Deterministic and stochastic internet-style networks with a single link, and one or two user under information delay

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***Abstract*—** In this paper we investigate the dynamics of the Internet-Style Network with delay using a single link, and one or two user under delay. We establish the existence of the Hopf bifurcation and the normal form. The stochastic system is associated to the deterministic model and the mean values and the square mean values of the variables for the linearized stochastic system are analyzed. The last part of the paper includes numerical simulations and conclusions.

***Keywords* —** deterministic dynamic economic model, economic growth, education product, human capital, stochastic dynamic economic model.

I. IN

GAME theory provides a natural framework for developing pricing and congestion control mechanisms for the Internet. Users on the network can be modeled as players in a congestion control game where they choose their strategies or in this the flow rates. Players are non-cooperative in terms of their demands for network resource, and have no specific information on other user strategies. A user's demand or utility for bandwidth is captured in a utility function and may not be bounded. To compensate for this, one can devise a pricing function, proportional to the bandwidth usage of a user, in order to preserve the network resources and to provide an incentive for the user to implement end-to-end congestion control [3]. A useful concept in such a non-cooperative congestion control game is finding the Nash equilibrium, where each player minimizes his own cost (or maximize payoff) given all other player's strategies. The non-cooperative congestion control game introduced in [2] is characterized by a cost function for each user that is defined as the difference of pricing and utility functions. The pricing function is proportional to the queuing delay experienced by the user, whereas the utility function that quantifies the user, demand for bandwidth belongs to a broad class of strictly increasing and strictly concave functions. In [7], [8], [9], the other Internet models are analyzed.

In this paper we will analyze the differential system which shapes one internet network with a single link, with single user and two user under information delay.

The rest of the paper is organized as follows. In section 2 the existence of a unique equilibrium of the system is established. We analyze the existence of the Hopf bifurcation considering r as bifurcation parameter. In section 3 we analyze the direction and stability of the Hopf bifurcation. In section 4 we analyze the stochastic model with delay associated for internet style for a single link with a single user. In section 5 we analyze the network model with a single link and two user and stability of the Hopf bifurcation. In section 6 for given values of the parameters the numerical simulations are given. In section 7 conclusions and future research are drawn.

II. THE EQUILIBRIUM POINT AND THE HOPF BIFURCATION FOR A SINGLE LINK AND SINGLE USER UNDER INFORMATION DELAY

The one internet network with single link, and single user under information delay is given by:

$$\begin{aligned} \dot{x}(t) &= U'(x(t)) - \alpha d(t-r) \\ \dot{d}(t) &= \tfrac{1}{c} x(t-r) - 1 \end{aligned} \tag{1}$$

where $x(t)$ is the user flow rate, $c > 0$ is the link capacity, $d(t-r)$ is the queuing delay, $r \geq 0$ is the delay between the user and the link $\alpha > 0$. The utility function $U(x)$ is

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assumed to be strictly increasing, differentiable, strictly concave and $U'(x) = \frac{dU(x)}{dx}$.

The first equation from (1) represents the dynamic system of a game where the cost (objective) function is given by:

$$J(x,d) = \alpha dx - U(x). \tag{2}$$

For system (1) the following affirmations hold:

Proposition 1:

1. The equilibrium point is (x_0, d_0), where:

$$x_0 = c,\ d_0 = \frac{1}{\alpha}U'(x_0). \tag{3}$$

2. With respect to the translation $x(t) = x_1(t) + x_0, d(t) = x_2(t) + d_0$, system (1) becomes:

$$\begin{aligned} \dot{x}_1(t) &= g(x_1(t)) - \alpha x_2(t-r) \\ \dot{x}_2(t) &= \frac{1}{c}x_1(t-r) \end{aligned} \tag{4}$$

where

$$g(x_1(t)) = U'(x_1(t) + x_0) - \alpha d_0 \tag{5}$$

3. The linearization of system (4) is:

$$\begin{aligned} \dot{u}_1(t) &= a_1(t)u_1 - \alpha u_2(t-r) \\ \dot{u}_2(t) &= \frac{1}{c}u_1(t-r) \end{aligned} \tag{6}$$

where $a_1 = g'(0) = U''(x_0)$.

4. The characteristic equation of (6):

$$\lambda^2 - a_1\lambda + \frac{\alpha}{c}e^{-2\lambda r} = 0. \tag{7}$$

Analyzing the roots of the characteristic equation with respect to r we obtain:

Proposition 2:

1. The roots of equation (7) are differentiable functions with respect to r.
2. If $r = 0$, the roots of equation (7) have a negative real part.

To establish the existence of the Hopf bifurcation we prove:

Proposition 3:

The characteristic equation (7) has the roots $\lambda_1 = i\omega_0$, $\lambda_2 = \overline{\lambda_1}$,

where:

$$\omega_0 = \sqrt{\frac{-a_1^2c^2 + \sqrt{a_1^4c^2 + 4\alpha^2}}{2c}} \tag{8}$$

for r_0 given by:

$$r_0 = -\frac{1}{2\omega_0}arctg\left(\frac{a_1}{\omega_0}\right). \tag{9}$$

Considering $\lambda = \lambda(r)$ in (7) and deriving it with respect to r we get:

$$\frac{d\lambda}{dr} = \lambda'(r) = \frac{2\alpha\lambda e^{-2\lambda r}}{2\lambda c - a_1c - 2\alpha r e^{-2\lambda r}}. \tag{10}$$

In (10) replacing r with r_0 given by (9) and ω with ω_0 given by (8), we obtain:

$$\mathrm{Re}\lambda'(r_0) = 2\alpha\omega_0^2\frac{a_1^2c^2 + 2\alpha\cos(2\omega_0 r_0)}{c^3(a_1 + 2r_0\omega_0^2)^2 + 4c\omega_0^2(2r_0a_1 - c)^2} \tag{11}$$

From (11) we have $\mathrm{Re}\,\lambda'(r_0) > 0$.

The above analysis can be summarized as follows:

Proposition 4:

Equation (7) has one Hopf bifurcation point at r_0, where r_0 is given by (9).

III. Direction and Stability of the Hopf Bifurcation for Internet Network with a Single Link, with a Single User under Information Delay

In this section, we study the direction, stability and the period of the bifurcating periodic solutions in system (4). The method we use is based on the normal form theory and the center manifold theorem introduced in [4], [8].

The notational convenience, let $r = r_0 + \mu$. Then $\mu = 0$ is the Hopf bifurcation value for system (4). System (4) can be rewritten as [10]:

$$\begin{aligned} \dot{x}_1(t) &= a_1x_1(t) - \alpha x_2(t-r) + \frac{1}{2}a_2x_1(t)^2 + \frac{1}{6}a_3x_1(t)^3 + \\ &+ O(|x_1(t)|^4) \\ \dot{x}_2(t) &= \frac{1}{c}x_1(t-r) \end{aligned} \tag{12}$$

where $a_1 = U^{'}(x_0)$, $a_2 = U^{''}(x_0)$, $a_3 = U^{'''}(x_0)$.

For $\phi = \phi(s,\mu)$, $s \in [-r, 0]$ with $\phi \in C([-r,0], C^2)$ we consider:

$$L_\mu\phi = A_1\phi(0) + B_1\phi(-r), \tag{13}$$

where A_1 and B_1 is given by:

$$A_1 = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix},\ B_1 = \begin{pmatrix} 0 & -\alpha \\ \frac{1}{c} & 0 \end{pmatrix}. \tag{14}$$

Let

$$F(\mu,\phi) = \begin{pmatrix} a_2\phi_1(0)^2 + a_3\phi_1(0)^2 + O(|\phi|)^4 \\ 0 \end{pmatrix} \tag{15}$$

For $\phi \in C^1([-r,0], C^2)$ we define:

$$A_1(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -r \le \theta < 0 \\ A_1\phi(0) + B_1\phi(-r), & \theta = 0 \end{cases} \tag{16}$$

and

$$R_1(\mu)\phi = \begin{cases} 0, & -r \le \theta < 0 \\ F(\mu,\phi), & \theta = 0 \end{cases} \tag{17}$$

System (12) can be rewritten as:

$$\dot{u}_t = A_1(\mu)u_t + R_1(\mu)u_t \tag{18}$$

where

$$u_t = u(t+\theta),\ \theta \in [-r,0].$$

For $\psi \in C^1([0,r],C^2)$ the adjunct operator $A_1^*(\mu)$ of $A_1(\mu)$ is defined as:

$$A_1^*(\mu)\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & 0 \le s < r \\ \psi(s)^T A_1 + \psi^T(r)B_1, & s = 0 \end{cases} \tag{19}$$

For $\phi \in C^1([-r,0],R^2)$, and $\psi \in C^1([0,r],R^2)$ we define a bilinear form by:

$$<\psi,\phi> = \overline{\psi}^T(0)\phi(0) - \int_{\theta=-r}^{0}\int_{s=0}^{\theta}\overline{\psi}^T(s-\theta)d\theta B\phi(s)ds \tag{20}$$

where $\psi(\theta) = B_1\delta(\theta+r)$, $\theta \in [-r,0]$, and δ is Dirac distribution.

In order to determine the Poincare normal form of the operator $A_1(\mu)$, we need to calculate the eigenvector q of $A_1(\mu)$ associated to the eigenvalue $\lambda_1 = i\omega_0$ and the eigenvector q^* of $A_1^*(\mu)$ associated to the eigenvalue $\lambda_2 = -i\omega_0$. We can easily verify that:

$$q(\theta) = v\exp(\lambda_1\theta),\quad \theta \in [-r,0] \tag{21}$$

where $v = (v_1, v_2)^T$ and

$$v_1 = c\lambda_1\ ,\ \ v_2 = \exp(\lambda_2 r_0) \tag{22}$$

is the eigenvector of $A_1(0)$ associated to the eigenvalue λ_1.

The eigenvector of $A_1^*(0)$ associated to λ_2 is given by:

$$q^*(\theta) = w\exp(\lambda_1\theta),\quad \theta \in [0,r_0] \tag{23}$$

where $w = (w_1, w_2)^T$ and

$$w_1 = \frac{\lambda_1}{\alpha}\exp(\lambda_2 r_0 w_2)\ ,\ \ w_2 = \alpha c \tag{24}$$

$$\alpha = \lambda_1 c\overline{v}_1\exp(\lambda_2 r_0) + \alpha c\overline{v}_2 + r_0\alpha\overline{v}_1\exp(\lambda_1 r_0) - \alpha c r_0\lambda_1\overline{v}_2 \tag{25}$$

Using (20), we can verify that $<\overline{q}^*,q>=0$, $<q^*,q>=1$.

In the following, we will follow the ideas and use the notation in [4]. Let:

$$z = <q^*,u_t>,\ w(t,\theta) = u_t(\theta) - 2\operatorname{Re}(z(t)q(\theta)) \tag{26}$$

Then

$$\dot{z}(t) = \lambda_1 z(t) + g(z(t),\overline{z}(t)) \tag{27}$$

where

$$g(z,\overline{z}) = \frac{1}{2}g_{20}z^2 + g_{11}z\overline{z} + \frac{1}{2}g_{02}\overline{z}^2 + \frac{1}{2}g_{21}z^2\overline{z}$$

$$\begin{aligned} g_{20} &= \overline{q}^*(0)^T F_{20},\quad g_{11} = \overline{q}^*(0)^T F_{11}, \\ g_{02} &= \overline{q}^*(0)^T F_{02},\quad g_{21} = \overline{q}^*(0)^T F_{21}. \end{aligned} \tag{28}$$

$$F_{20} = (a_2 v_1^2, 0)^T,\ F_{11} = (a_2 v_1 v_2, 0)^T,\ F_{02} = (a_2\overline{v}_1^2, 0)^T$$
$$F_{21} = (a_2(w_{20}^1\overline{v}_1 + 2w_{11}^1 v_1) + a_3 v_1^2\overline{v}_1, 0)^T$$

and

$$\begin{aligned} w(z,\overline{z}) &= \frac{1}{2}w_{20}z^2 + w_{11}z\overline{z} + \frac{1}{2}w_{02}\overline{z}^2 \\ w_{20}(\theta) &= -\frac{g_{20}}{\lambda_1}v\exp(\lambda_1\theta) - \frac{\overline{g}_{02}}{3\lambda_1}\overline{v}\exp(\lambda_2\theta) + \\ &\quad + E_1\exp(2\lambda_1\theta) \\ w_{11}(\theta) &= -\frac{g_{11}}{\lambda_1}v\exp(\lambda_1\theta) - \frac{\overline{g}_{11}}{\lambda_1}\overline{v}\exp(\lambda_2\theta) + E_2 \\ E_1 &= -(A_1 + \exp(\lambda_1 r_0)B_1 - 2\lambda_1 I)^{-1}F_{20} \\ E_2 &= -(A_1 + B_1)^{-1}F_{11} \end{aligned} \tag{29}$$

$$I = diag(1,1).$$

From (22), (23) and (28), we obtain:

$$\begin{aligned} g_{20} &= a_2 v_1^2\overline{w}_1,\ g_{11} = a_2 v_1 v_2\overline{w}_1,\ g_{02} = a_2\overline{v}_1^2\overline{w}_1 \\ E_1 &= -\frac{1}{D_1}(-2\lambda_1 a_2 v_1^2, -\frac{1}{c}\exp(\lambda_1 r_0)a_2 v_1^2)^T, \\ D_1 &= \frac{\alpha}{c}\exp(2\lambda_1 r_0) - 2\lambda_1(a_1 - 2\lambda_1) \\ E_2 &= (0, \frac{a_2}{\alpha}v_1 v_2)^T \end{aligned} \tag{30}$$

and

$$\begin{aligned} w_{20}^1(\theta) &= -\frac{g_{20}}{\lambda_1}v_1\exp(\lambda_1\theta) - \frac{\overline{g}_{02}}{3\lambda_1}\overline{v}_1\exp(\lambda_2\theta) + \\ &\quad + \frac{2\lambda_1 a_2 v_1^2}{D_1}\exp(2\lambda_1\theta) \\ w_{20}^2(\theta) &= -\frac{g_{20}}{\lambda_1}v_2\exp(\lambda_1\theta) - \frac{\overline{g}_{02}}{3\lambda_1}\overline{v}_2\exp(\lambda_2\theta) + \\ &\quad + \frac{a_2 v_1^2\exp(\lambda_1 r_0)}{D_1}\exp(2\lambda_1\theta) \end{aligned} \tag{31}$$

$$w_{11}^1(\theta) = -\frac{g_{11}}{\lambda_1} v_1 \exp(\lambda_1\theta) - \frac{\bar{g}_{11}}{\lambda_1} \bar{v}_1 \exp(\lambda_2\theta)$$

$$w_{11}^2(\theta) = -\frac{g_{11}}{\lambda_1} v_2 \exp(\lambda_1\theta) - \frac{\bar{g}_{11}}{\lambda_1} \bar{v}_2 \exp(\lambda_2\theta) + \\ + \frac{a_2 v_1 v_2}{\alpha}$$

$$g_{21} = \bar{w}_1 (a_2 (w_{20}^1(0)\bar{v}_1 + 2 w_{11}^1(0) v_1) + a_3 v_1^2 \bar{v}_1.$$

Using the theory of the normal form [4] we have the following formulas:

$$C_1(0) = \frac{i}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}$$

$$\mu_2 = -\frac{\operatorname{Re} C_1(0)}{\operatorname{Re} \lambda'(r_0)},$$

$$T = -\frac{\operatorname{Im} C_1(0) + \mu_2 \operatorname{Im} \lambda'(r_0)}{\omega_0}, \tag{32}$$

$$\beta_2 = 2\operatorname{Re} C_1(0).$$

Now we can state the main results of this section:

Proposition 5:

In the formulas (32), μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0 (< 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $r > r_0 (< r_0)$; β_2 determines the stability of the bifurcating periodic solutions: the solutions are orbitally stable (unstable) if $\beta_2 < 0 (> 0)$; and T determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T > 0 (< 0)$.

IV. The Mathematical Stochastic Model with Delay, Associated to the Internet Style Networks with Delay

Let $(\Omega, F_t, P), t \geq 0$ be a given probability space and $w(t) \in R$ be a scale Wiener process defined on Ω having independent stationary Gauss increments with $w(0) = 0$, $E(w(t)w(s)) = \min(t,s)$. The symbol E denotes the mathematical expectation. The sample trajectories of $w(t)$ are continuous, nowhere differentiable and have infinite variation on any finite time interval [5].

We are interested in knowing is the effect of the noise perturbation on system (1). The stochastic differential equation with delay is:

$$\begin{aligned} dx_1(t) &= (U'(x_1(t)) - \alpha x_2(t-r))dt + \sigma_1(x_1(t) - x_0)dw(t) \\ dx_2(t) &= (\frac{1}{c}(x_1(t-r)) - 1)dt + \sigma_2(x_2(t) - d_0)dw(t) \end{aligned} \tag{33}$$

where $\sigma_1 > 0, \sigma_2 > 0$, $w(t)$ is the scalar Wiener process and $x_1(t) = x(t,\omega)$, $x_2(t) = d(t,\omega)$ are the components of the process $x(t,\omega) = ((x(t,\omega), d(t,\omega))^T$ on the probability space.

Linearizing (33) around the equilibrium $(x_0, d_0)^T$ yields the linear differential equation with delay:

$$dy(t) = (A_1 y(t) + B_1 y(t-r))dt + C_1 u(t) dw(t) \tag{34}$$

where $y(t) = (y_1(t), y_2(t))^T$ and A_1, B_1 are given by (14) and $C_1 = diag(\sigma_1, \sigma_2)$.

Using the method from [6], [11], we analyze the first and the second moments of the solutions for (34) with respect to r.

Proposition 6:

1. For system (34), the moment of the solution is given by:

$$\dot{E}(y(t)) = A_1 E(y(t)) + B_1 E(y(t-r)). \tag{35}$$

2. The characteristic function for (35) is:

$$h(\lambda, r) = \lambda^2 - a_1\lambda + \frac{\alpha}{c} e^{-\lambda r}. \tag{36}$$

3. If $r = 0$, the roots of equation $h(\lambda, 0) = 0$ have a negative real part.
4. The equation $h(\lambda, r) = 0$ has one Hopf bifurcation point at r_0, where r_0 is given by (9).
5. If we denote by $E_1(t) = E(y_1(t))$, $E_2(t) = E(y_2(t))$ then

$$E_1(t) = v_1 z(t) + \bar{v}_1 \bar{z}(t), E_1(t) = v_2 z(t) + \bar{v}_2 \bar{z}(t) \tag{37}$$

where v_1, v_2 are given by (22), ω_0 is given by (8), r_0 is given by (9) and

$$\dot{z}(t) = i\omega_0 z(t),\ z(t) = x(t) + iy(t). \tag{38}$$

To examine the stability of the second moment of $y(t)$ for the linear stochastic differential equation with delay (34) we use Ito's rule to give the stochastic differential of $y(t)y(t)^T$ where $y(t) = (y_1(t), y_2(t))^T$.

Let $R(t,s) = E\{y(t)y^T(s)\}$ be the covariance matrix of the process $y(t)$ so that $R(t,t)$ satisfies:

$$\begin{aligned} \dot{R}(t,t) &= E\{dy(t)y^T(t) + y(t)dy^T(t) + C_1 y(t)y(t)^T C_1\} \\ &= A_1 R(t,t) + R(t,t)A_1^T + B_1 R(t,t-r) + R(t,t-r)B_1^T + \\ &+ C_1 R(t,t) C_1 \end{aligned} \tag{39}$$

where $C_1 = diag(\sigma_1, \sigma_2)$.

From (23) and $R_{ij}(t,s) = E(y_i(t)y_j(s)), i,j = 1,2$ we get:

Proposition 7:

1. The differential system (39) is given by:

$$\dot{R}_{11}(t,t) = (2a_1 + \sigma_1^2)R_{11}(t,t) - 2\alpha R_{12}(t,t-r)$$
$$\dot{R}_{22}(t,t) = \sigma_2^2 R_{22}(t,t) + \frac{2}{c}R_{12}(t,t-r)$$
$$\dot{R}_{12}(t,t) = (a_1 + \sigma_1\sigma_2)R_{12}(t,t) + \frac{1}{c}R_{11}(t,t-r) - \quad (40)$$
$$-\alpha R_{22}(t,t-r)$$

2. The characteristic function of (40) is given by:

$$l(\lambda,r) = (\lambda - 2a_1 - \sigma_1^2)(\lambda - \sigma_2^2)(\lambda - a_1 - \sigma_1\sigma_2) + \frac{2\alpha}{c}(2\lambda - 2a_1 - \sigma_1^2 - \sigma_2^2)e^{-2\lambda r} \quad (41)$$

Proof:

System (40) derives from (39) with A, B given by (13).

Let $R_{ij}(t,s) = e^{\lambda(t+s)}K_{ij}, i,j = 1,2$ where K_{ij} are constants. Replacing $R_{ij}(t,s)$ in (40) and setting the condition that the system we obtain should accept nontrivial solution, we get $l(\lambda,r) = 0$.

From (41) we have:

Proposition 8:

1. The characteristic equation $l(\lambda,r) = 0$ is given by:

$$\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 + (b_3\lambda + b_4)e^{-2\lambda r} = 0 \quad (42)$$

where

$$\begin{aligned} b_0 &= -\sigma_2^2(2a_1 + \sigma_1^2)(a_1 + \sigma_1\sigma_2), \\ b_1 &= (3a_1 + \sigma_1^2 + \sigma_1\sigma_2)\sigma_2^2 + (2a_1 + \sigma_1^2)(a_1 + \sigma_1\sigma_2) \\ b_2 &= -3a_1 - \sigma_1^2 - \sigma_1\sigma_2 - \sigma_2^2, \\ b_3 &= \frac{4\alpha}{c}, \\ b_4 &= -\frac{2\alpha(2a_1 + \sigma_1^2 + \sigma_2^2)}{c}. \end{aligned} \quad (43)$$

2 If $r = 0$, the characteristic equation $l(\lambda,0) = 0$ is given by:

$$\lambda^3 + b_2\lambda^2 + (b_1 + b_3)\lambda + b_0 + b_4 = 0 \quad (44)$$

3 If σ_1, σ_2 satisfy inequalities:

$$\begin{aligned} &b_2 > 0,\ b_1 + b_3 > 0,\ b_0 + b_4 > 0, \\ &b_2(b_1 + b_3) - (b_0 + b_4) > 0 \end{aligned} \quad (45)$$

then the roots of the equation (44) have a negative real part.

If σ_1, σ_2 satisfy (45) and $b_0 - b_4 < 0$, then the value $r = r_1$ is a Hopf bifurcation, where:

$$r_1 = \frac{r_4}{2\omega_1} = arctg\frac{(b_2b_3 - b_4)\omega_1^3 - (b_2b_0 - b_1b_4)\omega_1}{b_3\omega_1^4 + (b_4b_2^2 - b_1b_3)\omega_1^2 - b_0b_4} \quad (46)$$

and ω_1 is a positive real root of the equation:

$$\omega^6 + (b_2^2 - 2b_1)\omega^4 + (b_1^2 - 2b_2b_0 - b_3^2)\omega^2 + b_0^2 - b_4^2 = 0 \quad (47)$$

If we denote by:

$M_{ij}(t) = R_{ij}(t,t),\ i,j = 1,2$ then

$$\begin{aligned} M_{11}(t) &= v_{10}z(t) + \bar{v}_{10}\bar{z}(t), \\ M_{22}(t) &= v_{20}z(t) + \bar{v}_{20}\bar{z}(t), \\ M_{12}(t) &= v_{30}z(t) + \bar{v}_{30}\bar{z}(t) \end{aligned} \quad (48)$$

where

$$\begin{aligned} v_{10} &= \frac{-2\alpha \exp(\lambda_2 r_1)}{\lambda_1 - 2a_1 - \sigma_1^2}, \\ v_{20} &= \frac{2\exp(\lambda_2 r_1)}{c(2 - \sigma_2^2)}, \\ v_{30} &= 1 \end{aligned} \quad (49)$$

and

$\lambda_2 = -i\omega_1$ and $z(t)$ is the solution of equation

$$\dot{z}(t) = \lambda_1 z(t),\ z(t) = x(t) + iy(t) \quad (50)$$
$$\lambda_1 = i\omega_1.$$

V. A ANALYSIS OF NETWORK MODEL WITH A SINGLE LINK AND TWO USER UNDER INFORMATION DELAY

The internet network with single link and two users under information delay is given by:

$$\begin{aligned} \dot{x}_1(t) &= \alpha_1(U'(x_1(t) - d(t-r)) \\ \dot{x}_2 &= \alpha_2(U'(x_2(t) - d(t-r)) \\ \dot{d}(t) &= \frac{1}{c}(x_1(t-r) + x_2(t-r)) - 1 \end{aligned} \quad (51)$$

where $x(t) = (x_1(t), x_2(t))^T$ are the users flow rate, $c > 0$ is the link capacity, $d(t-r)$ is the queuing delay, $r \geq 0$ is the delay between the users and the the link, and $\alpha_1 > 0, i = 1,2$. The utility function $U(x)$ is assumed to be the strictly increasing, differentiable, strictly concave and $U'(x) = \frac{dU(x)}{dx}$.

For system (51) the following affirmations hold:

Proposition 9:

1. The equilibrium point is $(x_{10}, x_{20}, d_0)^T$ where:

$x_{10} = \frac{c}{2},\ x_{20} = \frac{c}{2}, d_0 = U'(x_{10})$.

2. With respect to the translation $x_1(t) = u_1(t) + x_{10}$, $x_2(t) = u_2(t) + x_{20}$, $d(t) = u_3(t) + d_0$, system (51) becomes:

$$\begin{aligned} \dot{u}_1(t) &= \alpha_1(U'(u_1(t)-d(t-r)) \\ u_2(t) &= \alpha_2(U'(u_2(t)-d(t-r)) \\ \dot{u}_3(t) &= \frac{1}{c}(u_1(t-r)-u_2(t-r)) \end{aligned} \quad (52)$$

2. The linearization of system (52) is:

$$\begin{aligned} \dot{u}_1(t) &= \alpha_1\rho_2 u_1(t) - \alpha_1 u_3(t-r) \\ \dot{u}_2(t) &= \alpha_2\rho_2 u_2(t) - \alpha_1 u_3(t-r) \\ \dot{u}_3(t) &= \frac{1}{c}u_1(t-r) + \frac{1}{c}u_2(t-r) \end{aligned} \quad (53)$$

where:

$\rho_2 = U''(x^*),\ x^* = x_{10} = x_{20}$.

3. The characteristic function of (53) is given by:

$$\begin{aligned} h(\lambda,\tau) &= \lambda^3 - (\alpha_1+\alpha_2)\rho_2\lambda^2 + \alpha_2\rho_2^2\lambda + \\ &+ \frac{1}{c}((\alpha_1+\alpha_2)\lambda - 2\alpha_1\alpha_2\rho_2)e^{-2\lambda r} \end{aligned} \quad (54)$$

Analyzing the roots of the equation $h(\lambda,\tau)=0$ with respect to r we obtain:

Proposition 10:

1. If $r=0$ the equation $h(\lambda,\tau)=0$, has roots with a negative real part.
2. If $r \neq 0$, exist r_0, given by:

$$r_0 = \frac{1}{2\omega_0} arctg \frac{\rho_2(\omega_0^3 - \alpha_1\alpha_2\rho_2^2\omega_0) - (\alpha_1+\alpha_2)\rho_2\omega_0^3}{2\alpha_1\alpha_2\rho_2^2\omega_0^2(\alpha_1+\alpha_2) + (\alpha_1+\alpha_2)\omega_0(\omega_0^3 - \alpha_1\alpha_2\rho_2^2\omega_0)} \quad (55)$$

where ω_0 is a positive real root of the equation:

$$\begin{aligned} & c^2\omega^6 + (c^2(\alpha_1+\alpha_2)^2\rho_2^2 - 2c^2\alpha_1\alpha_2\rho_2^2)\omega^4 + \\ & + (c^2\alpha_1^2\alpha_2^2\rho_2^4 - (\alpha_1+\alpha_2)^2)\omega^4 - 4\alpha_1^2\alpha_2^2\rho_2^2 = 0 \end{aligned} \quad (56)$$

so that for $r \in [0,\tau_0)$, equation $h(\lambda,r)=0$ has roots with a negative real part.

Considering $\lambda = \lambda(r)$ in $h(\lambda,r)=0$, and deriving it with respect to r we get:

$$\frac{d\lambda(r)}{dr} = \frac{2\lambda((\alpha_1+\alpha_2)\lambda - 2\alpha_1\alpha_2\rho_2)e^{-2\lambda r}}{c(3\lambda^2 - 2(\alpha_1+\alpha_2)\rho_2\lambda + \alpha_1\alpha_2\rho_2^2) + (\alpha_1+\alpha_2 - 2r(\alpha_1+\alpha_2)\lambda + 4\alpha_1\alpha_2\rho_2)e^{-2\lambda r}} \quad (57)$$

Let:

$$M = \mathrm{Re}\left(\left.\frac{d\lambda(r)}{dr}\right|_{\lambda=i\omega_0,\, r=r_0}\right),\ N = \mathrm{Im}\left(\left.\frac{d\lambda(r)}{dr}\right|_{\lambda=i\omega_0,\, r=r_0}\right) \quad (58)$$

From (56) and (57) result:

Proposition 11:

If $r=0$ the equation $h(\lambda,\tau)=0$, has one Hopf bifurcation point at r_0, where r_0 is given by(55).

We study the direction, stability and period of the bifurcating periodic solutions in system (52). The method we use is based on the normal form theory and the center manifold theorem introduced in [4], [8].

The notational convenience, let $r = r_0 + \mu$, with $\mu \geq 0$. Then $\mu = 0$ is the Hopf bifurcation value for system (52). System (52) can rewritten as:

$$\begin{aligned} \dot{u}_1(t) &= \alpha_1\rho_2 u_1(t) - \alpha_1 u_3(t-r) + \frac{1}{2!}\alpha_1\rho_3 u_1(t)^2 + \\ &+ \frac{1}{3!}\alpha_1\rho_4 u_1(t)^3 + O(u_1(t))^4 \\ \dot{u}_2(t) &= \alpha_2\rho_2 u_2(t) - \alpha_{21} u_3(t-r) + \frac{1}{2!}\alpha_2\rho_3 u_2(t)^2 - \\ &- \frac{1}{3!}\alpha_2\rho_4 u_2(t)^3 + O(u_2(t))^4 \\ \dot{u}_3(t) &= \frac{1}{c}u_1(t-r) + \frac{1}{c}u_2(t-r) \end{aligned} \quad (59)$$

where

$\rho_3 = U'''(x^*), \rho_4 = U^{IV}(x^*)$.

For $\phi = \phi(\theta,\mu)$, $\theta \in [-r,0]$, with $\phi \in C([-r,0],C^2)$. We consider

$$L_\mu \phi = A_2\phi(0) + B_2\phi(-r) \quad (60)$$

where A_2 and B_2 is given by:

$$A_2 = \begin{pmatrix} \alpha_1\rho_2 & 0 & 0 \\ 0 & \alpha_2\rho_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 & -\alpha_1 \\ 0 & 0 & -\alpha_2 \\ \frac{1}{c} & \frac{1}{c} & 0 \end{pmatrix} \quad (61)$$

Let

$$F(\mu,\phi) = \begin{pmatrix} \frac{1}{2!}\alpha_1\rho_3\phi_1(0)^2 + \frac{1}{3!}\alpha_1\rho_4\phi_1(0)^3 \\ \frac{1}{2!}\alpha_{21}\rho_3\phi_{21}(0)^2 + \frac{1}{3!}\alpha_{21}\rho_4\phi_1(0)^3 \\ 0 \end{pmatrix} \quad (62)$$

For $\phi \in C^1([-r,0],C^2)$, we define:

$$A_2(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -r \leq \theta \leq 0 \\ A_2\phi(0) + B_2\phi(-r), & \theta = 0 \end{cases}$$

$$B_2(\mu)\phi = \begin{cases} 0, & -r \leq \theta \leq 0 \\ F(\mu\theta), & \theta = 0. \end{cases}$$

For $\psi \in C^1([-r,0],C^2)$ the adjunct operator $A_2^*(\mu)$ of $A_2(\mu)$ is defined as:

$$A_2^*(\mu)\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & -r \leq s \leq 0 \\ \psi(s)^T A_2 + \psi(s)^T B_2, & s = 0. \end{cases}$$

For ϕ and ψ we define a bilinear form by:

$$<\psi,\phi> = \bar{\psi}^T(0)\phi(0) - \int_{\vartheta=-r}^{\theta}\int_{s=0}^{\theta} \bar{\psi}^T(s-\theta)d\eta_2(\theta)\phi(s)ds \quad (63)$$

where

$$\eta_2(\theta)=\begin{cases} 0, & \theta=0 \\ B_2\delta(\theta+r), & \theta\in[-r,0]\end{cases}$$

and $\delta(\phi+r)$ is Dirac distribuition.

In order to determine the Poincare normal form of the operator $A_2(\mu)$, we need to calculate the eigenvector q of $A_2(\mu)$ associated to the eigenvalue $\lambda_1=i\omega_0$ and the eigenvector q^* of $A_2^*(\mu)$ associated to eigenvalue $\lambda_2=-i\omega_0$. We can easily verify that

$$q(\theta)=v\exp(\lambda_1\theta),\ \theta\in[-r,0]$$

where $v=(v_1,v_2,v_3)^T$ and

$$\begin{aligned} v_1&=-\alpha_1(\lambda_1-\alpha_2\rho_2)\exp(\lambda_2 r)\\ v_2&=-\alpha_2(\lambda_1-\alpha_2\rho_2)\exp(\lambda_2 r)\\ v_3&=(\lambda_1-\alpha_1\rho_{2)}(\lambda_1-\alpha_2\rho_2)\end{aligned} \tag{64}$$

The eigenvector of $A_2^*(\mu)$ associated to eigenvalue λ_2 is given by:

$q^*(s)=w\exp(\lambda_2 s), s\in[0,r_0]$,

where $w=(w_1,w_2,w_3)^T$ and

$$\begin{aligned} w_1&=\frac{\exp(\lambda_1 r_0)}{c(\lambda_2-\alpha_1\rho_2)}w_3\\ w_2&=\frac{\exp(\lambda_1 r_0)}{c(\lambda_2-\alpha_2\rho_2)}w_3\\ w_3&=\frac{(\lambda_1-\alpha_1\rho_{2)}(\lambda_1-\alpha_2\rho_2)}{f}\end{aligned} \tag{65}$$

$$f=e^{\lambda_2 r_0}(\bar v_1-r_0\alpha_1 e^{\lambda_1 r_0}\bar v_3)(\lambda_1-\alpha_2\rho_2)+e^{\lambda_2 r_0}(\bar v_2-\alpha_2 r_0 e^{\lambda_1 r_0}\bar v_3)(\lambda_1-\alpha_1\rho_2)+$$

$$+(c\bar v_3+r_0 e^{\lambda_1 r_0}\bar v_1+r_0 e^{\lambda_1 r_0}\bar v_2)(\lambda_1-\alpha_1\rho_2)(\lambda_1-\alpha_2\rho_2)$$

Using (63), we can verify that $<q^*,q>=1$, $<\bar q^*,q>=<q^*,\bar q>=0$, $<\bar q^*,q>=1$.

Next, we will follow the ideas and use the notations in [4]. Let

$$z=<q^*,u_t>,\quad w(t,\theta)=u_t(\theta)-2\mathrm{Re}(z(t)q(\theta)) \tag{66}$$

Then

$$\dot z(t)=\lambda_1 z(t)+g(z(t),\bar z(t))$$

where

$$g(z,\bar z)=\frac{1}{2}g_{20}z^2+g_{11}z\bar z+\frac{1}{2}g_{02}\bar z^2+\frac{1}{2}g_{21}z^2 z$$

$$\begin{aligned} g_{20}&=\bar q^*(0)^T F_{20},\quad g_{11}=\bar q^*(0)^T F_{11},\\ g_{02}&=\bar q^*(0)^T F_{02},\quad g_{21}=\bar q^*(0)^T F_{21},\end{aligned} \tag{67}$$

$$\begin{aligned} F_{20}&=(\alpha_1\rho_1 v_1^2,\ \alpha_2\rho_3 v_2{}^2,0)^T,\\ F_{11}&=(\alpha_1\rho_1 v_1\bar v_1,\ \alpha_2\rho_3 v_1\bar v_1,0)^T,\\ F_{02}&=(\alpha_1\rho_3\bar v_1^2,\ \alpha_2\rho_3\bar v_2{}^2,0)^T,\end{aligned} \tag{68}$$

$$F_{21}=(\alpha_1(\rho_3(2v_1w_{111}+w_{120}\bar v_1)+\rho_4 v_1^2\bar v_1),$$

$$\alpha_2(\rho_3(2v_2w_{211}+w_{220}\bar v_2)+\rho_4 v_2^2\bar v_2),0)^T.$$

$$w(z,\bar z)=\frac{1}{2}w_{20}z^2+w_{11}z\bar z+\frac{1}{2}w_{02}\bar z^2$$

$$\begin{aligned} w_{20}&=-\frac{g_{20}}{2\lambda_1}v-\frac{g_{02}}{3\lambda_1}\bar v+E_1\\ w_{11}&=-\frac{g_{11}}{\lambda_1}v-\frac{\bar g_{11}}{\lambda_1}\bar v+E_2\end{aligned} \tag{69}$$

The vectors $E_1=(E_{11},E_{12},E_{13})^T$, $E_2=(E_{21},E_{22},E_{23})^T$ are given by:

$$\begin{aligned} &E_{11}=\frac{\rho_3}{2\rho_2}(v_1\bar v_1-v_2\bar v_2), E_{12}=-E_{11}, E_{13}=\frac{\rho_3}{2}(v_1\bar v_1-v_2\bar v_2),\\ &E_{21}=\frac{\rho_3}{2\rho_2}(v_1{}^2-v_2{}^2), E_{22}=-E_{21}, E_{23}=\frac{\rho_3}{2}\exp(\lambda_2 r_0)(v_1{}^2-v_2{}^2),\end{aligned} \tag{70}$$

From (68), (69), (70) result:

$$\begin{aligned} g_{20}&=\alpha_1\rho_3 v_1^2\overline{w_1}+\alpha_2\rho_3 v_2^2\overline{w_2},\\ g_{11}&=\alpha_1\rho_3 v_1\overline{v_1 w_1}+\alpha_2\rho_3 v_2\overline{v_2 w_2},\\ g_{02}&=\alpha_1\rho_3\overline{v_1^2 w_1}+\alpha_2\rho_3\overline{v_2^2 w_2},\\ g_{21}&=(\alpha_1(\rho_3(2v_1w_{111}+w_{120}\bar v_1)+\rho_4 v_1^2\bar v_1)\overline{w_1}+\\ &\quad+\alpha_2(\rho_3(2v_2w_{211}+w_{220}\bar v_2)+\rho_4 v_2^2\bar v_2)\overline{w_2}.\end{aligned} \tag{71}$$

The parameters $C_1(0)$, μ_2, T given by (32) with $g_{20}, g_{02}, g_{21}, g_{11}$ and (71).

The stochastic perturbation of for the system (51) is given by:

$$\begin{aligned} dx_1(t)&=\alpha_1(U'(x_1(t)-d(t-r))dt+\sigma_1(x_1(t)-x_{10})dw(t)\\ dx_2(t)&=\alpha_2(U'(x_{21}(t)-d(t-r))dt+\sigma_2(x_2(t)-x_{20})dw(t)\\ dx_3(t)&=\frac{1}{c}(x_1(t-r)+x_2(t-r))dt+\sigma_3(x_3(t)-d_0)dw(t)\end{aligned} \tag{72}$$

where $\sigma_i>0,\ i=1,2,3.$

For the stochastic differential system with delay a similar study can be done.

VI. NUMERICAL SIMULATION

In what follows, we consider $U(x)=u\ln(x+1)$. Using Maple 14, for: $u=100$, $c=2$, $\alpha=300$, the equilibrium point is $x_0=2$, $d_0=0{,}11$. For this values we have

$\omega_0 = 6{,}2247$, $r_0 = 0.085$, $\beta_2 = -0.159$, $\mu_2 = 0.003$, $T = 0.029$. The limit cycle is supercritical, the solutions are orbitally stable, the period increases.

The orbit $(t, x(t))$ is given in Fig. 1, orbit $(t, d(t))$ is given in Fig. 2, orbit $(x(t - r_0), x(t))$ in Fig. 3 and the orbit $(d(t - r_0), d(t))$ in Fig 4.

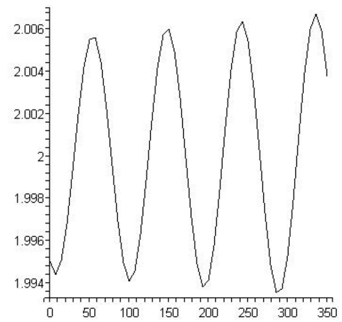

Fig.1 The orbit $(t, x(t))$

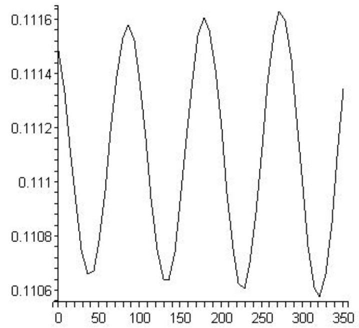

Fig.2 The orbit $(t, d(t))$

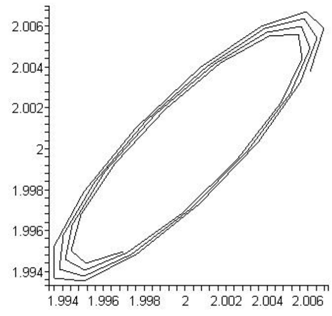

Fig.3 The orbit $(x(t - r_0), x(t))$

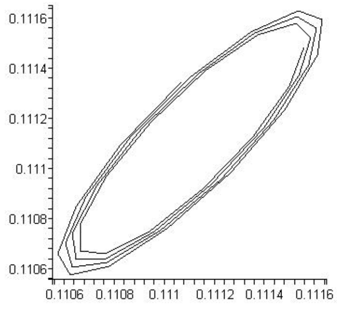

Fig.4 The orbit $(d(t - r_0), d(t))$

For $\sigma_1 = -0.1$, $\sigma_2 = -1.2$, $u = 100$, $c = 2$, $\alpha = 300$, using the Euler stochastic method, the figures Fig.5, Fig.6, Fig.7, present the orbits $(t, x_1(t, \omega))$, $(t, x_2(t, \omega))$, $(x_1(t, \omega), x_2(t, \omega))$ of the system (33).

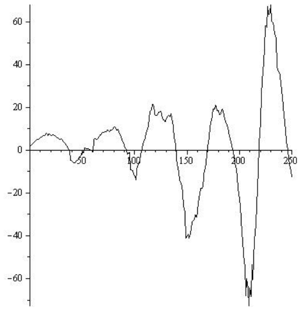

Fig.5 The orbit $(t, x_1(t, \omega))$

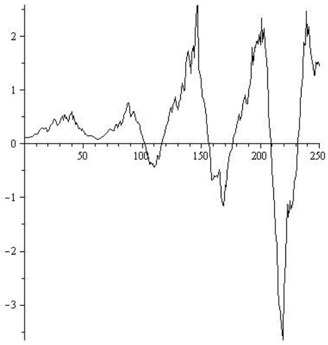

Fig.6 The orbit $(t, x_2(t, \omega))$

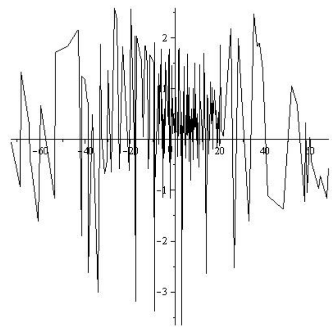

Fig.7 The orbit $(x_1(t, \omega), x_2(t, \omega))$

For $\sigma_1 = 0.3$, $\sigma_2 = 0.3$ we obtain $\omega_1 = 20{,}127$, $r_1 = 0.01182$, and the figures Fig. 8, fig.9, fig.10 show the orbits $(t, M_{11}(t))$, $(t, M_{22}(t))$, $(t, M_{12}(t))$.

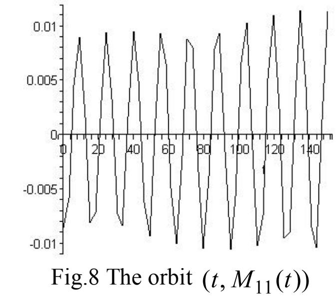

Fig.8 The orbit $(t, M_{11}(t))$

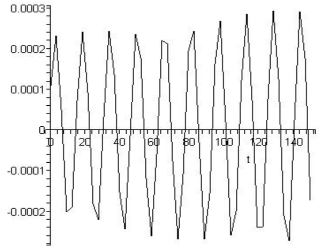

Fig.9 The orbit $(t, M_{22}(t))$

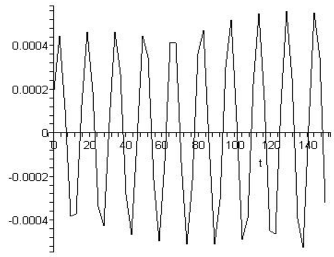

Fig.10 The orbit $(t, M_{12}(t))$

In what follows, we consider $U(x)=u\ln(x+1)$, $\alpha_1 = 2$, $\alpha_2 = 5$, $c = 50$, $u = 5$ the equilibrium point is $x_{10} = 25$, $x_{20} = 25$ $d_0 = 0{,}1923$. For this values we have $\omega_0 = 0{,}8942$, $r_0 = 1{,}7402$, $\beta_2 = 0{,}4150$, $\mu_2 = -0.000082$, $T = -0.0000017$. The limit cycle is subcritical, the solutions are orbitally unstable, the period decreases.

The orbit $(t, x_1(t,\omega))$ is given in Fig. 11, orbit $(t, x_2(t,\omega))$ is given in Fig. 12, and the orbit $(t, x_3(t,\omega))$ in Fig 13.

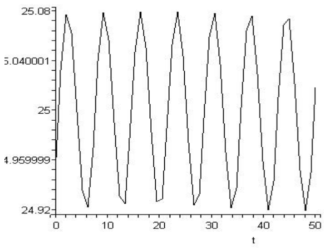

Fig.11 The orbit $(t, x_1(t,\omega))$

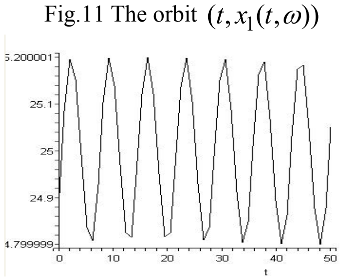

Fig.12 The orbit $(t, x_2(t,\omega))$

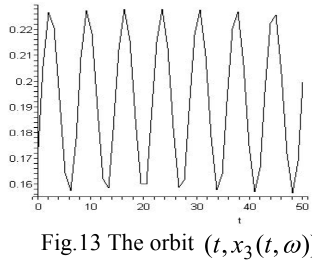

Fig.13 The orbit $(t, x_3(t,\omega))$

Numerical methos for ordinary differential equations are also given in [12].

VII. CONCLUSIONS

In this paper, we have examined the deterministic model for a network with a single link and with two users with delay. The time delay is determined for which a Hopf bifurcation takes place. The direction and the stability of the Hopf bifurcation are analyzed. The stochastic model is associated to the deterministic model. For this model the mean values and the square mean values of the linearized stochastic are analyzed. It is proved that there is a value of the delay for which a Hopf bifurcation takes place. The theoretical results are also justified by the numerical simulations. An analysis of the network model with a single link and multiple users will be done in a future paper.

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