Redefining Chaos: Devaney-chaos for Piecewise Continuous Dynamical Systems

Byungik Kahng

Abstract— One of the most widely accepted definition of chaos is the one by Devaney, which we will call *Devaney-chaos*. The purpose of our research is to investigate how the first two characteristic properties of Devaney-chaos are affected by the presence of the discontinuity, and subsequently, what kind of *adjustments* must be made to improve Devaney-chaos so that it can be applied to discontinuous dynamical systems as well as continuous systems. Under the afore-mentioned adjustments, we prove that the first two adjusted conditions of Devaney-chaos can be successfully used to characterize the complex orbit-behavior of piecewise continuous dynamical systems. Also, we show that the straightforward application of unadjusted Devaney-chaos is too inclusive when the system is discontinuous, consequently necessitating the afore-mentioned adjustments. We use the classification theorems of the singularities of the invertible planar piecewise isometric dynamical systems as the main tools.

Keywords— Devaney-chaos, Piecewise continuous dynamical system, Piecewise isometric dynamical system, Singularity.

I. Introduction

THIS paper is an extended and revised version of [28], which was written for the author's plenary lecture in the 15th American Conference on Applied Mathematics, Houston, Texas, April 30 – May 2, 2009, from which it won the best paper award. However, [28] is somewhat incomplete as a stand-alone-publication, because relatively large quantity of materials had to be left out due to the page-limit enforced by the conference. One of the two main rationales for this extended version is to include these topics, and consequently, to improve [28].

There is the second – and more important – reason, however. The author ignored the invitation to write the extended version by the conference organizers and the editors of the affiliated journals until now, because his research toward additional new results on this topic had still been going on. Now, with the addition of some fundamentally new results (Definition VI.2 and Theorem VI.3), the author thinks it is finally appropriate to write the extended and revised version of [28].

Some refinements on his old definitions and theorems were also done. In particular, it is worth noting the newly refined definitions of piecewise continuity, countably piecewise continuity, and topologically almost continuity (Definition VI.1). Not only did they enabled more precise treatment of our main results, but also, opened up other interesting problems to be studied for future research projects.

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This is an extended and retitled version of *On Devaney's Definition of Chaos for Discontinuous Dynamical Systems* (612-297), which won the best paper award in 15th WSEAS American Conference on Applied Mathematics, Houston, Texas, April 30 – May 2, 2009 [28].

For more effective description of the inclusion of the new materials and the improvements of old results, as explained in the previous paragraphs, the author put more appropriate title that summarizes his work more accurately. Even so, this paper remains largely as an extended and revised version of [28].

Acknowledgments. *The author thanks his Ph.D thesis advisor, Julian Palmore, for introducing him to Devaney-chaos. He also thanks Keith Burns for providing a different perspective on the shuffling-phenomenon, which lead to more precise treatment. Finally, he thanks Peter Ashwin and Arek Goetz for introducing him to piecewise isometric dynamical systems, and Xinchu Fu and Miguel Mendes for some of the ideas that help him extending his results to piecewise continuous dynamical systems.*

II. Background Information and Main Results

ONE of the most useful and widely accepted definition of chaos is the one by Devaney [9], which we will call *Devaney-chaos* (Definition III.1). Roughly speaking, Devaney-chaos consists of three conditions, *(1) the sensitive dependence upon the initial condition*, *(2) the topological transitivity*, and *(3) the dense distribution of the periodic orbits*. The third condition is often omitted for being too stringent. The precise definition of Devaney-chaos will be given in Definition III.1.

The concept of Devaney-chaos was developed for the iterative dynamics of continuous maps on continua. The purpose of our research is to study the effects of the discontinuity on the first two conditions of Devaney's definition of chaos, and subsequently improve Devaney's definition to the one that befits discontinuous dynamical systems as well as continuous systems. Figure II.1 exemplifies the effects of the discontinuity we just mentioned. It illustrates the *periodic sets* (white) and the *aperiodic sets* (dark) of selected examples of the discontinuous dynamical systems, or more precisely, selected examples of bounded invertible planar piecewise isometric dynamical systems.

There are a number of alternative ways to define the *chaos*, besides Devaney-chaos. An incomplete list of more popular methods include, *Lyapunov-chaos* (positive Lyapunov exponent) [2], [15], [39], [42], [45], *topological chaos* (positive topological entropy) [39], [42], *Smale-chaos* (homeomorphic to a Bernoulli shift dynamics)[1] [2], [15], and *Li-Yorke-chaos* (the existence of non-trivial scrambled set) [13], [31], [32], [43]. Providing the precise definitions of all these competing

[1]More common method is to look for the presence of Smale's horse-shoe-map, but the horse-shoe-map is homeomorphic to a Bernoulli shift [2], [15].

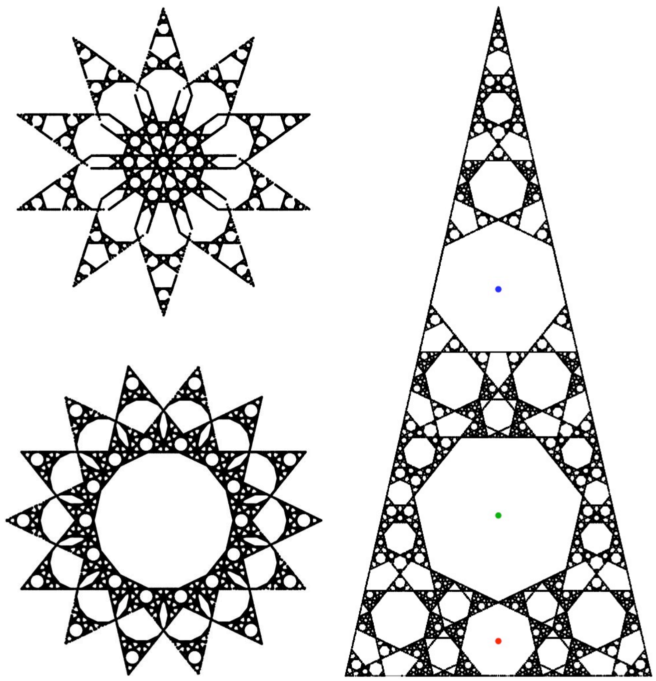

Fig. II.1. Periodic Sets (white) and Aperiodic Sets (dark) of Selected Examples of Bounded Invertible Planar Piecewise Isometric Dynamical Systems with Rational Rotations.

notions of chaos and their comparative analyses is not our concern here. The focus of our attention is Devaney-chaos. We review how it is defined in general, study how it is affected by the discontinuity, and show how it must be adjusted in the presence of the discontinuity.

Devaney-chaos is more inclusive than most of the competing notions of chaos, especially when the dynamics includes discontinuity, or more precisely speaking, *singularity*. Relatively recent discoveries by Goetz and Buzzi on the discontinuous dynamical systems include that the piecewise isometric dynamical systems, which are partly inspired by the digital signal processing and Hamiltonian dynamics, can generate complicated orbit structure, even though their Lyapunov-exponents and topological entropies are 0 [7], [16], [19]. Consequently, neither Lyapunov-chaos nor topological chaos can be applied to explain the complex behavior of the piecewise isometric dynamics. Goetz also proved that Smale-chaos fails to apply as well [18]. Devaney-chaos, on the other hand, proved to be a useful tool, *at least for some special cases*, as exemplified by some of the author's contributions to this topic [24], [25], [26], [27], [28], [29].

The author's success on using Devaney's definition of chaos to explain the complex behavior of discontinuous dynamical systems, however, was due in no small part by the *special conditions*, which were put in rather arbitrarily and have few justifiable rationales except that they produce pretty pictures, such as Figure III.1, Figure III.2, Figure III.3, and to a certain degree, Figure V.1 and Figure V.2, as well. This limitation was exposed by the author in [25], where he proved that the dynamics in the *exceptional set*[2] is not necessarily *topologically transitive* (Definition III.1), and therefore, not Devaney-chaotic if considered in the entire exceptional set. Also, similar properties on different systems were found, almost at the same time, independently from the author and also from each other [22], [37].

Despite the setback, the efforts to study the *chaotic behavior* generated by the discontinuity continued, even though most of the literature avoided using the term *chaos* or *topological transitivity*. Instead, they concentrated more toward the *minimal invariant sets* and *invariant decomposition*, which are intimately tied to the *ergodic spectrum* and the *unique ergodicity* of the dynamics, besides the topological transitivity [1], [5], [6], [35], [38]. Interestingly, this direction of development is precisely the opposite of the orientation of the earlier literature on this topic, which came primarily from digital signal processing and informatics [8], [14], [30], [33], [41]. That is, the *chaos* generated by the discontinuity, or more precisely, the *information overflow*, was their center of attention, even though the term *chaos* was not properly defined for their purpose.

In particular, Lowenstein recently proved that the minimality and the unique ergodicity holds under the invariant decomposition, for a substantially broad class of discontinuous dynamical systems [35]. In the viewpoint of Devaney-chaos, this results by Lowenstein implies the topological transitivity in each minimal invariant set. If appropriate adjustment can be made to the other condition, the *sensitive dependence upon the initial condition*, therefore, we will have a valid notion of Devaney-chaos in each minimal invariant set, which we can now call the *chaotic set* [28], [29], of the discontinuous dynamical system.

The basic goal of our research is to make the appropriate adjustment to the sensitive dependence part of Devaney-chaos for the discontinuous dynamical systems and study some of its common properties. In this paper, we do this first to invertible planar piecewise isometric dynamical systems through Definition V.4, which is well-defined and as strong as possible because of Theorem IV.4, Theorem V.1 and Theorem V.2. Afterwards, we extend our results to the invertible piecewise isometric dynamical systems in arbitrary dimension, and then toward the non-invertible cases. Ultimately, we extend our results to piecewise continuous iterative dynamical systems, through Definition VI.1, Definition VI.2 and Theorem VI.3.

III. Devaney's Definition of Chaos

IN [9], whose first edition was published in 1989, Devaney defined the chaos using three conditions, *(1) the sensitive dependence upon the initial condition*, *(2) the topological transitivity*, and *(3) the dense distribution of the periodic orbits*. However, most of the other literature admits only the first two conditions. See, for instance, [2], [15], [39], [42], [45], for the alternative descriptions of Devaney-chaos. We,

[2] In [25], the author used the term *singular set* instead. In this paper, he is following more refined terminology as in his more recent publication [29], among others.

too, will follow the latter convention and concentrate our efforts only to the first two conditions[3].

Definition III.1 (Devaney's Definition of Chaos)**.** *Let* (X, d) *be a metric space. Then, a map* $f : X \to X$ *is said to be* **Devaney-chaotic** *on* X *if it satisfies the following conditions.*

(1) f *has* **sensitive dependence on initial conditions***. That is, there exists a certain* $\delta > 0$ *such that, for any* $x \in X$ *and* $\epsilon > 0$*, there exist some* $y \in X$ *where* $d(x, y) < \epsilon$ *and* $m \in \mathbb{N}$ *so that* $d(f^m(x), f^m(y)) > \delta$*.*

(2) f *is* **topologically transitive***. That is, for any pair of open sets* $U, V \subset X$*, there exists a certain* $m \in \mathbb{N}$ *such that* $f^m(U) \cap V \neq \emptyset$*.*

Devaney's definition of chaos makes sense only for the iterative dynamical systems on continua. From Definition III.1, one can immediately see that no map is Devaney-chaotic if X is a discrete space. By taking $\epsilon = \frac{1}{2}$, for instance, one can prove that no map in $\mathbb{Z}$ can have the sensitive dependence upon the initial condition. For the rest of this paper, therefore, we will assume that the metric space X, on which the iterative dynamics takes place, is a continuum.

The purpose of this article is to study Devaney-chaos for discontinuous maps. Unlike the discreteness of the space, the discontinuity of the map affects Devaney's chaos conditions rather unevenly. In some cases, the conditions of Devaney-chaos are not affected by the presence of the discontinuity. Consider the following example.

Example III.2 (Symmetric Uniform Piecewise Elliptic Rotation Maps on $\mathbb{T}^2$)**.** *Let* $X = [0, 1)^2 \subset \mathbb{R}^2$ *and* $\theta \in \mathbb{R}$*. Define* $f : X \to X$ *by*

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y + 1 & \textit{mod 1} \\ x + 2y\cos\theta - \cos\theta & \textit{mod 1} \end{pmatrix}. \qquad \text{(III.1)}$$

Then, f *is known to be Devaney-chaotic in its aperiodic set, if* $\theta = k\pi/4$*,* $(k = 1, 3)$ *or* $\theta = k\pi/5$*,* $(k = 1, 2, 3, 4)$*.*

Proof. The proof for the first case, $\theta = k\pi/4$, $(k = 1, 3)$, is given in [24], and that of the second case is presented in [27]. See, also, [1] for the former case and [20], [34], [36] for the latter case. They do not deal with Devaney-chaos directly, but they study the dynamical systems similar to those of [24] and [27], respectively[4]. □

The *periodic sets* (white polygonal regions) and the *aperiodic sets* (dark set) of the iterative dynamics of the **symmetric uniform piecewise elliptic rotation map (SUPER map)** given by the equality (III.1) for $\theta = \pi/4$ and $\theta = 2\pi/5$ are presented in Figure III.1 and Figure III.2, respectively. Even though the SUPER map is not exactly a piecewise isometry by itself, it is easy to make it a piecewise isometry (or piecewise pure rotation) by tilting it appropriately, as in Figure III.3.

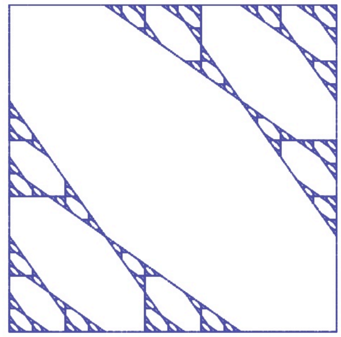

Fig. III.1. $\theta = \frac{\pi}{4}$.

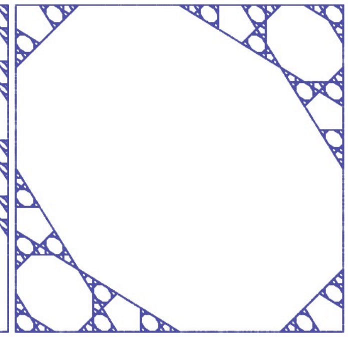

Fig. III.2. $\theta = \frac{2\pi}{5}$.

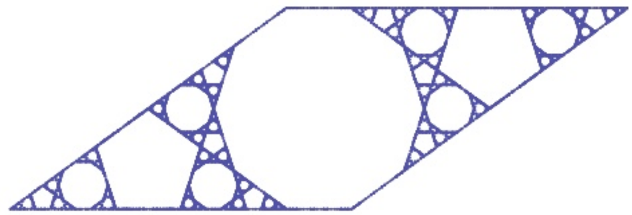

Fig. III.3. $\theta = \frac{\pi}{5}$.

The iterative dynamics of the SUPER map (Example III.2), was originated, in part, from the information overflow analysis problems in digital signal processing [8], [14], [26], [30], [33], [41]. Most engineering literature about the SUPER map implicitly assumes the *chaotic behavior* in the aperiodic set [8], [14], [30], [33], [41]. Interestingly, though, only a few special cases of SUPER maps are known to be Devaney-chaotic in their entire aperiodic set, while most of the others are proven to be otherwise [25], [35], [37], [38]. All these *failures*, however, come from the topological transitivity, while our focus of attention for this paper is the sensitive dependence.

In other occasions, as we will see, for instance, in the following example, Definition III.1 appears to be too inclusive.

Example III.3. *Let* $X \subset \mathbb{C}$ *be the set given by* $X = \phi(\mathbb{R})$*, where*

$$\phi(\theta) = (-1)^{[\theta]}(1 - e^{-2\pi i\theta(-1)^{[\theta]}}).$$

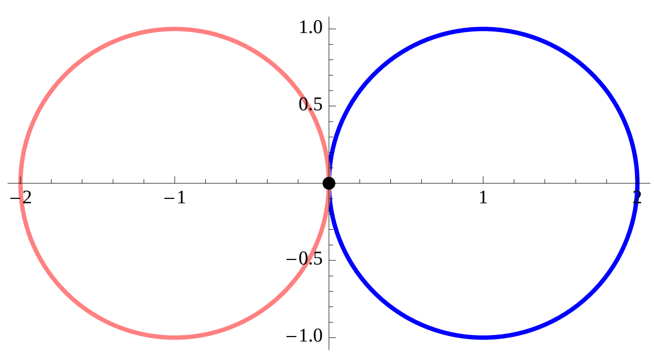

Fig. III.4. $X = \phi(\mathbb{R}) \subset \mathbb{C}$.

[3]In fact, our decision to exclude the third condition, the dense distribution of the periodic orbits, is driven partly by necessity. The characterization of the periodic orbits is a delicate matter for many discontinuous dynamical systems and most of them are left unsolved. We leave this aspect as a topic of future research.

[4][1] proves the minimality, which implies the topological transitivity of Devaney's definition of chaos (Definition III.1), but the rest of Devaney-chaos was not studied in [1]. It pays more attention to symbolic coding and the unique ergodicity instead. Likewise, [20], [34], [36] do not directly deal with Devaney-chaos, even though they study the systems similar to those of [27].

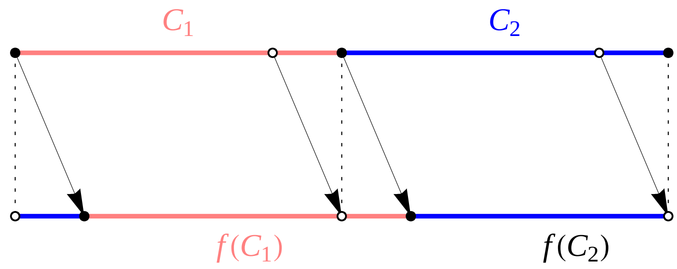

Fig. III.5. $\sigma : \theta \mapsto \theta + \rho \pmod 2$.

Here, $[\theta]$ is the largest integer not exceeding θ (the floor function). Figure III.4 depicts $X \subset \mathbb{C}$. Let $\rho \in \mathbb{R}$ be an irrational number. Define a map $f : X \to X$ by

$$f(\phi(\theta)) \mapsto \phi(\theta + \rho).$$

Then f is continuous everywhere in X except at the singular point $z = 0$, at which f is double-valued. Furthermore, f is Devaney-chaotic, according to Definition III.1.

Proof. The map $f : X \to X$ has two continuous pieces and one singular point. When $2k - 1 < \theta < 2k$ where $k \in \mathbb{Z}$, we get $z = (-1)^{[\theta]}(1 - e^{-2\pi i\theta(-1)^{[\theta]}}) = -1 + e^{2\pi i\theta}$, which is the punctured circle $\{z \in \mathbb{C} : |z+1| = 1, z \neq 0\}$ (the left hind side circle of Figure III.4). When $2k < \theta < 2k + 1$ where $k \in \mathbb{Z}$, on the other hand, we get $z = (-1)^{[\theta]}(1 - e^{-2\pi i\theta(-1)^{[\theta]}}) = 1 - e^{-2\pi i\theta}$, which yields $\{z \in \mathbb{C} : |z - 1| = 1, z \neq 0\}$ (the right hand side circle of Figure III.4). When $\theta \in \mathbb{Z}$, we get the singular point, $z = 0$, which is doubly defined by ϕ.

Figure III.5 and Figure III.6 illustrate the dynamics of f. Because of the periodicity of ϕ, we can decompose f by $f = \phi \circ \sigma$, where $\sigma(\theta) = \theta + \rho \pmod 2$, which is homeomorphic to the rotation map $e^{i\pi\theta} \mapsto e^{i(\theta+\rho)}$. Because ρ is irrational, any $\{\sigma^n(\theta) : n \in \mathbb{N}\}$ is dense in $[0, 2]$. Taking a certain θ_0 such that $\{\sigma^n(\theta_0) : n \in \mathbb{N}\} \cap \mathbb{Z} = \emptyset$, we get a point $z_0 = \phi(\theta_0)$ such that $\{f^n(z_0) : n \in \mathbb{N}\}$ is dense in X, and thus, the topological transitivity of f follows.

Figure III.4 and Figure III.6 depict the sensitive dependence on the initial condition around the singularity $z = 0$. A pair of points in the left hand side circle that are separated by the singularity are *cut off* and then sent to drastically different places. The most extreme case is the singular point (the black point in Figure III.4), in which case, the point itself is separated and sent to two different positions (the black points in Figure III.6). This observation applies to every pair of points. Because of the irrational rotation, every pair of points in either circle must be separated by the singularity $z = 0$ after a certain number of iterations of f. In the next application of f, they are separated by the singularity. $\square$

Recall that the map f in Example III.3 has a singularity at $z = 0$, at which f is double-valued. Alternatively, we can regard f discontinuous but single valued at $z = 0$ by taking either one of the two branches. It is easy to see, however, these considerations make difference almost nowhere.

Remark III.4. *Even though we just proved that the map $f : X \to X$ of Example III.3 satisfies the conditions of Devaney-*

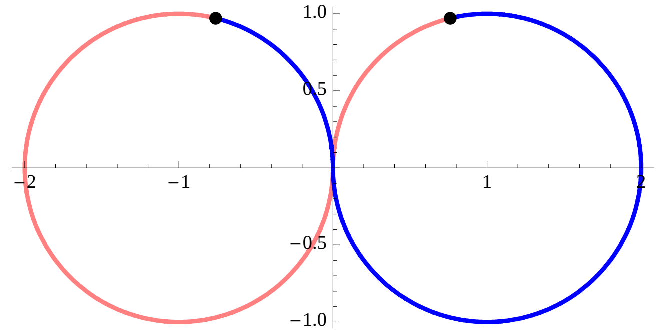

Fig. III.6. $f(z) = f(\phi(\theta)) = \phi(\theta + \rho)$.

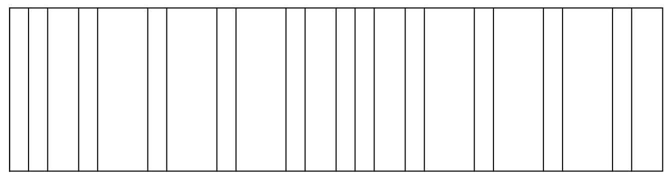

Fig. III.7. The aperiodic set of σ after 10 iterations.

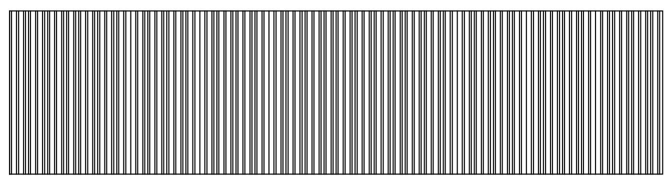

Fig. III.8. The aperiodic set of σ after 100 iterations.

chaos, calling this map chaotic, is somewhat problematic, in a sense that the dynamics is too trivial. For instance, compare the visualization of the aperiodic set of the map $\sigma : \theta \mapsto \theta + \rho \pmod 2$, which is the straightened-out version of that of f in X, presented in Figure III.7 and Figure III.8, with those of non-trivial discontinuous dynamics presented in Figure III.1 and Figure III.2. In fact, if we identify the left hand side and the right hand side loops of Figure III.4 by $-1 + e^{2\pi i\theta} \equiv 1 - e^{-2\pi i\theta}$, then the quotient map $\tilde{f} : \tilde{X} \to \tilde{X}$ is reduced to a rotation map $\tilde{f} : [-1 + e^{2\pi i\theta}] \mapsto [-1 + e^{2\pi i(\theta+\rho)}]$.

IV. Singularities of Invertible Planar Piecewise Isometric Dynamical Systems

Roughly speaking, a *piecewise isometry* is a discontinuous map on a union of finitely many regions (which we call, *atoms*), in each of which the map is an isometry (which we call, the *isometric component*). There are a number of different ways to define it. In this paper, we will follow the author's method as appeared in [29].

Definition IV.1 (Multiple-valued Map)**.** *Let X and Y be sets. Also, let $\mathscr{P}(X)$ and $\mathscr{P}(Y)$ be their power sets. A set function $f : \mathscr{P}(X) \to \mathscr{P}(Y)$ is called a* **multiple-valued map** *from X to Y if $f(A) = \bigcup\{f(\{x\}) : x \in A\}$ for all $A \subset X$.*

Rather than defining a piecewise isometry as a discontinuous map, we define it as a kind of multiple valued map.

Definition IV.2 (Piecewise Isometry)**.** *Let $\{P_1, \cdots, P_n\}$ be a collection of mutually disjoint connected open regions in $\mathbb{R}^2$ with piecewise smooth boundary, and let $X = \bigcup_{i=1}^n \bar{P}_i$. A multiple-valued map $f : X \to \mathbb{R}^2$ is called a* **piecewise isometry** *subordinate to $\{P_1, \cdots, P_n\}$, if there exist isometries $f_i : \bar{P}_i \to \mathbb{R}^2$, $i \in \{1, \cdots, n\}$ such that $f(x) = \{f_i(x) : x \in \bar{P}_i\}$. Here, each P_i and f_i are called an* **atom** *and an* **isometric component** *of f, respectively. We say f is* **bounded***, if each P_i is bounded. We say f is a* **polygon exchange***, if each P_i is polygonal. We say f is a* **piecewise rotation***, if each $f_i : \bar{P}_i \to \mathbb{R}^2$ is a rotation. Finally, we say f is* **invertible***, if $f(P_i) \cap f(P_j) = \emptyset$ whenever $i \neq j$.*

Note that the a piecewise isometry can be multiple valued only on the common boundary $\partial P_i \cap \partial P_j$, in which case, we will say that the piecewise isometry has the **singularity** there. Besides this distinction, there is essentially no difference between the singularity and the discontinuity, so we will use the term singularity in favor of discontinuity from now on.

Piecewise isometries are important for our purpose because of the simplicity. That is, every non-trivial behavior (including the chaotic behavior) of a piecewise isometric dynamics must come from the singularity (discontinuity), because the isometric components alone contribute nothing. In this paper, we restrict ourselves to the planar piecewise isometries, in which case each isometric component must be a composition of a rotation, a translation, and possibly an inversion.

Piecewise isometric dynamical systems often exhibit complex orbit behavior, some of which are visualized in Figure II.1, Figure III.1, Figure III.2, Figure III.3, and also in Figure V.1 and Figure V.2. These apparent *fractal patterns* are generated by the singularity. The dark regions are given by the forward and the backward iterates of the singularity, which we call the *singular set* and denote Σ. Upon taking the closure, $\bar{\Sigma}$, we get the set of points whose orbits get arbitrarily close to the singularity, which we call the *exceptional set*. More precisely, we define the singular set and the exceptional set as follows.

Definition IV.3 (Singular Set and Exceptional Set)**.** *Let $\{P_1, \cdots, P_n\}$, $\{f_1, \cdots, f_n\}$ and M be as in Definition IV.2. Suppose that $f : M \to M$ is an invertible piecewise isometry subordinate to $\{P_1, \cdots, P_n\}$, with the isometric components $\{f_1, \cdots, f_n\}$. Let*

$$\Sigma^+ = \{x \in M : \ f \text{ is multiple-valued at } x\},$$
$$\Sigma^- = \{x \in M : \ f^{-1} \text{ is multiple-valued at } x\}.$$

We call the set,

$$\Sigma = \bigcup_{k=0}^{\infty} \left(f^k(\Sigma^+) \cup f^{-k}(\Sigma^-) \right),$$

the **singular set** *of f. We call its closure $\bar{\Sigma}$, the* **exceptional set** *of f.*

The interior of the white regions consists of the points whose orbits stay away from the singularity, and it is tedious but not difficult to prove that they turn out to be the periodic sets [23]. For this reason, we call the exceptional set $\bar{\Sigma}$ alternatively as the **aperiodic set**, even though it may contain some degenerate periodic points[5]. We will use both of these two terms from this point on.

Inside the periodic set, the dynamics of the piecewise isometric system is rather trivial. Unaffected by the singularity, the dynamics there must be an iteration of a composition of a translation, a rotation, and possibly an inversion. In the aperiodic set, on the other hand, the piecewise isometric dynamics can be quite complicated. In some cases, it is known that the piecewise isometric dynamical systems are Devaney-chaotic (in traditional sense as in Definition III.1) in their aperiodic sets [24], [27]. In other occasions, it is known that the aperiodic set consists of more than one invariant sets in which the dynamics is Devaney-chaotic [25]. We will call such invariant sets, the *chaotic sets* (Definition **??**). In many cases, the chaotic sets exhibit *riddling* [25]. Finally, it is also possible that the dynamics turns out to be too trivial to be called chaotic, as we saw in Example III.3.

We aim to find a way to distinguish the trivial cases such as Example III.3 from the non-trivial cases. We begin with investigating the behavior of the map on the singularity.

Theorem IV.4 (Isometric Continuation Theorem)**.** *Let $f : X \to X$ be a bounded invertible planar piecewise isometry. Let $f_1, \cdots, f_n$ be the isometric components of f in the atoms $P_1, \cdots, P_n$ respectively. Suppose that f has the* **cutting singularity** *on a curve segment $S_{ij} = \partial P_i \cap \partial P_j$, that is, $f_i(S_{ij}) \cap f_j(S_{ij})$ has length 0. Then, the quotient map $\tilde{f}$ on the quotient space $\tilde{X}$ given by the following identification is continuous in $\tilde{P}_i \cup \tilde{P}_j$.*

$$p \equiv q \iff \begin{cases} \textit{either} & p = q; \\ \textit{or} & p = f_i(x), q = f_j(x), \\ & \exists i, j \in \{1, \cdots, n\}, x \in S_{ij}; \\ \textit{or} & p = f_j(x), q = f_i(x), \\ & \exists i, j \in \{1, \cdots, n\}, x \in S_{ij}. \end{cases}$$

In other words, the discontinuity on S_{ij} disappears.

Proof. See the author's paper, [29]. □

We call the identification ($\equiv$) of Theorem IV.4, the **patch-up identification**, and the process of making $\tilde{f} : \tilde{X} \to \tilde{X}$ out of $f : X \to X$, the **isometric continuation**. Some singularity completely disappears under the isometric continuation, but not in general. What it does, typically, is to *postpone* the singularity by one iteration.

Let us take Figure IV.1 and Figure IV.2 for examples. The piecewise isometry f on the rhombus of Figure IV.1 consists of three isometric components, f_{-1}, f_0, f_{+1}, and three atoms, P_{-1}, P_0, P_{+1}. f_0 rotates P_0, f_{+1} rotates P_{+1} and then translates it upward, and f_{-1} is applied similarly to P_{-1} (Figure IV.2).

By taking the isometric continuation, which is the natural identification of the top edge E_T and the bottom edge E_B

[5]The degenerate periodic points, if they exist, must always be on Σ [23]. For this reason, some people call $\bar{\Sigma} \setminus \Sigma$ the aperiodic set. The trade off in this case is that $\bar{\Sigma} \setminus \Sigma$ does not contain all the aperiodic points.

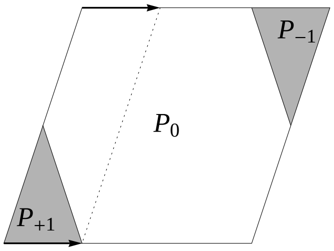

Fig. IV.1. Before.

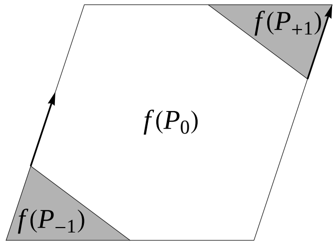

Fig. IV.2. After.

of the rhombus, we can remove the discontinuity at $\partial P_0 \cap \partial P_{+1}$ and at $\partial P_0 \cap \partial P_{-1}$. Moreover, applying the isometric continuation also to f^{-1}, we can identify the slanted edges as well, and consequently, we get $\tilde{X} = \mathbb{T}^2$.

The singularity does not go away completely in general, however. As we can see from the splitting and the sliding of the arrow in Figure IV.1 and Figure IV.2, the cutting singularity evolves to another type of singularity, which we call the *sliding singularity*. Note that $f_0(E_T) \cap f_{+1}(E_B)$ has positive length, and thus, the condition of Theorem IV.4 no longer applies. Consequently, the isometric continuation stops here. The periodicity and the singularity structure generated by this sliding singularity is depicted in Figure III.3, and to a certain degree, in Figure III.1 and Figure III.2 as well. The precise definition of the sliding singularity is rather complicated and quite unnecessary for our purpose. Interested readers may refer to [29]. The key point here is that a singularity can evolve to another type of singularity upon taking the isometric continuations.

Besides the cutting and the sliding singularities, there is another type of singularity that we must consider, the *shuffling singularity*. The precise definition of the shuffling singularity is also quite complicated. Understanding the shuffling singularity is essential to improving Devaney's definition of chaos for discontinuous dynamics, however. Therefore, despite its complexity and tediousness, we present it here in the following abbreviated form. See [29] for the full detail.

Definition IV.5 (Shuffling Singularity)**.** *Let $\{P_1, \cdots, P_n\}$, $\{f_1, \cdots, f_n\}$, X and f be as in Theorem IV.4. We say f has the* **shuffling singularity** *on a segment $S_1 \subset \partial P_{i_1} \cap \partial P_{j_1}$, if there are a finite number of segments $S_k \subset \partial P_{i_k} \cap \partial P_{j_k}$, $k \in \{1, 2, \cdots, r\}$ and a positive integer $m \in \mathbb{Z}^+$ that satisfy the following conditions.*

(1) *For every S_k, $k \in \{1, \cdots, r\}$, there exists a certain S_l, $l \in \{1, \cdots, r\}$ such that $\tilde{f}^m \circ f_{i_k}(S_k) = \tilde{f}^m \circ f_{j_l}(S_l)$, upon taking appropriate branches of the isometric continuation $\tilde{f}^m$.*

(2) *$S_k \cap S_l$ has length 0, if $k \neq l$.*

(3) *Each S_k is the maximal segment with respect to the inclusion that satisfies (1) and (2).*

Here is rather trivial example of a shuffling singularity, as illustrated in Figure IV.3 and Figure IV.4.

Example IV.6. *Let $X = [a_0, a_4] \times [b, c] = \bar{P}_1 \cup \cdots \cup \bar{P}_4 \subset \mathbb{R}^2$, where $P_k = (a_{k-1}, a_k) \times (b, c)$, $k \in \{1, 2, 3, 4\}$, as depicted in Figure IV.3 and let $f : X \to X$ be a piecewise translation*

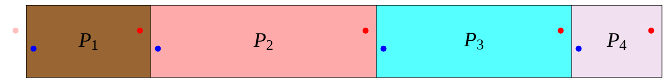

Fig. IV.3. $X = \bar{P}_1 \cup \cdots \cup \bar{P}_4$.

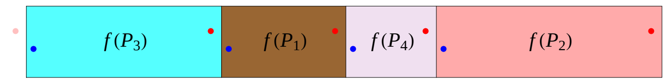

Fig. IV.4. $f(X) = \overline{f(P_1)} \cup \cdots \cup \overline{f(P_4)}$.

as depicted in Figure IV.4. Then, each vertical segment of ∂P_k is a shuffling singularity of f.

Proof. Without the isometric continuation, each common boundary edge $\partial P_i \cap \partial P_j$ is a cutting singularity, while the far left hand side edge $\{a_1\} \times [b, c]$ and the far right hand side edge $\{a_4\} \times [b, c]$ are non-singular. Upon taking the isometric continuation given by the patch-up identification that identifies $\{a_1\} \times [b, c]$ and $\{a_4\} \times [b, c]$, we get the quotient map $\tilde{f} : \tilde{X} \to \tilde{X}$, where $\tilde{X}$ is can be regarded as a band, $\{(\cos(2\pi ix/(a_2 - a_1)), \sin(2\pi ix/(a_2 - a_1)), y) : (x, y) \in X\}$. We can easily see that $\tilde{f}$ gains the continuity on $\partial\tilde{P}_2 \cap \partial\tilde{P}_3 \subset \tilde{X}$, but loses the continuity on $\partial\tilde{P}_1 \cap \partial\tilde{P}_4 \subset \tilde{X}$, consequently yielding the shuffling singularity in every $\partial\tilde{P}_i \cap \partial\tilde{P}_j$ as illustrated by the red and blue dots of Figure IV.3 and Figure IV.4. Note that the light red and light blue dots outside X in Figure IV.4 and Figure IV.3 stand for the copies of the red and blue dots at the other side of the strip under the patch-up identification. □

Note that the shuffling of the atoms in Example IV.6 can be regarded as a slight generalization of the exchange of the adjacency in Example III.3. Indeed, it is not difficult to prove that the singularity of Example III.3 is also a shuffling singularity, without any isometric continuation. Even though the map $f : X \to X$ of Example IV.6 cannot be reduced to a trivial map as that of Example III.3 (Remark III.4), both of the iterative dynamical systems share some important characteristics. One of them, of course, is the shuffling singularity, as we just pointed out. Also, if at least one ratio $(a_i - a_{i-1})/(a_j - a_{j-1})$ of Example IV.6 is irrational for some i, j, $i \neq j$, then the whole space turns out to be the aperiodic set, as depicted by Figure IV.5 and Figure IV.6. Compare these pictures with Figure III.7 and Figure III.8 on Example III.3. Compare, also, with the complicated and aesthetically beautiful orbit structure depicted in Figure II.1, Figure III.1, Figure III.2, Figure III.3, and to a certain degree, Figure V.1 and Figure V.2 as well, all of which originated from non-shuffling singularities.

Do the shuffling singularity contribute indeed nothing toward the complex periodicity and singularity structure? At this moment, this question remains unanswered. The author cautiously conjectures so, however. What he does know for sure is that the shuffling singularity does not contribute toward the *chaotic dynamics* according to his adjusted definition of Devaney-chaos (Definition VI.2).

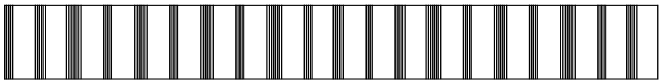

Fig. IV.5. The singular set of f after 50 iterations.

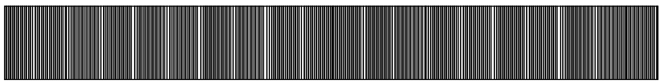

Fig. IV.6. The singular set of f after 150 iterations.

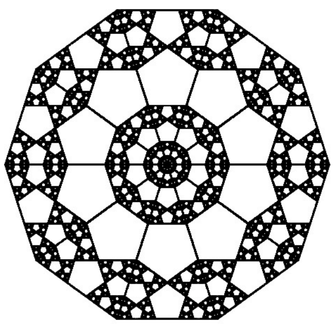

Fig. V.1. The entire singular set.

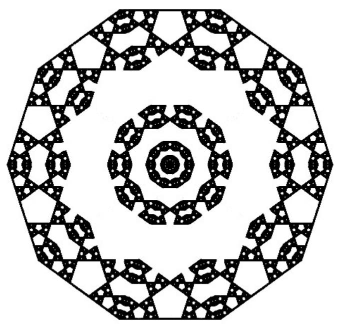

Fig. V.2. Essential singularity only.

V. Devaney-chaos for Invertible Planar Piecewise Isometric Dynamical Systems

In this section, we will use the different types of singularities of the piecewise isometric dynamical systems that we introduced in the pervious section to re-define Devaney-chaos. The following theorem justifies this argument, by guaranteeing that there is no other types of singularities to consider.

Theorem V.1 (The Classification Theorem)**.** *Suppose that $\{P_1, \cdots, P_n\}$, $\{f_1, \cdots, f_n\}$, X and f are as in Theorem IV.4, and let $S \subset \partial P_i \cap \partial P_j$. Then, we must have one the the following.*

(1) *For some $m \in \mathbb{N}$, $\tilde{f}^m \circ f_i = \tilde{f}^m \circ f_j$. That is, the singularity disappears after finitely many applications of the isometric continuation. In other words, the singularity is* **removable**.

(2) *f has the shuffling singularity on S.*

(3) *f has the non-shuffling sliding singularity on S. We call such singularity the* **essential singularity**.

Proof. See [29]. □

Figure V.1 and Figure V.2 exemplify the difference between the essential and non-essential singularities. Figure V.1 illustrates the entire singularity structure of the piecewise isometric system discussed in [44], while Figure V.2 depicts only the essential singularity structure. Note that only the line segments between adjacent periodic sets are removed. The majority of the aperiodic set remain the same, and so does the Hausdorff dimension, which is $\frac{\log 6}{\log(2+\sqrt{5})} \approx 1.24114397$. See [19], [27] for the precise calculation of the Hausdorff dimension.

The key idea behind this project was the realization that the shuffling singularity is non-chaotic.

Theorem V.2 (Shuffling Singularity Theorem)**.** *Suppose that $\{P_1, \cdots, P_n\}$, $\{f_1, \cdots, f_n\}$, X and f are as in Theorem IV.4. Let $S_1 \subset \partial P_{i_1} \cap \partial P_{j_1}, \cdots, S_r \subset \partial P_{i_r} \cap \partial P_{j_r}$ and $m \in \mathbb{N}$ be as in Definition IV.5. Let x_k and y_k be the points sufficiently close to S_k where $k \in \{1, \cdots, r\}$, such that each x_k and y_k are positioned in the same geometrical location from S_k as x_1 and y_1 are from S_1. Suppose further that each x_k and y_k are separated up to $m+1$ iterations only by S_k, where the crossing point is $c_k = \overline{x_k y_k} \cap S_k$. Then, for every $k \in \{1, \cdots, r\}$, there exists a unique $l \in \{1, \cdots, r\}$ that satisfies the following conditions.*

(1) *$\tilde{f}^m \circ f_{i_k}(c_k) = \tilde{f}^m \circ f_{j_l}(c_l)$. That is, the two points merge to the same point after $m+1$ iterations, upon taking appropriate branches.*

(2) *The curve segment $\overline{f^{m+1}(x_k) f^{m+1}(y_l)}$ is a line segment that runs through the (common) point $\tilde{f}^m \circ f_{i_k}(c_k) = \tilde{f}^m \circ f_{j_l}(c_l)$.*

(3) *$\mu_1\left(\overline{x_k y_k}\right) = \mu_1\left(\overline{f^{m+1}(x_k) f^{m+1}(y_l)}\right)$. That is, the length of the line segment is preserved.*

Let us call these points **conjugate points**.

Proof. Figure V.3, Figure V.4 and Figure V.5 explain the main idea behind this theorem. The complete proof is rather long and tricky. See [29] for detail. □

Remark V.3. *Theorem V.2 tells us that the shuffling singularity shuffles not only the curve segments of the singular set, but also their one-sided neighborhoods as well. Consequently, the piecewise isometric dynamics around the shuffling singularity merely shuffles the x_k's and y_k's. Hence, even though $d(f(x_k), f(y_k))$ suddenly gets large, the minimal distances $\min\{d(f(x_k), g(y_l)) : 0 \le l \le r\}$ and $\min\{d(f(x_l), g(y_k)) : 0 \le l \le r\}$ remain the same as $d(x_k, y_k)$. Consequently, the shuffling singularity does not contribute toward the sensitive dependence upon the initial condition.*

Figure IV.3 and Figure IV.4 illustrate the idea behind Remark V.3. Here, the original points x_k's and y_k's, and the image points $f(x_k)$'s and $f(y_k)$'s are represented as the red and blue dots near the singular edges[6].

Remark V.3 allows us to draw the following conclusion, the adjusted definition of Devaney-chaos that can be used to the discontinuous dynamical systems as well. Theorem V.2 and Remark V.3 justifies the adjustment and Theorem V.1 guarantees that this improvement is as strong as possible.

Finally, we are ready to state the adjusted definition of Devaney-chaos for invertible planar piecewise isometric dynamical systems. As we pointed out above, the validity of this definition is based upon Theorem V.1 and Theorem V.2.

Definition V.4 (Devaney-chaos for Invertible Planar Piecewise Isometric Dynamical Systems)**.** *Let $f : X \to X$ be an invertible planar piecewise isometry and let $Y \subset X$ be invariant under f, that is, $f(Y) = Y$. Then, the iterative*

[6]The light dots outside the strip represent the copies of the points in the other side of the strip. Recall the the outer edges can be identified through the isometric continuation.

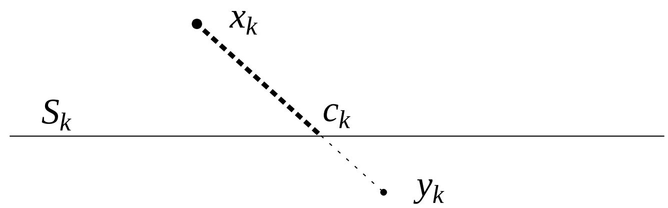

Fig. V.3. x_k and y_k before iteration.

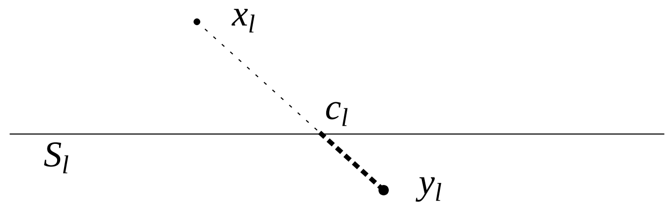

Fig. V.4. x_l and y_l before iteration.

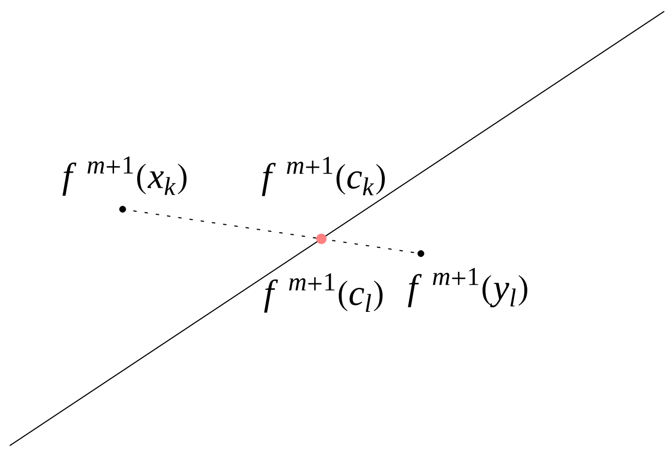

Fig. V.5. After iteration.

dynamics of $f : Y \to Y$ *is said to be* **Devaney-chaotic** *in the* **chaotic set** Y*, if it satisfies the following conditions.*

(1) f *has* **sensitive dependence on initial conditions** *in* Y*. That is, there exists a certain* $\delta > 0$ *such that, for any* $x_0 \in Y$ *and* $\epsilon > 0$*, there exist some* $y_0 \in Y$ *where* $d(x, y) < \epsilon$ *and* $m \in \mathbb{N}$ *such that*

$$\min\{d(f^{m_k}(x_k), f^{m_k}(y_j)) : 0 \leq k, j \leq r\} > \delta. \tag{V.1}$$

where x_k*'s and* y_l*'s are the conjugate points of* x_0 *and* y_0*, respectively.*

(2) f *is* **topologically transitive** *(Definition III.1).*

VI. Devaney-chaos for Piecewise Continuous Iterative Dynamical Systems

Thanks to Theorem IV.4, Theorem V.1 and Theorem V.2, the adjusted Devaney-chaos that we defined in the previous section (Definition V.4) is solidly established and as strong as possible. The trade-off is, of course, that Definition V.4 applies only to the invertible planar piecewise isometric dynamical systems. In this section, we will consider three types of extension. The first is toward arbitrary dimension, and the second is beyond the invertibility. The third is the consideration of piecewise continuity, instead of piecewise isometry.

Extending our results to invertible piecewise isometric dynamical systems in arbitrary dimension is not too difficult, because the conjugate points that we defined in Theorem V.2 can be easily extended to arbitrary dimension. All we need to do is to set up the system of vector equations, $\overrightarrow{c_k x_k} = \overrightarrow{c_l x_l}$ and $\overrightarrow{c_k y_k} = \overrightarrow{c_l y_l}$, to define the conjugate points x_l and y_l of x_k and y_k, respectively, where S_k and S_l in Figure V.3 and Figure V.4 are now hyper-planes. The extension of Theorem V.2 is, therefore, easily achieved, and thus, the extension of Definition V.4 follows. We do not know, however, if this extended notion is Devaney-chaos is as strong as possible yet, because Theorem V.1 applies only to the 2-dimensional cases at this moment. We leave it as a possible topic for future research.

Going beyond the invertibility is more delicate matter. If the piecewise isometry is allowed to be non-invertible, then some $f_i(P_i)$ and $f_j(P_j)$ may overlap, and thus, the conjugacy relation and the conclusion of Theorem V.2 cannot be achieved. Indeed, it is known that non-invertible piecewise isometric dynamics can have attracting and repelling behavior due to the discontinuity [3], [4], [10], [11], [12], [16], [17], [21], [40]. It is necessary, however, to screen out the *shuffling-like* behavior, or Devaney-chaos for this case will become too inclusive. Our solution is to provide the adjusted definition of Devaney-chaos without using the conjucacy relation, as in Definition VI.2. This idea turned out to be useful for piecewise continuous iterative dynamics as well, by the way.

Going from piecewise isometry to piecewise continuity is another difficult step. Because the discontinuous maps in general can be extremely complicated, let us confine ourselves to the discontinuous maps that are on the verge of continuity. They are, *piecewise continuous maps*, *countably piecewise continuous maps* and *topologically almost continuous maps*. The last is easy to define. We say a map $f : X \to X$ is **topologically almost continuous** if its discontinuity set has measure 0 with respect to every Baire measure of X. The piecewise continuity and countably piecewise continuity, on the other hand, are somewhat tricky to define precisely.

Definition VI.1 (Continuity Partition)**.** *Let* X *be a topological space and let* $f : X \to X$ *be an endomorphism. We define the* **continuity partition** $\mathscr{P}_f(X)$ *of* $f : X \to X$ *as the collection* $\{P_i \subset X : i \in I\}$ *of mutually disjoint connected sets that partitions the whole space, i.e.* $X = \bigcup_{i \in I} P_i$*, such that each* P_i *is a maximal connected set in which* f *is continuous, i.e. if there is a connected set that strictly contains* P_i*, then it has at least one discontinuous point. We say* f *is* **piecewise continuous** *with respect to a measure* μ *in* X*, if there is a finite subset* J *of the index set* I *such that* $\mu(\bigcup_{i \in (I \setminus J)} P_i) = 0$*. Also, we say* f *is* **countably piecewise continuous** *with respect to* μ*, if there is a countable subset* J *of* I *such that* $\mu(\bigcup_{i \in (I \setminus J)} P_i) = 0$*. We say* f *is* **topologically piecewise continuous / topologically countably piecewise continuous***, if* f *is piecewise continuous / countably piecewise continuous with respect to every Baire measure. Finally, we say* f *is* **piecewise continuous / countably piecewise continuous** *with*

finite partition / countable partition *if its continuity partition is a finite partition / countable partition.*

The reason that Definition VI.1 became so complicated is because the boundaries of the atoms can get rather complicated. For instance, it must be possible to consider the atoms with fractal boundaries (say Koch snowflake), because such atoms can be homeomorphically deformed to much simpler regions.

In particular, when the iterative dynamical system is given by a piecewise continuous map with finite partition, it is quite possible that some singularities do not contribute toward the *chaotic behavior*, in a sense that the orbits do not disperse away when bundled together with finite number of other orbits. Although it is difficult to define *conjucacy* and *shuffling* when the distance and angle are no longer preserved, we can still extend the key ideas of Devaney-chaos as follows. As usual, we consider only the case that the whole space of the dynamics is a continuum.

Definition VI.2 (Devaney-chaos for Piecewise Continuous Dynamical Systems)**.** *Let (X,d) be a metric space and $f : X \to X$ be a piecewise continuous endomorphism with finite partition, $\mathscr{P}_f(X) = \{P_1, \cdots, P_n\}$. Let Y be an invariant subset of X under f, that is, $f : Y \to Y$. Then, the iterative dynamical system of $f : Y \to Y$ is said to be* **Devaney-chaotic** *on the* **chaotic set** *Y, if it satisfies the following conditions.*

(1) *f has* **sensitive dependence on initial conditions***. That is, there exists a certain $\delta > 0$ such that, for any $\epsilon > 0$ and any finite set $\{x_1, \cdots, x_r\}$, where each x_k belong to distinct P_{i_k}, we can find some finite set $\{y_1, \cdots, y_r\} \subset Y$ and positive integers $m_1, \cdots, m_r$ such that*

$$\min\{d(f^{m_k}(x_k), f^{m_k}(y_j)) : 0 \le j \le r\} > \delta. \tag{VI.1}$$

(2) *f is* **topologically transitive** *(Definition III.1).*

Now that we proposed the adjusted definition of Devaney-chaos, we must prove that our definition is indeed well-defined.

Theorem VI.3. *Let (X,d) be a metric space and $f : X \to X$ be a continuous endomorphism. Then, the iterative dynamical system $f : X \to X$ is Devany-chaotic according to Definition III.1 if and only if it is Devaney-chaotic according to Definition VI.2. On the other hand, suppose that $f : X \to X$ is an invertible planar piecewise isometry and $Y \subset X$ is invariant under f, that is, $f(Y) = Y$. Then, the iterative dynamics of $f : Y \to Y$ is Devaney-chaotic in Y according to Definition V.4 if it is Devaney-chaotic according to Definition VI.2.*

Proof. The first equivalence is easy to see. If $f : X \to X$ is continuous, then there is only one continuity partition, so $r = 1$ in Definition VI.2, which is easily reduced to the classical definition of Devaney-chaos, Definition III.1. The second equivalence follows from the fact that the sensitive dependence condition of Definition VI.2 is more restrictive than that of Definition V.4. It is because the selection of x_k's are arbitrary, instead of being confined to conjugate points only. Therefore, Devancy-chaos according to Definition VI.2 implies Devaney-chaos according to Definition V.4. □

At this moment, we do not know for sure if Definition VI.2 is as strong as possible, as was the case for Definition V.4, because the theorems that correspond to Theorem IV.4, Theorem V.1 and Theorem V.2 are not established yet. For the time being, therefore, we leave this part as a possible future research problem.

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