse can be proved analogously (by steps in reverse order). □

Consider now a feedback system as in Fig. 1 with a plant affected by distributed delays.

Theorem 10. (The Nyquist criterion for LTI-TDSs with distributed delays). Let the plant and the controller have transfer functions as in (47) with distributed delays (and possibly lumped ones) and let the control system be of the scheme as in Fig. 1. Let quasipolynomials $a(s)$ and $p(s)$ have no root on the imaginary axis, i.e. $a(s)\neq 0, p(s)\neq 0$ for any imaginary $s=\mathrm{j}\omega$, $\omega\in\mathbb{R}$, and define the denominator $m_{ap}(s)$ of $G_O(s)$ as in (57). Then

a) If $m_{ap}(s)$ is a retarded quasipolynomial with

$$\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg} m_{ap}(s)=l\pi/2 \tag{68}$$

then the closed-loop system is asymptotically stable if

$$\underset{s=\omega\mathrm{j},\omega\in[0,\infty)}{\Delta\arg}(1+G_O(s))=(n-l-2\overline{n}_U)\frac{\pi}{2}=\overline{n}_{U,ap}\pi \tag{69}$$

holds where *n* is the highest *s*-power in $m_{ap}(s)$, $\overline{n}_U$ is the number of common zeros of the numerator and denominator of $G_O(s)$ in $\mathbb{C}^+$ and $\overline{n}_{U,ap}$ stands for the number of unstable zeros of $m_{ap}(s)$ which are not included in the numerator of $G_O(s)$.

b) If $m_{ap}(s)$ is a neutral quasipolynomial with (57) and (58) satisfying (13), then the closed-loop system is asymptotically and strongly stable if (69) holds.

Proof. Consider a general case for retarded LTI-TDSs. Formulation b) of Theorem 10 can be proved in a similar way.

Let the numerator and denominator (i.e. $m_{ap}(s)$) of $G_O(s)$ have exactly $\overline{n}_U$ common zeros in $\mathbb{C}^+$. From (48) it arises that these roots are zeros of $m(s)$ as well, hence, they are not the system poles since are canceled just by $m_{ap}(s)$.

Thus, all number $n_{U,ap}$ of unstable zeros of $m_{ap}(s)$ is given by (16) as

$$\begin{aligned}&n_{U,ap}=\overline{n}_U+\overline{n}_{U,ap}=\left(\frac{n}{2}-\frac{\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg} m_{ap}(s)}{\pi}\right)\\ &\quad\Rightarrow \underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg} m_{ap}(s)=\left(n-2\left(\overline{n}_U+\overline{n}_{U,ap}\right)\right)\frac{\pi}{2}\end{aligned} \tag{70}$$

and those of $m(s)$

$$\begin{aligned}&\overline{n}_U=\left(\frac{n}{2}-\frac{\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg} m(s)}{\pi}\right)\\ &\quad\Rightarrow \underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg} m(s)=(n-2\overline{n}_U)\frac{\pi}{2}\end{aligned} \tag{71}$$

From (47), (48), (68), (70) and (71) we have finally

$$\begin{aligned}&\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg} m(s)/m_{ap}(s)=\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg}(1+G_0(s))\\ &=\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg} m(s)-\underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\Delta\arg} m_{ap}(s)=(n-l-2\overline{n}_U)\frac{\pi}{2}=\overline{n}_{U,ap}\pi\end{aligned} \tag{72}$$

□

Clearly, Theorem 7 holds true as well. Note that the common unstable zeros of $m_{ap}(s)$ and $m(s)$ are not taken as poles of $G_O(s)$.

V. Examples

A. *Retarded LTI-TDS with lumped delays*

Let the retarded LTI-TDS plant be described by the transfer function

$$G(s)=\frac{b(s)}{a(s)}=\frac{\exp(-1.1s)}{s-5\exp(-s)} \tag{73}$$

and consider utilization of a proportional controller $q=q_0$.

The controlled system is unstable which is clear from the Mikhaylov plot $a(\mathrm{j}\omega)$ displayed in Fig. 3 (a detailed zoom to the origin of the complex plane is added) since the overall phase shift (the argument change) is $-5\pi/2$, i.e. $l=-5$. In other words, the plant has three unstable poles because of Proposition 1. The task is to find the appropriate range of q_0 so that the closed loop is asymptotically stable.

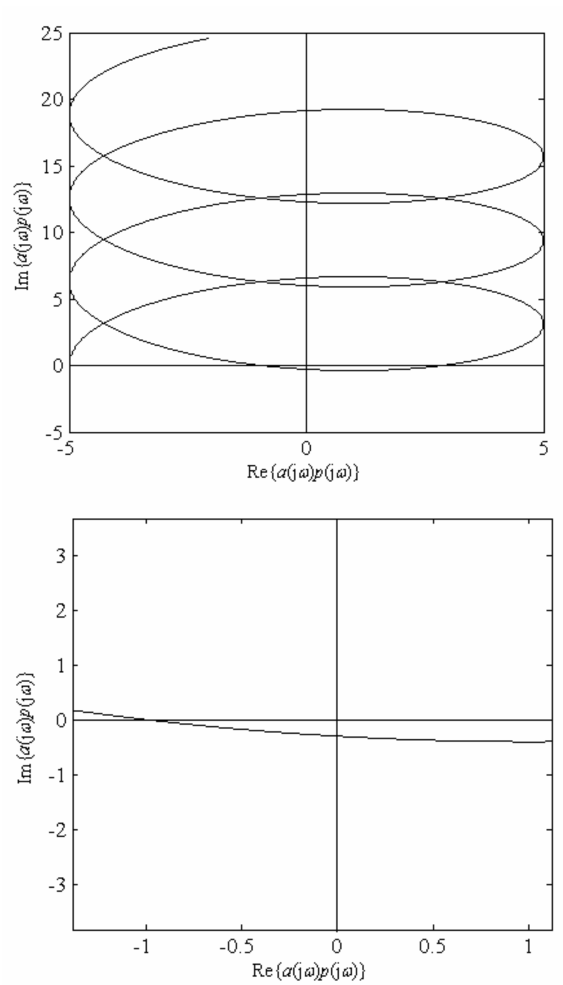

Fig. 3 Mikhaylov plot of $a(s)$ from (73) (a) and a detail of the vicinity of the origin (b)

Hence, the closed-loop characteristic quasipolynomial reads

$$m(s)=s-5\exp(-s)+q_0\exp(-1.1s) \tag{74}$$

According to Remark 1, one can calculate the set of frequencies as $\Omega_1=\{0.953, 4.741, 6.702, 10.385, 12.498, \ldots\}$ and easily verify that the critical frequency satisfying definition (29) is $\omega_C=0.953$ which gives rise to the critical gain $q_C=5.803$. Since $\sin(1.048)=0.867$ and $\cos(1.048)=0.5$, Theorem 5 results in the stabilizing interval

$$5<q_0<5.803 \tag{75}$$

Set e.g. $q_0=5.4$ and display the Nyquist plot of the open loop, see Fig. 4. The overall phase shift around the point -1 is 3π. Because $n=1$, the closed loop is stable according to Theorem 6 (or Theorem 7).

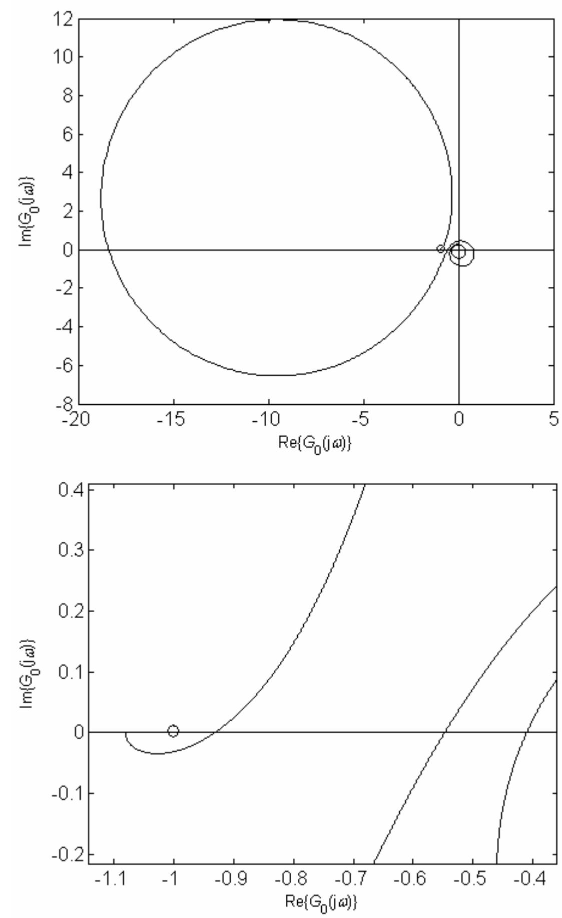

Fig. 4 Nyquist plot of $G_O(s)$ for plant (73) and a proportional controller $q_0=5.4$ (a) and a detail of the vicinity of the critical point -1 (b)

B. *Neutral LTI-TDS with lumped delays*

Let the neutral LTI-TDS plant be described by the transfer function

$$G(s) = \frac{b(s)}{a(s)} = \frac{3s+1}{(1+0.5\exp(-s))s^2 - s + 1} \tag{76}$$

and consider utilization of a proportional controller $q = 2$. The open loop transfer function denominator

$$m_{ap}(s) = a(s) = (1+0.5\exp(-s))s^2 - s + 1 \tag{77}$$

is strongly stable since (13) holds (i.e. the controlled system is stable as well). However, the system is not asymptotically stable, because

$$\underset{s=\omega j, \omega \in [0,\infty)}{\Delta \arg} m_{ap}(s) \in (-\pi - \Phi_{ap}, -\pi + \Phi_{ap}) \tag{78}$$

where

$$\Phi_{ap} = \arcsin(0.5) = \pi/6 \tag{79}$$

see Fig. 5. Since $n = 2$ and the "main" part of $\Delta \arg m_{ap}(s)$ equals $-\pi$ (i.e. $l = -2$), number of unstable poles from (16) is 2 (i.e. a complex conjugate pair).

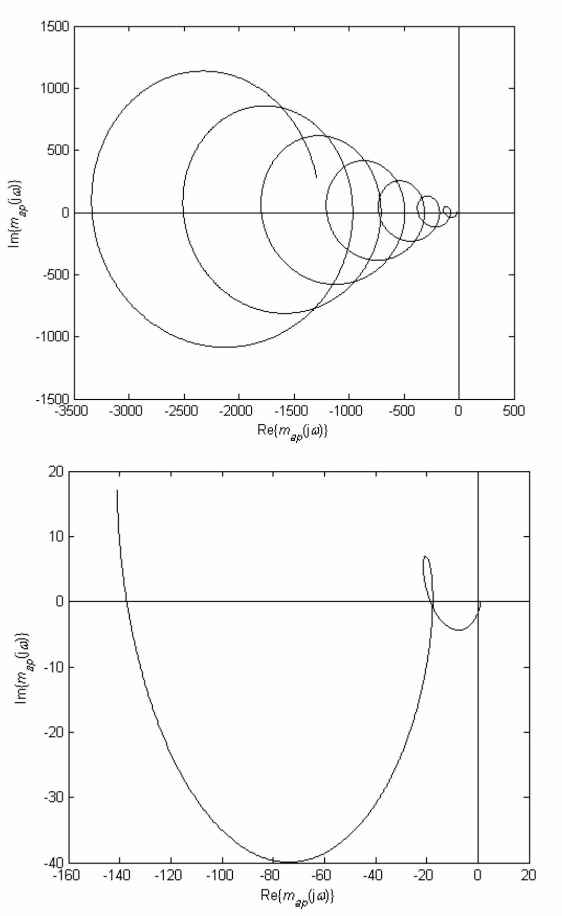

Fig. 5 Mikhaylov plot of $m_{ap}(s)$ from (77) (a) and a detail of the vicinity of the origin (b)

The Nyquist plot of $G_O(s)$ is displayed in Fig. 6.

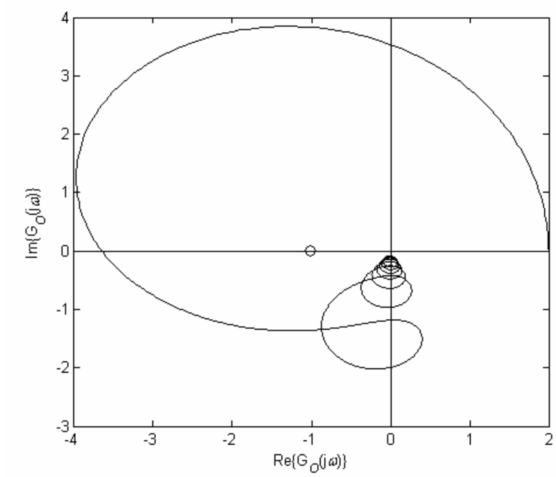

Fig. 6 Nyquist plot of $G_O(s)$ for plant (76) and a proportional controller $q_0 = 2$

According to Theorem 8, the closed loop system is asymptotically (and strongly) stable, since

$$\underset{s=j\omega, \omega \in [0,\infty)}{\Delta \arg} (1 + G_0(s)) = 2\pi \tag{80}$$

which also agrees with the precept about the number of unstable poles.

VI. Conclusion

This contribution has presented a study about the asymptotic and neutral stability of LTI-TDSs. In the first part of the paper, a basic overview about the stability and the argument principle for LTI-TDs has been presented. A revision of our results about the asymptotic stability of retarded quasipolynomials has been introduced in the second part. The Nyquist criteria for retarded and neutral systems based on the argument principle for a simple feedback loop have followed. Both lumped and distributed delays have been taken into account in theorems. It was i.a. verified that the obligatory statement about the number of open-loop unstable poles holds for these systems as well.

In the future research, other feedback control systems can be utilized which give rise to rather more complex criteria

Some of the presented results have been clarified by simulation examples.

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