Multi-dimensional Mathematical Models of Intensive Steel Quenching for Sphere. Exact and Approximate Solutions

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***Abstract*—**In this paper we develop mathematical models for 3-D and 1-D hyperbolic heat equations and construct their analytical solutions for the determination of the initial heat flux for rectangular and spherical samples. Some solutions of time inverse problems are obtained in closed analytical form. We use approximate analytical solutions on the basis of conservative averaging method and compare the difference between polynomial approximations of exact solutions. Some numerical results are given for a silver ball. The influence of relaxation time on solution, linearity of classical and hyperbolic heat equation, linear and non-linear boundary conditions are investigated.

***Keywords*—**Intensive quenching, Hyperbolic Heat equation, Direct problem, Inverse problem, Exact solution, Fredholm integral equation, Conservative averaging method.

I. Introduction

CONTRARY to the traditional method of steel quenching in oil or polymer solutions, the intensive quenching process uses environmentally friendly highly agitated water or low concentration of water/mineral salt solutions and very fast cooling rates are applied [1]-[7]. Experiments show that classical heat conduction equation doesn't stand when we try to model process of rapid cooling [8]. We propose to use hyperbolic heat equation for more realistic description of the intensive quenching process (especially for process initial stage) [9], [10], [11].

Complete bibliography on hyperbolic heat conduction equation can be found in [12].

In our previous papers we have constructed analytical exact and approximate [9], [10] solutions for intensive quenching processes. Here we consider few other models and construct solutions for direct and inverse problems of hyperbolic heat conduction equation. Since there is water involved in the process, we have to solve nonlinear boundary condition case. Here are given both approximate (on the basis of the conservative averaging method, see [13], [14]), and exact (on the basis of Green function method, see [15]-[18]) solutions.

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II. Mathematical Formulation of 3-D Problem and Solutions for Parallelepiped

In this section we give the mathematical statement for direct and time inverse problems.

A. Mathematical Statement of Full 3-D Problem for Parallelepiped

The non-dimensional temperature field fulfils hyperbolic heat equation (telegraph equation):

$$\tau_r \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} = a^2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right),$$
$$x \in (0,l), y \in (0,b), z \in (0,w), \tag{1}$$
$$t \in (0,T], \quad a^2 = \frac{k}{c\rho}.$$

Here c is the specific heat capacity, k - the heat conduction coefficient, ρ - the density, τ_r - the relaxation time. It represents the time lag needed to establish resulting heat flux when temperature gradient is suddenly imposed [9], [10], [11].

It is a natural assumption that planes $x = 0, y = 0, z = 0$ are symmetry surfaces of the sample:

$$\left.\frac{\partial V}{\partial x}\right|_{x=0} = 0, \tag{2}$$

$$\left.\frac{\partial V}{\partial y}\right|_{y=0} = 0, \tag{3}$$

$$\left.\frac{\partial V}{\partial z}\right|_{z=0} = 0. \tag{4}$$

On all other sides of the steel part we have heat exchange with environment. Although the method proposed here is applicable for non-homogeneous environment temperature, for simplicity we consider models of constant environment temperature

$\Theta_0 = 0$.

This restriction gives following homogeneous third type boundary conditions on the all three outer sides:

$$\left.\left(\frac{\partial V}{\partial x} + \beta V\right)\right|_{x=l} = 0, \beta = \frac{h}{k}, \tag{5}$$

$$\left.\left(\frac{\partial V}{\partial y} + \beta V\right)\right|_{y=b} = \left.\left(\frac{\partial V}{\partial z} + \beta V\right)\right|_{z=w} = 0. \tag{6}$$

h denotes heat exchange coefficient.

The initial conditions are assumed in form:

$$V\big|_{t=0} = V_0(x, y, z), \tag{7}$$

$$\left.\frac{\partial V}{\partial t}\right|_{t=0} = W_0(x, y, z). \tag{8}$$

From the practical point of view the condition (8) is unrealistic. The initial heat flux must be determined theoretically. As additional condition we assume that the temperature distribution and the distribution of heat fluxes at the end of process are given (known):

$$V\big|_{t=T} = V_T(x, y, z), \tag{9}$$

$$\left.\frac{\partial V}{\partial t}\right|_{t=T} = W_T(x, y, z). \tag{10}$$

As the first step we use well known substitution:

$$V(x, y, z, t) = \exp\left(-\frac{t}{2\tau_r}\right) U(x, y, z, t). \tag{11}$$

Then the differential equation (1) transforms into differential equation without the first time derivative:

$$\frac{\partial^2 U}{\partial t^2} = a_\tau^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) + \frac{1}{4\tau_r^2} U,$$

$$x \in (0, l), y \in (0, b), z \in (0, w), t \in (0, T], \tag{12}$$

$$a_\tau^2 = a^2 / \tau_r.$$

The initial and boundary conditions take form:

$$U\big|_{t=0} = V_0(x, y, z), \tag{13}$$

$$\left.\frac{\partial U}{\partial t}\right|_{t=0} = W_0(x, y, z) + \frac{V_0(x, y, z)}{2\tau_r}, \tag{14}$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0, \tag{15}$$

$$\left.\frac{\partial U}{\partial y}\right|_{y=0} = 0, \tag{16}$$

$$\left.\frac{\partial U}{\partial z}\right|_{z=0} = 0, \tag{17}$$

$$\left.\left(\frac{\partial U}{\partial x} + \beta U\right)\right|_{x=l} = 0, \tag{18}$$

$$\left.\left(\frac{\partial U}{\partial y} + \beta U\right)\right|_{y=b} = 0, \tag{19}$$

$$\left.\left(\frac{\partial U}{\partial z} + \beta U\right)\right|_{z=w} = 0. \tag{20}$$

Additional conditions (9), (10) transform as follow:

$$U\big|_{t=T} = \exp\left(\frac{T}{2\tau_r}\right) V_T(x, y, z), \tag{21}$$

$$\left.\frac{\partial U}{\partial t}\right|_{t=T} = \exp\left(\frac{T}{2\tau_r}\right)\left[W_T(x, y, z) + \frac{V_T(x, y, z)}{2\tau_r}\right]. \tag{22}$$

B. Exact Solution of Direct 1-D Problem

We will start with a formulation of the mathematical model for a steel part which is thin in y, z directions – one-dimensional model:

$w \ll l, b \ll l.$

Then in accordance with conservative averaging method [13], [14] we introduce following integral averaged value:

$$u(x, t) = (bw)^{-1} \int_0^b dy \int_0^w U(x, y, z, t) dz. \tag{23}$$

Assuming the simplest approximation by constant in the y, z directions, we obtain 1-D differential equation with the source term:

$$\frac{\partial^2 u}{\partial t^2} = a_\tau^2 \frac{\partial^2 u}{\partial x^2} - cu, x \in (0, l), t \in (0, T],$$

$$c = \left[\beta\left(\frac{1}{b} + \frac{1}{w}\right) - \frac{1}{4\tau_r^2}\right]. \tag{24}$$

Initial conditions (13), (14) for the differential equation (12) are as follow:

$$u\big|_{t=0} = u_0(x), \tag{25}$$

$$u_0(x) = (bw)^{-1} \int_0^b dy \int_0^w V_0(x, y, z) dz,$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = v_0(x), v_0(x) = w_0(x) + \frac{u_0(x)}{2\tau_r}, \tag{26}$$

$$w_0(x) = (bw)^{-1} \int_0^b dy \int_0^w W_0(x,y,z)dz.$$

The boundary conditions remain in the same form:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \tag{27}$$

$$\left.\left(\frac{\partial u}{\partial x} + \beta u\right)\right|_{x=l} = 0. \tag{28}$$

Solution of this one-dimensional direct problem (24)-(28) is well known, see [15]-[18]:

$$u(x,t) = \frac{\partial}{\partial t}\int_0^l u_0(\xi)G(x,\xi,t)d\xi + \int_0^l v_0(\xi)G(x,\xi,t)d\xi. \tag{29}$$

The Green function has representation [15]:

$$G(x,\xi,t) = \sum_{i=1}^{m-1} \frac{\varphi_i(x)\varphi_i(\xi)\sinh\left(t\sqrt{|a_\tau^2\lambda_i^2 + c|}\right)}{\|\varphi_i\|^2 \sqrt{|a_\tau^2\lambda_i^2 + c|}} + \sum_{i=m}^{\infty} \frac{\varphi_i(x)\varphi_i(\xi)\sin\left(t\sqrt{a_\tau^2\lambda_i^2 + c}\right)}{\|\varphi_i\|^2 \sqrt{a_\tau^2\lambda_i^2 + c}}, \tag{30}$$

$$\varphi_i(x) = \cos(\lambda_i x),\ \|\varphi_i\|^2 = \frac{l}{2} + \frac{\beta}{2\left(\lambda_i^2 + \beta^2\right)}.$$

Here the natural number m in the both sums is given by inequalities:

$$a_\tau^2\lambda_i^2 + \left[\beta\left(\frac{1}{b} + \frac{1}{w}\right) - \frac{1}{4\tau_r^2}\right] < 0, i = \overline{1, m-1},$$

$$a_\tau^2\lambda_i^2 + \left[\beta\left(\frac{1}{b} + \frac{1}{w}\right) - \frac{1}{4\tau_r^2}\right] > 0, i = \overline{m, \infty}.$$

The eigenvalues λ_i are roots of the transcendental equation:

$$\lambda \tan(\lambda l) = \beta. \tag{31}$$

C. Solution of Time inverse 1-D Problem

As we mentioned earlier, from the experimental point of view initial condition (22) is unrealizable and the $v_0(x)$ must be calculated theoretically. The differentiation of solution (29) gives:

$$\frac{\partial}{\partial t}u(x,t) = \int_0^l u_0(\xi)\frac{\partial^2}{\partial t^2}G(x,\xi,t)d\xi + \int_0^l v_0(\xi)\frac{\partial}{\partial t}G(x,\xi,t)d\xi. \tag{32}$$

The additional conditions (21) and (22) at the end of the process regarding the function $u(x,t)$ are as follow:

$$u|_{t=T} = u_T(x), u_T(x) = \exp\left(\frac{T}{2\tau_r}\right)\tilde{v}_T(x),$$
$$\tilde{v}_T(x) = (bw)^{-1}\int_0^b dy \int_0^w V_T(x,y,z)dz, \tag{33}$$

Respectively

$$\left.\frac{\partial u}{\partial t}\right|_{t=T} = v_T(x), \tag{34}$$

$$v_T(x) = \exp\left(\frac{T}{2\tau_r}\right)\left[\frac{\tilde{v}_T(x)}{2\tau_r} + w_T(x)\right], \tag{35}$$

$$w_T(x) = (bw)^{-1}\int_0^b dy \int_0^w W_T(x,y,z)dz. \tag{36}$$

If both additional conditions are known we introduce new time argument by formula

$$\tilde{t} = T - t. \tag{37}$$

The main differential equation (24) remains its form:

$$\frac{\partial^2 u}{\partial \tilde{t}^2} = a_\tau^2 \frac{\partial^2 u}{\partial x^2} - cu, x \in (0,l), \tilde{t} \in (0,T]. \tag{38}$$

The boundary conditions (27), (28) remain the same. Both additional conditions transform to initial conditions for the equation (38):

$$u|_{\tilde{t}=0} = u_T(x),\ \left.\frac{\partial u}{\partial \tilde{t}}\right|_{\tilde{t}=0} = -v_T(x). \tag{39}$$

The solution of direct problem (38), (27), (28) and (39) is similar with the solution (29):

$$u(x,\tilde{t}) = \frac{\partial}{\partial \tilde{t}}\int_0^l u_T(\xi)G(x,\xi,\tilde{t})d\xi - \int_0^l v_T(\xi)G(x,\xi,\tilde{t})d\xi. \tag{40}$$

For the heat flux we have an expression:

$$\frac{\partial}{\partial \tilde{t}}u(x,\tilde{t}) = \int_0^l u_T(\xi)\frac{\partial^2}{\partial \tilde{t}^2}G(x,\xi,\tilde{t})d\xi - \int_0^l v_T(\xi)\frac{\partial}{\partial \tilde{t}}G(x,\xi,\tilde{t})d\xi. \tag{41}$$

From formula (41) immediately follows a nice explicit representation for the initial heat flux:

$$v_0(x) = \int_0^l u_T(\xi) \frac{\partial^2}{\partial \tilde{t}^2} G(x,\xi,\tilde{t})\Big|_{\tilde{t}=T} d\xi - \\ -\int_0^l v_T(\xi) \frac{\partial}{\partial \tilde{t}} G(x,\xi,\tilde{t})\Big|_{\tilde{t}=T} d\xi. \quad (42)$$

In previous paper [11] we have used the Green function for classical (parabolic) heat equation, but here we used Green function for the wave (hyperbolic) equation.

III. Mathematical Statement of Problem for Sphere

A. 3-D Problem for Sphere

We examine a problem for spherical sample hence it is useful to apply the Spherical coordinate system. Hyperbolic heat equation is in form

$$\tau_r \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} = a^2 (\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial V}{\partial}) + \frac{1}{r^2 \sin^2\theta}(\frac{\partial^2 V}{\partial \varphi^2}) \\ + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial V}{\partial \theta})) + F; \\ 0 < r < R;\ W_0(\varphi) < \theta < W_1(\varphi);\ 0 < \varphi < 2\pi; \quad (43)$$

Initial conditions:

$$V(r;\theta,\varphi,0) = V_0(r;\theta,\varphi), \\ \frac{\partial V(r;\theta,\varphi,0)}{\partial t} = V_1(r;\theta,\varphi) \quad (44)$$

and boundary conditions

$$r^2 \frac{\partial V}{\partial r}|_{r\to 0} \to 0, \\ \left(kr^2 \frac{\partial V}{\partial r} + h_1 V\right)|_{r=R} = g_1(\theta,\varphi,t), \\ V|_{\varphi=+0} = V|_{\varphi=2\pi-0}, \\ \frac{\partial V}{\partial \varphi}|_{\varphi=+0} = \frac{\partial V}{\partial \varphi}|_{\varphi=2\pi-0}, \\ \left(\sin\theta \frac{\partial V}{\partial \theta} - h_2 V\right)|_{\theta=W_0(\varphi)} = g_2(r,\varphi,t), \\ \left(\sin\theta \frac{\partial V}{\partial \theta} + h_3 V\right)|_{\theta=W_1(\varphi)} = g_3(r,\varphi,t). \quad (45)$$

B. Time inverse 1-D Problem for Sphere

We can apply conservative averaging method to reduce problems dimensions. If we assume that 1-D problem can describe the process – i. e. process does not depend on values of θ and φ - our problem is in form

$$\tau_r \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} = a^2 (\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r}) + F(r,t), \\ r \in (0,R), t \in (0,T), R < \infty, \\ k\frac{\partial V}{\partial r} + h_1 V = \Theta_1(t), r = R, t \in [0,T], \\ \frac{\partial V}{\partial r} = 0, r = 0, \\ V(r,0) = V^0(r),\ r \in [0,R]. \quad (46)$$

As mentioned previously, initial heat flux cannot be determined experimentally so we use temperature distribution at the end of process:

$$V(x,T) = V_T(x), x \in [0,R] \quad (47)$$

IV. Application of Conservative Averaging Method for Time Inverse Hyperbolic Heat Conduction Problem

In this part we consider 1-D spherical hyperbolic heat equation:

$$\tau_r \frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t} = a^2 \frac{1}{r}\frac{\partial^2}{\partial r^2}(rV) \quad (48)$$

Transformation

$$U = rV \quad (49)$$

allows us to go to Cartesian coordinates. We compare the solution of this equation with classical parabolic heat equation.

A. Original Problem

We start with the formulation of the one-dimensional mathematical model for intensive steel quenching without heat losses:

$$\tau_r \frac{\partial^2 U}{\partial t^2} + \frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2} + f(x,t), \\ x \in (0,H), t \in (0,T), H < \infty, \quad (50)$$

$$\left(-k\frac{\partial U}{\partial x} + hU\right) = h\Theta(t), x = 0, t \in [0,T], \quad (51)$$

$$\frac{\partial U}{\partial x} = 0, x = H, \quad (52)$$

$$U = U^0(x), t = 0, x \in [0,H]. \quad (53)$$

The initial heat flux

$$\frac{\partial U}{\partial t} = V_0(x), x \in [0,H],\ t = 0 \quad (54)$$

can't be measured experimentally and must be calculated. As an additional condition we assume experimentally realizable condition – the temperature distribution at the end of process is given:

$$U(x,T) = U_T(x), x \in [0,H]. \quad (55)$$

B. The Approximate Solution by Conservative Averaging Method

By applying conservative averaging method to the problem (11)-(20) we obtain the integral average temperature $u_0(t)$ and following boundary problem for ordinary differential equation:

$$\tau_r \frac{du_0^2}{dt^2} + \frac{du_0}{dt} + \frac{h}{c_H} u_0 = \frac{h}{c_H}\Theta(t) + f(t), \quad (56)$$

$$c_H = c\rho H,$$

$$u_0(0) = u_0^0, u_0(T) = u_T. \quad (57)$$

We are interested to determine

$$v_0 = \frac{du_0(0)}{dt}. \quad (58)$$

To solve this problem we split it in two sub-problems:

$$u_0(t) = \bar{u}(t) + \bar{w}(t). \quad (59)$$

First sub-problem has homogeneous main equation:

$$\tau_r \frac{d\bar{u}}{dt^2} + \frac{d\bar{u}}{dt} + \frac{h}{c_H}\bar{u} = 0 \quad (60)$$

together with non-homogeneous initial conditions:

$$\bar{u}(0) = u_0^0, \ \frac{d\bar{u}(0)}{dt} = v_0. \quad (61)$$

This problem can be solved in traditional way and its solution is:

$$\bar{u}(t) = e^{-\frac{t}{2\tau_r}}(u_0^0 e^{\frac{t}{2\tau_r}\beta} + ((\frac{u_0^0}{2\tau_r}(1-\frac{1}{\beta}) - \frac{v_0}{\beta})\sinh(\frac{t}{2\tau_r}\beta)).$$

Here $\beta = \frac{1}{2\tau_r}\sqrt{1-4\tau_r\frac{h}{c_H}}$. (62)

The second sub-problem has non-homogeneous main equation and homogeneous initial conditions:

$$\tau_r \frac{d\bar{w}^2}{dt^2} + \frac{d\bar{w}}{dt} + \frac{h}{c_H}\bar{w} = \frac{h}{c_H}\Theta(t) + f(t) \quad (63)$$

$$\bar{w}(0) = \frac{d\bar{w}(0)}{dt} = 0.$$

The solution of this problem has a following form:

$$\bar{w}(t) = e^{-\frac{t}{2\tau_r}}\int_0^t q(t-\tau)\Phi(\tau)d\tau,$$

$$\Phi(t) = \tau_r^{-1}\left[\gamma\Theta(t) + f(t)\right]e^{\frac{t}{2\tau_r}}, \gamma = \frac{h}{c_H}.$$

Here $q(t)$ is solution of the differential equation (63) with special initial conditions:

$$q(0) = 0, \ \frac{dq(0)}{dt} = 1,$$

i.e. $q(t) = \beta^{-1}\sinh(\beta t)$.

Hence:

$$\bar{w}(t) = \frac{2\tau_r}{\beta}e^{-\frac{t}{2\tau_r}}\int_0^t \sinh(\frac{t-\omega}{2\tau_r}\beta)(e^{\frac{\omega}{2\tau_r}}\frac{h}{c\cdot\rho}\Theta(\omega))d\omega$$

Consequently, we have finally obtained the solution of the problem (56), (57) as:

$$\begin{aligned} u_0(t) &= e^{-\frac{t}{2\tau_r}}[u_0^0 e^{\frac{t}{2\tau_r}\beta} + \\ &+(\frac{1}{2\tau_r}u_0^0 - u_0^0\frac{1}{\beta 2\tau_r} - \frac{v_0}{\beta})\sinh(\frac{t}{2\tau_r}\beta)] + \\ &+\frac{2\tau_r}{\beta}e^{-\frac{t}{2\tau_r}}\int_0^t \sinh(\frac{(t-\omega)}{2\tau_r}\beta)(e^{\frac{\omega}{2\tau_r}}\cdot\frac{h}{c\cdot\rho}\Theta(\omega))d\omega. \end{aligned} \quad (64)$$

As the last step we use the additional information – condition (9), i.e. the known value at the end of the process. This information allows us to express unknown second initial condition in closed and simple form:

$$\begin{aligned} v_0 &= \frac{\beta}{\sinh(\frac{T\beta}{2\tau_r})}\times \\ &(u_T\cdot e^{\frac{T}{2\tau_r}} + u_0^0\cdot e^{\frac{T}{2\tau_r}\beta} + \frac{u_0^0}{2\tau_r}(1-\frac{1}{\beta})\sinh(\frac{T\beta}{2\tau_r})) + \\ &\frac{2\tau_r}{\sinh(\frac{T\beta}{2\tau_r})}\times \\ &\int_0^T \sinh(\frac{(T-\omega)}{2\tau_r}\beta)\left(\frac{h}{c\cdot\rho}\Theta(\omega) + f(\omega)\right)e^{\frac{\omega}{2\tau_r}}d\omega. \end{aligned} \quad (65)$$

We can increase the order of the approximation for the solution of the original problem (11)-(20) by the representation with polynomial of second degree and exponential approximation. Linear approximation reduces to approximation with constant. Approximation with second degree polynomial [13], [14]:

$$U(x,t) = u_0(t) + \frac{x}{R}u_1(t) + \left(\frac{x}{R}\right)^2 u_2(t).$$

We use boundary conditions to determine $u_1(t)$ and $u_2(t)$. The integration over interval $x \in [0, H]$ of the main equation practically gives the same ordinary differential equation (56). The only difference is in the same coefficient at two terms:

$$\begin{aligned} &\tau_r\frac{du_0^2}{dt^2} + \frac{du_0}{dt} + \frac{\delta}{R}u_0 = \delta\Theta(t) + f(t), \\ &\delta = \frac{2kh}{c\rho(hR+2k)}. \end{aligned} \quad (66)$$

The additional conditions remain the same. It means that we can use formulae obtained above, replacing the parameters β, γ by following expressions:

$$\beta = \frac{1}{2\tau_r}\sqrt{1-\frac{4\cdot\tau_r\cdot h}{c_H(1+\frac{hR}{2k})}}, \gamma = \frac{h}{c_H\left(1+\frac{hH}{2k}\right)} \quad (67)$$

Exponential approximation:

$U(x,t) = u_0(t) + (e^{-x}-1)u_1(t) + (1-e^x)u_2(t).$

Differential equation is in for

$$\tau_r \frac{du_0^2}{dt^2} + \frac{du_0}{dt} + \frac{hku_0}{2c\rho R(k\sinh R + h\cosh R - h)} =$$

$$= \frac{hk\Theta(t)}{2c\rho(k\sinh R + h\cosh R - h)} + f(t).$$

Alike previous case, difference is only in parameters β and γ:

$$\beta = \frac{1}{2\tau_r}\sqrt{1-\frac{4\cdot\tau_r\cdot h}{c_H(2(\sinh R + \frac{h}{k}(\cosh R - 1))}},$$

$$\gamma = \frac{h}{c_H(2(\sinh R + \frac{h}{k}(\cosh R - 1))}. \quad (68)$$

τ_r	v_0
0.2	-4499.270429
0.5	-1799.270299
1.5	-600.2998684

We have obtained solution of well posed problem in closed form. This solution can be used as initial approximation for integrated over $x \in [0, H]$ equation.

Conservative averaging method can be applied to problems with non linear BC. Condition for nucleate boiling ($m \in [3; 3\frac{1}{3}]$):

$$k\frac{\partial U}{\partial x} + \beta^m [U - \Theta_B(t)]^m = 0, x = R, t \in [0, T].$$

One dimensional nonlinear BC case is solved numerically.

V. Results

We solved several problems and obtained numerical results using Maple and COMSOL Multiphysics. Modelling is done for a silver ball, r=0.02m, temperature at the beginning of the process (t=0) is 600°C, and at the end (t=T) 0°C.

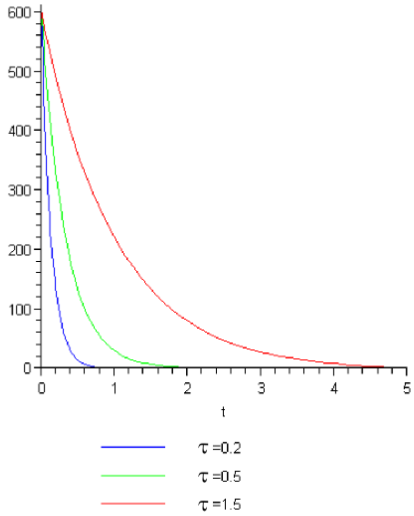

Fig. 1 Dependence on τ_r value

Smaller relaxation time values correspond to faster cooling and greater heat fluxes.

If we compare results obtained by approximation with constant, second degree polynomial and exponential approximation (Fig. 2.a, 2.b.), we see that they are close.

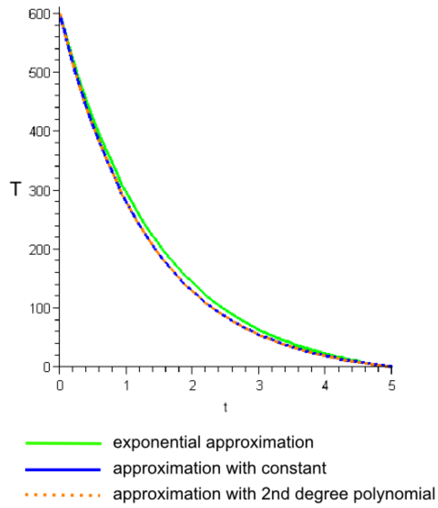

Fig. 2.a Comparison of results. t=2 sec

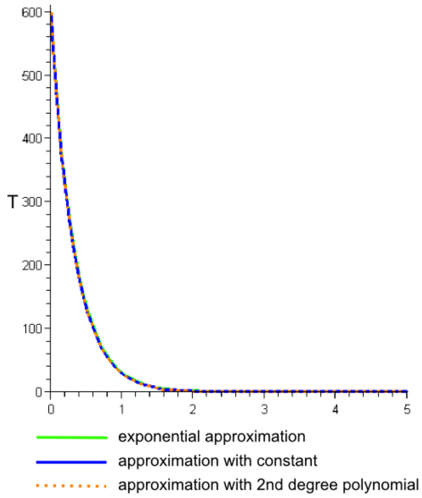

Fig. 2.b Comparison of results. t=0.5 sec

The smaller the τ_r value the closer the approximate solutions, so we can conclude that it is sufficient to use approximation with constant.

When initial heat flux v_0 is changed a little: $\pm\varepsilon$, the solution also changes (Fig. 3.1, 3.b). It is important to calculate v_0 very precisely. Note that time interval [1; 5] is observed.

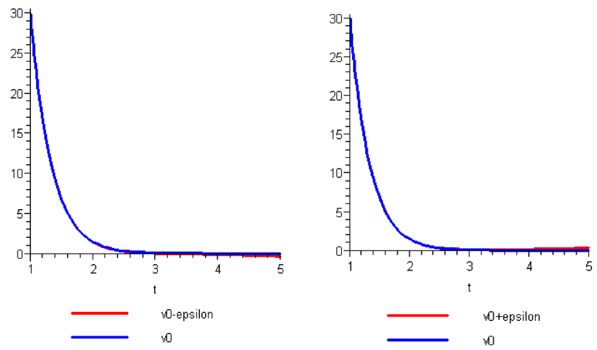

Fig. 3.a Differences in solutions when initial heat flux is changed. ε=0.01

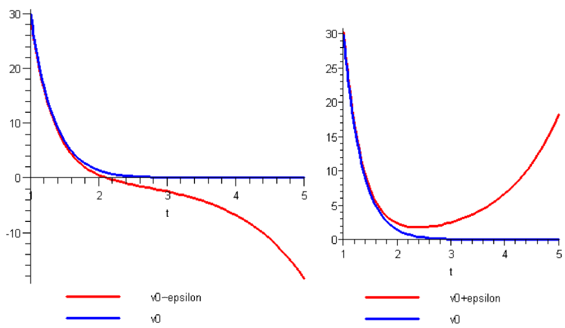

Fig. 3.b Differences in solutions when initial heat flux is changed. ε=0.5

If we compare solutions of classic (parabolic) and hyperbolic heat conduction problems, using nonlinear boundary condition case, we obtain graphic in Figure 4.

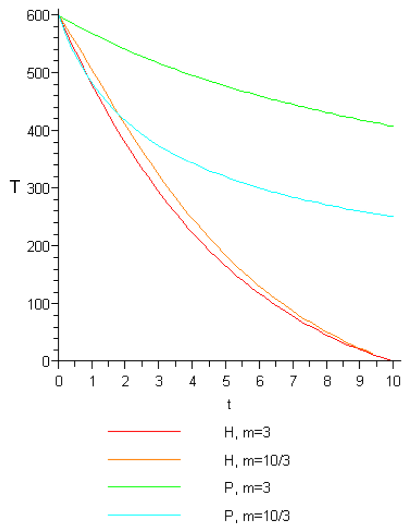

Fig. 4. Nonlinear BC case

We examined temperature on the radius. As you can see, the temperature on radius is not monotony. It means that the form of boundary condition on the surface can vary:

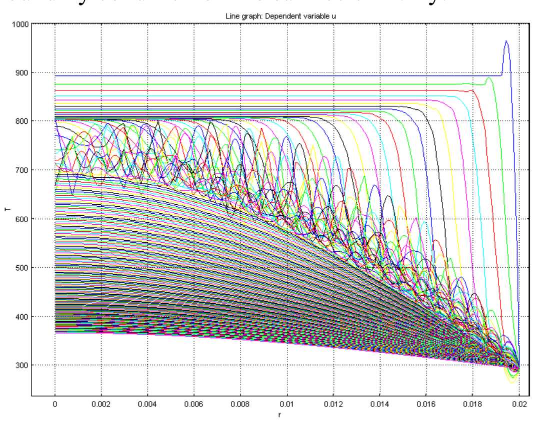

Fig. 5. Temperature distribution on radius

It is very clear that at the beginning of the process hyperbolic term is extremely important but later process is described by classic heat equation. Fig. 4 presents temperature's wave-like nature in the intensive steel quenching process.

It is possible to define precise points were temperature is computed:

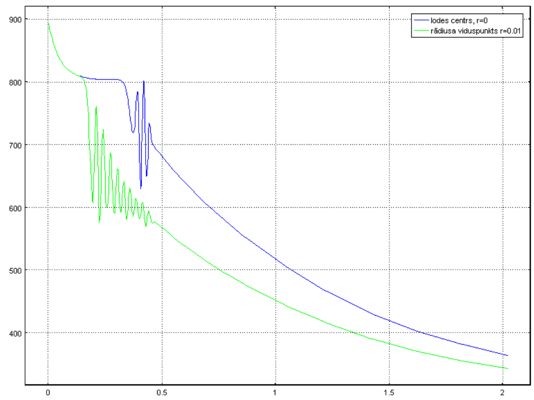

Fig. 6. Temperature changes at the centre (r=0) and r=0.01m

VI. CONCLUSIONS

We have constructed some solutions for time inverse problems for hyperbolic heat equation with linear and nonlinear boundary conditions. The solutions for determination of initial heat flux are obtained in closed analytical form. Numerical results are obtained and examined for spherical sample.

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