Non-Isothermal Steady Flow of Power-Law Fluids between Parallel Plates

Gabriella Bognár, János Kovács

Abstract— We study the shear flow of a non-Newtonian fluid between flat parallel plates in relative tangential motion with temperature dependent viscosity. The temperature and velocity distributions are investigated for a layer of fluid with Nahme type rheological equation, located between two plates and the upper plate moves with constant velocity. The existence and multiplicity results are examined for the solutions to the corresponding boundary value problems. An application of this result for experimentally determined material parameters is given.

Keywords—Non-Newtonian fluid, Nahme-type model, Ostwald-de Waele power law model, Polymer melts, Temperature dependent viscosity

I. Introduction

In many fields, such as food industry, drilling operations, the fluids either synthetic or natural, are mixtures of water, particle, oils, red cells and other long chain molecules. This combination imparts non-Newtonian characteristics to the resulting liquids. The viscosity function varies non-linearly with the shear rate. The practice of non-Newtonian fluids includes extrusion of polymer fluids, colloidal and suspension solutions, molten plastics and many others. Due to the diversity of fluids in nature many models have been proposed to describe their behavior. One particular non-Newtonian model which has been widely studied is the Ostwald-de Waele power-law model.

Viscous heating can play an important role in the channel flow dynamics of fluids with a temperature-dependent viscosity such as polymers and silicate melts. Exact solutions are given for flows of power-law fluids, with heat generation and temperature dependent viscosity, in three situations, namely pressure flow through a circular tube, shear flow between rotating concentric cylinders and shear flow between parallel plates by Martin [14].

It is well-defined that nonisothermal flows of fluids with strong temperature dependence viscosity can lead undesirable instability in technological processes. These phenomena can be mathematically described by nonlinear governing equations ([8], [10], [12], [15], [17]-[22]).

The polymer melt that is investigated in this work is a commercial grade low density polyethylene BRALEN RB 03-23 of Slovnaft Petrochemicals [25], s.r.o., further referred to as LDPE. BRALEN RB 03-23 which is designed for production of heavy duty and shrink films of thickness 0,07 - 0,25 mm. It is well suited for blow moulding of various containers, pipes, sheets and profiles extrusion and also for injection moulding. In this paper we focus our investigations to the physical quantities characterizes polyethylene flow in a capillary. First, the material characterization is given to obtain the necessary data. A quantitative description of the rheological behavior of polymer melts is crucial in understanding the relation between processing and product properties. As an intermediate step between the well-defined rheometrical flows and complicated industrial processing flows, simplified, experimentally accessible, inhomogeneous flows that exhibit a combination of transient shear and elongational deformation are investigated. The detailed analysis of these flows allows the assessment of constitutive models and numerical predictions for prototype industrial flows. One of the main problems in constitutive modeling is to obtain a correct description of the transient nonlinear behavior in elongation and shear flows simultaneously. Well-known and widely used models, such as the non-Newtonian power-law model yield satisfactory results.

The non-Newtonian viscosity depends strongly both on velocity gradient and on temperature.

We consider an incompressible homogeneous fluid with constant density and the fluid viscosity is temperature-dependent. The power-law viscosity function will be applied.

These phenomena can be mathematically described by nonlinear governing equations ([9], [13], [23], [25], [26]).

For steady viscometric flows of temperature dependent Newtonian fluids exact solutions have been given by Nahme [16] and Kearsley [11]. These authors adopted an exponential form for viscosity-temperature dependence. Bird and Turian [6] have investigated heat generation in a Newtonian fluid in a

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cone-plate viscometer and applied Nahme's result to validate the solution.

Considering thermal boundary conditions, simultaneously developing steady laminar flow of viscous non-Newtonian fluid flowing between parallel plates was investigated numerically in [7].

To discuss these problems we investigate simultaneously the equations of motion and energy. These equations can be solved numerically by e.g., the Runge-Kutta method. In fact, the analytical solutions prove to have two-valued cases, which would probably not have been suspected by applying numerical techniques alone.

The aim of this paper is to establish exact solutions for a class of steady shear flows of power-law fluid between two infinite flat parallel plates in relative tangential motion. We examine the existence, the multiplicity and the parameter dependence of the solutions to the corresponding nonlinear second order differential equation. Rheological parameters obtained by experiments are applied to our calculations.

II. Mathematical Model

We consider an incompressible homogeneous fluid with constant density and the fluid viscosity is temperature-dependent.

In this paper, we apply the Ostwald-de Waele power-law formula for non-Newtonian viscosity ([2], [3], [4], [5], [19], [20]) when the connection between the shear stress and strain rate is given by

$$\tau_{xy} = \mu \frac{dv_x}{dy}, \tag{1}$$

where

$$\mu = \eta_0 \left| -\frac{dv_x}{dy} \right|^{n-1}. \tag{2}$$

In (2) $n > 0$ is called the power-law index. The case $n < 1$ is referred to pseudoplastic or shear-thinning fluid, the case $n > 1$ is known as dilatant or shear-thickening fluid. The Newtonian fluid is a special case, where the power-law index $n = 1$. We assume that the power-law constant n is not dependent on the temperature. In (2) constants η_0 and n are characteristic of each polymer and each polymer solution.

Although, the Arrhenius-type law of viscosity-temperature dependence relationship

$$\eta_0 = \eta_A e^{\frac{A}{RT}}$$

is more general and adequate to describe the polymer viscosities, for simplicity in this study we assume the Nahme-type exponential approximation when the temperature dependence of η_0 is assumed by

$$\eta_0 = \overline{\eta} e^{-\frac{A(T-T_2)}{RT_2^2}}, \tag{3}$$

where T denotes the temperature, A is the activation energy (a rheological factor), R is the universal gas constant and $\overline{\eta}$ is the viscosity value at the reference temperature T_2. In this paper we assume that the thermal conductivity k and density ρ of the fluid do not change appreciably with temperature and pressure.

Consider that a layer of non-Newtonian fluid is located between two parallel plates $y = 0$ and $y = h$. The upper plates moves in the direction of the positive x axis with constant velocity V. The fluid adheres to the walls $y = 0$ and $y = h$ having constant temperatures T_1 and T_2 ($T_1 > T_2$), respectively. The equation of motion and the equation of energy can be written [14]

$$\frac{d}{dy}\left(\eta_0 \left(\frac{d\overline{v}}{dy} \right)^n \right) = 0,$$

$$k \frac{d^2T}{dy^2} + \eta_0 e^{-n\theta} \left(\frac{d\overline{v}}{dy} \right)^{n+1} = 0$$

or in dimensionless form

$$\frac{d}{d\eta}\left(e^{-n\theta} \left(\frac{dv}{d\eta} \right)^n \right) = 0, \tag{4}$$

$$\frac{d^2\theta}{d\eta^2} + \overline{\eta} \frac{h^2}{nk} \frac{A}{RT_2^2} \left(\frac{V}{h} \right)^{n+1} e^{-n\theta} \left(\frac{dv}{d\eta} \right)^{n+1} = 0 \tag{5}$$

with dimensionless variables

$$v = \frac{\overline{v}}{V},$$

$$\theta = \frac{A(T - T_2)}{nRT_2^2},$$

$$\eta = \frac{y}{h}.$$

The system (4), (53) should be completed by one of the two boundary conditions:

Case i. Plate temperatures prescribed

$$\eta = 0\ :\quad v(0) = 0,\ \theta(0) = \theta_a, \tag{6}$$

$$\eta = 1\ :\quad v(1) = 1,\ \theta(1) = 0, \tag{7}$$

where

$$\theta_a = \frac{A}{nR}\frac{T_1 - T_2}{T_2^2}.$$

Case ii. The moving and stationary plates with the same temperature and

$$\eta = 0\ :\quad v(0) = 0,\ \theta'(0) = 0, \tag{8}$$

$$\eta = 1\ :\quad v(1) = 1,\ \theta(1) = 0. \tag{9}$$

Integrating (4) and denoting the constant of integration by α one gets

$$\frac{dv}{d\eta} = \alpha^{1/n}\, e^{\theta}. \tag{10}$$

Applying (10) the energy equation (5) can be rewritten

$$\frac{d^2\theta}{d\eta^2} + \gamma e^{\theta} = 0, \tag{11}$$

with

$$\gamma = \eta \frac{-h^2}{nk}\frac{A}{RT_2^2}(\alpha)^{1+1/n}. \tag{12}$$

III. Exact solutions

Theorem *Consider the boundary value problem (11), (8), (9). All solutions are concave on* $[0,1]$. *There exists a value* $\gamma^* > 0$ *such that*

for each $\gamma \in (0, \gamma^*)$, *there are two solutions,*

for $\gamma = \gamma^*$ *there is a unique solution, and*

for $\gamma \in (\gamma^*, \infty)$ *there is no solution.*

Proof: Taking into consideration (11) we see that

$$\frac{d^2\theta}{d\xi^2} < 0 \quad \text{for} \quad \xi \in [0,1]$$

and the concavity is obvious.

Let us substitute

$$\theta' = w(\theta)$$

to (11) one can obtain the first order differential equation

$$w'\, w + \gamma e^{\theta} = 0.$$

It is possible to separate the variables and one has

$$\theta' = -\sqrt{C_1 - 2\gamma e^{\theta}}, \quad C_1 = const.$$

A second integration yields the solution of the form

$$\frac{\sqrt{C_1}}{2}\eta + C_2 = cosh^{-1}\left(\frac{1}{\sqrt{\frac{2\gamma}{C_1}e^{\theta}}}\right),$$

then

$$e^{\theta} = \frac{C_1}{2\gamma}\frac{1}{cosh^2\left(\frac{\sqrt{C_1}}{2}\eta + C_2\right)},$$

$$\theta(\eta) = ln\left(\frac{C_1}{2\gamma}\right) - 2\,ln\,cosh\left(\frac{\sqrt{C_1}}{2}\eta + C_2\right).$$

Then one can get

$$\frac{dv}{d\eta} = \alpha^{1/n}\frac{C_1}{2\gamma}\frac{1}{cosh^2\left(\frac{\sqrt{C_1}}{2}\eta + C_2\right)},$$

$$v(\eta) = \frac{\alpha^{\frac{1}{n}}}{\gamma}\sqrt{C_1}\,tanh\left(\frac{\sqrt{C_1}}{2}\eta + C_2\right) + C_3.$$

Since

$$\theta'(\eta) = -\sqrt{C_1}\,tanh\left(\frac{\sqrt{C_1}}{2}\eta + C_2\right),$$

then we obtain from the second condition in (8) that

$$C_2 = 0.$$

Applying

$$\theta(0)=\theta_a$$

to $\theta(\eta)$ we have

$$e^{\theta_a}=\frac{C_1}{2\gamma}.$$

Applying this relation we determine for each θ_a if there exists a value of γ for which the boundary condition $\theta(1)=0$ holds. Hence, for

$$\theta(\eta)=\theta_a-2\ln\cosh\left(\frac{\sqrt{2\gamma e^{\theta_a}}}{2}\eta\right), \qquad (13)$$

we have

$$e^{\theta_a}=\cosh^2\left(\frac{\sqrt{2\gamma e^{\theta_a}}}{2}\right)$$

from which

$$\gamma=\frac{e^{-\theta_a}}{2}\left[\ln\left(\frac{1+\sqrt{1-e^{-\theta_a}}}{1-\sqrt{1-e^{-\theta_a}}}\right)\right]^2. \qquad (14)$$

Equation (14) defines the values of γ for each θ_a when the problem (11), (8), (9) has a solution. Fig.1 represents the dependence of γ on θ_a. It shows that there exists a unique γ^* for which (11), (8), (9) has exactly one solution and for smaller values of γ there are precisely two solutions. The approximate maximum value of γ is $\gamma^*=0.88$ at $\theta_a=1.18$.

The velocity distribution corresponding to temperature distribution (13) with

$$v(0)=0$$

has the form

$$v(\eta)=\frac{\alpha^{\frac{1}{n}}}{\gamma}\sqrt{2\gamma e^{\theta_a}}\tanh\left(\frac{\sqrt{2\gamma e^{\theta_a}}}{2}\eta\right).$$

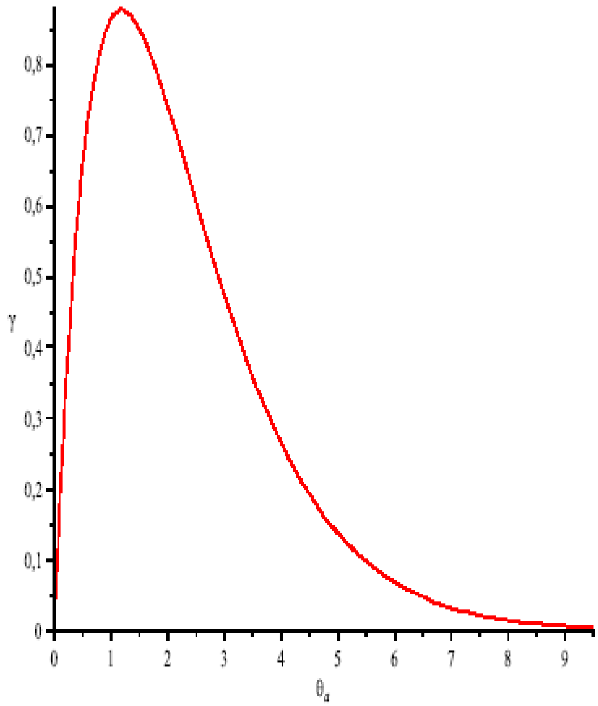

Fig.1. Dependence of γ on θ_a

The constant of integration α can be obtained from condition

$$v(1)=1,$$

therefore

$$\alpha=\left[\sqrt{\frac{2e^{\theta_a}}{\gamma}}\tanh\left(\frac{\sqrt{2\gamma e^{\theta_a}}}{2}\right)\right]^{-n}.$$

Next we consider the boundary value problem (11), (6), (7). Similarly as in the previous case we have

$$\theta(\eta)=\ln\left(\frac{C_1}{2\gamma}\right)-2\ln\cosh\left(\frac{\sqrt{C_1}}{2}\eta+C_2\right),$$

$$v(\eta)=\frac{\alpha^{\frac{1}{n}}}{\gamma}\sqrt{C_1}\tanh\left(\frac{\sqrt{C_1}}{2}\eta+C_2\right)+C_3.$$

From the boundary conditions (6), one gets a system of four transcendental equations for the constants of integration C_1, C_2, C_3, and α

$$e^{\theta_a} = \frac{C_1}{2\gamma} cosh^{-2} C_2,$$

$$1 = \frac{C_1}{2\gamma} cosh^{-2}\left(\frac{\sqrt{C_1}}{2} + C_2\right),$$

$$C_3 = -\frac{\alpha^{\frac{1}{n}}}{\gamma}\sqrt{C_1}\, tanh\, C_2,$$

$$1 = \frac{\alpha^{\frac{1}{n}}}{\gamma}\sqrt{C_1}\left[tanh\left(\frac{\sqrt{C_1}}{2} + C_2\right) - tanh\, C_2\right]. \quad (15)$$

It is remarkable that a simple geometry produces a very cumbersome result. If we restrict our attention to the case of equal plate temperatures, i.e., $\theta_a = 0,$ then we have some simplifications in the formulas as

$$C_2 = -\frac{\sqrt{C_1}}{4}.$$

Then, we have

$$\theta'(\eta) = -\sqrt{C_1}\, tanh \frac{\sqrt{C_1}}{2}\left(\eta - \frac{1}{2}\right),$$

that means

$$\theta'\left(\frac{1}{2}\right) = 0,$$

and

$$\theta\left(\frac{1}{2}\right) = e^{\theta_b} = \frac{C_1}{2\gamma}.$$

The calculations for θ in the previous case on $[0, 1]$ can similarly be performed here on $[1/2, 1]$. Applying the boundary condition

$$v(0) = 0$$

for

$$v(\eta) = \frac{\alpha^{\frac{1}{n}}}{\gamma}\sqrt{C_1}\, tanh \frac{\sqrt{C_1}}{2}\left(\eta - \frac{1}{2}\right) + C_3$$

we get

$$C_3 = \frac{\alpha^{\frac{1}{n}}}{\gamma}\sqrt{C_1}\, tanh \frac{\sqrt{C_1}}{4}.$$

That implies

$$\theta(\eta) = ln\left(\frac{C_1}{2\gamma}\right) - 2\, ln\, cosh \frac{\sqrt{C_1}}{2}\left(\eta - \frac{1}{2}\right),$$

$$v(\eta) = \frac{\alpha^{\frac{1}{n}}}{\gamma}\sqrt{C_1}\left[tanh \frac{\sqrt{C_1}}{2}\left(\eta - \frac{1}{2}\right) + tanh \frac{\sqrt{C_1}}{4}\right].$$

Moreover, α can be determined from condition

$$v(1) = 1\ :$$

therefore

$$1 = 2\frac{\alpha^{\frac{1}{n}}}{\gamma}\sqrt{C_1}\, tanh \frac{\sqrt{C_1}}{4},$$

$$\alpha = \left[\frac{2}{\gamma}\sqrt{C_1}\, tanh \frac{\sqrt{C_1}}{4}\right]^{-n}.$$

IV. Determination of Rheological Parameters for LDPE

In this section the experimental determination of the material parameters is discussed.

The rheological properties of the polymer BRALEN RB 03-23 [27] were measured by dynamic viscosity using a Physica UDS 200 type Universal Dynamic Spectrometer. The dynamic viscosity measurements were performed at different temperatures (170, 180, 190, 200 and 210°C) in the range of angular frequency 0.1-600 s^{-1}. Complex viscosity (η*), was determined as characteristic quantities. Two intervals of angular frequencies are used as the Ostwald-de Waele power-law model can be fitted well only in narrow intervals of the temperature. These phenomena will be exhibited in Table 2 and Table 3.
Table 1 exhibits the determination of power exponent *n* in the Ostwald-de Waele power-law model (2). It seems that *n* does

depend on the temperature but in our calculations we shall assume that the power-law constant n is not dependent on the temperature. Therefore, the average value will be applied in the further calculations.

ω=600…0.1 s^{-1}		ω=600…59.7 s^{-1}	
Temperature [°C]	n	Temperature [°C]	n
150	0.36519	150	0.27759
150	0.36716	150	0.27761
160	0.38631	160	0.29587
160	0.38449	160	0.29325
170	0.40066	170	0.29552
170	0.40306	170	0.30220
180	0.41773	180	0.31369
180	0.41786	180	0.31529
190	0.43397	190	0.33341
190	0.43354	190	0.33132
200	0.44517	200	0.34128
200	0.44786	200	0.33930
210	0.45845	210	0.35441
210	0.45842	210	0.35631
Average	0.415705	Average	0.316218

Table 1.

On the base of average value of n with formulas (1) and (2) average η_0 can be evaluated. The dependence of η_0 on the temperature is represented in Fig.2.

	ω=600…0,1 s^{-1} (n=0.4157)	
Temperature [°C]	Average η_0	Standard deviation η_0
170	962	11%
170	980	11%
180	744	11%
180	768	11%
190	617	12%
190	637	12%
200	546	13%
200	527	14%
210	487	15%
210	479	15%

Table 2.

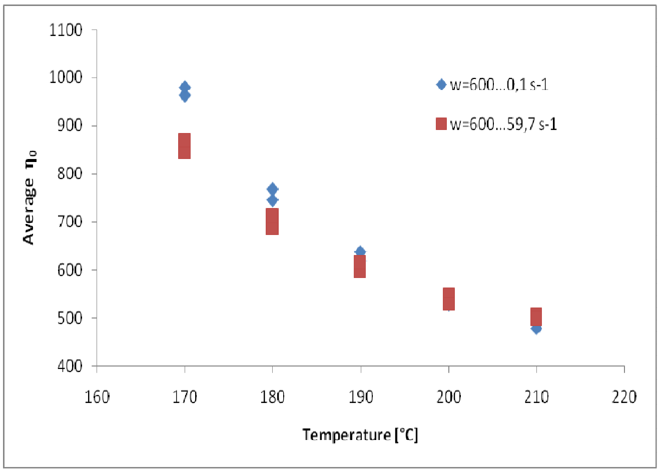

Fig.2.

ω=600…59.7 s^{-1} (n=0.3162)		
Temperature [°C]	Average η_0	Standard deviation η_0
170	846	9%
170	868	8%
180	688	8%
180	711	7%
190	599	6%
190	616	6%
200	546	6%
200	530	6%
210	506	5%
210	498	4%

Table 3.

The activation energy is determined according to equality (3) (see Table 4.). The dependence of activation energy on the temperature is exhibited in Fig.3.

	ω=600…0.1 s^{-1}	ω=600…59.7 s^{-1}
Temperature [°C]	Activation energy [J/mol]	Activation energy [J/mol]
180	40919	33125
190	31935	24070
200	27810	21741
210	19492	12886

Table 4.

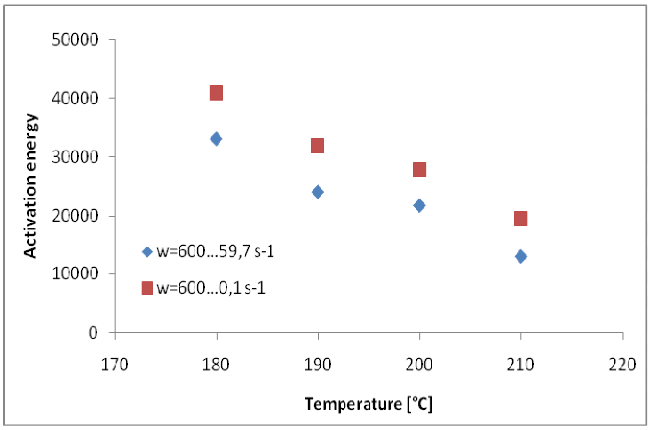

Fig. 3.

Now after calculations, we summarize the parameters necessary for the determination of γ (12) in Table 6-7.

ω=600…0.1 s^{-1}					
h [mm]	T_2 [°C]	η_0	A [J/mol]	n	k W/(m K)
0.5	180	971.449	40919	0.4157	0.33
0.5	190	756.157	31935	0.4157	0.33
0.5	200	627.152	27810	0.4157	0.33
0.5	210	536.616	19492	0.4157	0.33

Table 6.

ω=600…59.7 s^{-1}					
h [mm]	T_2 [°C]	η_0	A [J/mol]	n	k W/(m K)
0.5	180	856.914	33125	0.3162	0.33
0.5	190	699.604	24070	0.3162	0.33
0.5	200	607.580	21741	0.3162	0.33
0.5	210	537.854	12886	0.3162	0.33

Table 7.

In the literature for the heat conductivity $k = 0.33\ W/m\,K$ was found [24]. Using the rheological parameters we want to check the multiplicity of solutions to (11), (6) and (7) for the examined LDPE.

Temperature [°C]	ω=600…0.1 s^{-1} γ	ω=600…59.7 s^{-1} γ
180	0.43	0.50
190	0.53	0.61
200	0.61	0.73
210	0.70	0.84

Table 8.

According to the numerical calculations from system (15) with $\theta_a = 0$ for $T_1 = T_2$ one can determine the values of γ for the corresponding temperature values. We note that system (15) is reduced to

$$\sqrt{C_1} = -4C_2,$$
$$\cosh C_2 = -\frac{4C_2}{\sqrt{2\Omega\alpha^{1+1/n}}},$$
$$C_2 \tanh C_2 = \frac{\Omega}{8}\alpha,$$
$$C_3 = \frac{4C_2}{\Omega\alpha}\tanh C_2.$$

We have seen from Table 8. that the values of γ are smaller than γ^*. Then we can conclude that in all cases there are two solutions for the differential equation (11) with boundary conditions (6), (7).

List of symbols:

x *axis*	
y *axis*	
k *heat conductivity*	[W/m K]
h *distance of plates*	[mm]
n *power-law index*	[-]
A *activation energy*	[J/mol]
R *gas constant*	8.314 [J/mol K]
ω *angular frequency*	[s^{-1}]
T_1 *plate temperature*	[°C]
T_2 *plate temperature*	[°C]
T *mass temperature*	[°C]
ρ *density*	[g/cm^3]

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