Flow and heat transfer over an unsteady stretching sheet in a micropolar fluid with prescribed surface heat flux

N. Bachok, A. Ishak, and R. Nazar

***Abstract*—** The unsteady laminar flow of an incompressible micropolar fluid over a stretching sheet with prescribed surface heat flux is investigated. The governing partial differential boundary layer equations are first transformed into ordinary differential equations before being solved numerically by a finite-difference method. The effects of the unsteadiness parameter, material parameter and Prandtl number on the flow and heat transfer characteristics are studied. It is found that the surface shear stress and the heat transfer rate at the surface are higher for micropolar fluids compared to Newtonian fluids.

***Keywords*—**Unsteady flow, Heat transfer, Stretching sheet, Micropolar fluid, Fluid mechanics.

I. Introduction

THE fluid dynamics due to a stretching surface is important in manufacturing processes. Examples are numerous and they include the aerodynamic extrusion of plastic sheets, the boundary layer along a liquid film in condensation processes, paper production, glass blowing, metal spinning and drawing plastic films. The thermal fluid flow problems have been extensively studied numerically, theoretically as well as experimentally (see [1,2]). The quality of the final product depends on the rate of heat transfer at the stretching surface. Crane [3] first obtained an elegant analytical solution to the boundary layer equations for the problem of steady two-dimensional flow due to a stretching surface in a quiescent incompressible fluid. Gupta and Gupta [4] extended the problem posed by Crane [3] to a permeable sheet and obtained closed-form solution, while Grubka and Bobba [5] studied the thermal field and presented the solutions in terms of Kummer's functions. The 3-dimensional case has been considered by Wang [6]. Chen [7] studied the case when buoyancy force is taken into consideration, and Magyari and Keller [8] considered exponentially stretching surface. The heat transfer over a stretching surface with variable surface heat flux has been considered by Char and Chen [9], Lin and Cheng [10], Elbashbeshy [11] and very recently by Ishak et al. [12].

Manuscript received July 28, 2010. This work was supported in part by the IPTA Fundamental Research Grant.

N. Bachok is with the Department of Mathematics and Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia (phone: 603-8946-6849; fax: 603-8943-7958; e-mail: norfifah@math.upm.edu.my).

A. Ishak and R. Nazar are with the School of Mathematical Science, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor, Malaysia (e-mail: anuarishak@yahoo.com, rmn72my@yahoo.com).

All of the above mentioned studies dealt with stretching sheet where the flows were assumed to be steady. The unsteady flows due to a stretching sheet have received less attention; a few of them are those considered by Devi et al. [13], Andersson et al. [14], Nazar et al. [15], and very recently by Ishak et al. [16]. In Ref. [15], the similarity transformation introduced by Williams and Rhyne [17] was used, which transformed the governing partial differential equations with three independent variables to two independent variables, which are more convenient for numerical computations.

Motivated by the above investigations, in this paper we present the unsteady flow and heat transfer characteristics caused by a stretching sheet immersed in a micropolar fluid. The governing partial differential equations are transformed into ordinary ones using similarity transformation, before being solved numerically by the Keller-box method. The results obtained are then compared with those of Elbashbeshy [11] and the series solution for the steady-state flow case to support their validity.

II. Problem Formulation

Consider an unsteady, two-dimensional laminar flow of an incompressible micropolar fluid over a stretching sheet. At time $t=0$, the sheet is impulsively stretched with velocity $U_w(x,t)$ along the x-axis, keeping the origin fixed in the fluid of ambient temperature T_∞. The stationary Cartesian coordinate system has its origin located at the leading edge of the sheet with the positive x-axis extending along the sheet, while the y-axis is measured normal to the surface of the sheet. The boundary layer equations may be written as [12,16]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\left(\frac{\mu+\kappa}{\rho}\right)\frac{\partial^2 u}{\partial y^2}+\frac{\kappa}{\rho}\frac{\partial N}{\partial y}, \tag{2}$$

$$\rho j\left(\frac{\partial N}{\partial t}+u\frac{\partial N}{\partial x}+v\frac{\partial N}{\partial y}\right)=\gamma\frac{\partial^2 N}{\partial y^2}-\kappa\left(2N+\frac{\partial u}{\partial y}\right), \tag{3}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{4}$$

subject to the boundary conditions

$$u = U_w, \quad v = 0, \quad N = -m\frac{\partial u}{\partial y}, \quad \frac{\partial T}{\partial y} = -\frac{q_w}{k} \quad \text{at} \quad y = 0,$$
$$u \to 0, \quad N \to 0, \quad T \to T_\infty \quad \text{as} \quad y \to \infty, \tag{5}$$

where m is the boundary parameter with $0 \le m \le 1$ [18], u and v are the velocity components in the x- and y-directions, respectively, T is the fluid temperature in the boundary layer, N is the microrotation or angular velocity, and j, γ, μ, κ, ρ, and α are the microinertia per unit mass, spin gradient viscosity, dynamic viscosity, vortex viscosity, fluid density and thermal diffusivity, respectively. It is assumed that the stretching velocity $U_w(x,t)$ and the surface heat flux $q_w(x,t)$ are of the forms

$$U_w(x,t) = \frac{ax}{1-ct}, \quad q_w(x,t) = \frac{bx}{1-ct} \tag{6}$$

where a, b and c are constants with $a > 0$, $b \ge 0$ and $c \ge 0$ (with $ct < 1$), and both a and c have dimension time^{-1}. It should be noted that at $t = 0$ (initial motion), Eqs. (1) – (4) describe the steady flow over a stretching surface. These particular forms of $U_w(x,t)$ and $q_w(x,t)$ have been chosen in order to be able to devise a new similarity transformation, which transforms the governing partial differential equations (1) – (4) into a set of ordinary differential equations, thereby facilitating the exploration of the effects of the controlling parameters (see Andersson et al. [14]).

As was shown by Ahmadi [19], the spin-gradient viscosity γ can be defined as

$$\gamma = (\mu + \kappa/2)\, j = \mu(1 + K/2)\, j, \tag{7}$$

where $K = \kappa/\mu$ is the dimensionless viscosity ratio and is called the material parameter. Relation (6) is invoked to allow the field of equations predicts the correct behavior in the limiting case when the microstructure effects become negligible and the total spin N reduces to the angular velocity [19,20].

The continuity equation (1) is satisfied by introducing a stream function ψ such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. The momentum, angular momentum and energy equations can be transformed into the corresponding ordinary differential equations by the following transformation:

$$\eta = \left(\frac{U_w}{\nu x}\right)^{1/2} y, \quad \psi = (\nu x U_w)^{1/2} f(\eta),$$
$$N = U_w\left(\frac{U_w}{\nu x}\right)^{1/2} h(\eta), \quad \theta(\eta) = \frac{k(T - T_\infty)}{q_w}\left(\frac{U_w}{\nu x}\right)^{1/2}, \tag{8}$$

where η is the similarity variable. The transformed nonlinear ordinary differential equations are:

$$(1+K) f''' + ff'' - f'^2 + Kh' - S\left(f' + \frac{1}{2}\eta f''\right) = 0 \tag{9}$$

$$\left(1 + \frac{K}{2}\right)h'' + f h' - f' h - K(2h + f'') - S\left(\frac{3}{2}h + \frac{1}{2}\eta h'\right) = 0 \tag{10}$$

$$\frac{1}{\Pr}\theta'' + f\theta' - f'\theta - S\left(\theta + \frac{1}{2}\eta\theta'\right) = 0 \tag{11}$$

where primes denote differentiation with respect to η, $Pr = \nu/\alpha$ is the Prandtl number and $S = c/a$ is the unsteadiness parameter. The boundary conditions (5) now become

$$f(0) = 0, \quad f'(0) = 1, \quad h(0) = -mf''(0), \quad \theta'(0) = -1,$$
$$f'(\eta) \to 0, \quad h(\eta) \to 0, \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty. \tag{12}$$

The quantities of physical interest are the values of $f''(0)$ and $1/\theta(0)$ which represent the skin friction coefficient and the heat transfer rate at the surface, respectively. Thus, our task is to investigate how the governing parameters S, m, K and Pr influence these quantities.

We note that when $K = 0$ (viscous fluid) and $S = 0$ (steady flow), the problem under consideration reduces to a steady-state flow, where the closed-form solution for the flow field and the solution for the thermal field in terms of Kummer's functions are respectively given by

$$f(\eta) = 1 - e^{-\eta}, \tag{13}$$

$$\theta(\eta) = \frac{1}{\Pr} e^{-\eta \Pr} \frac{M(\Pr - 1, \Pr + 1, -\Pr e^{-\eta})}{M(\Pr - 1, \Pr, -\Pr)}, \tag{14}$$

where $M(a,b,z)$ denotes the confluent hypergeometric function [21], with

$$M(a,b,z) = 1 + \sum_{n=1}^{\infty} \frac{a_n}{b_n}\frac{z^n}{n!},$$
$$a_n = a(a+1)(a+2)\cdots(a+n-1),$$
$$b_n = b(b+1)(b+2)\cdots(b+n-1).$$

By using Eqs. (13) and (14), the skin friction coefficient $f''(0)$ and the surface temperature $\theta(0)$ can be shown to be given by

$$f''(0) = -1, \tag{15}$$

$$\theta(0) = \frac{1}{\Pr}\frac{M(\Pr - 1, \Pr + 1, -\Pr)}{M(\Pr - 1, \Pr, -\Pr)}. \tag{16}$$

The nonlinear ordinary differential equations (9) – (12) have been solved numerically by a finite-difference scheme known as the Keller-box method, as described in the book by Cebeci and Bradshaw [22], which is very familiar to the present authors (see Bachok et al. [23,24] and Bachok and Ishak [25,26]).

III. SOLUTION PROCEDURE

(i) Finite-difference method

To solve the transformed differential Eqs. (9) – (11) subjected to the boundary conditions (12), Eqs. (9) – (11) are first converted into a system of seven first-order equations, and the difference equations are then expressed using central differences. For this purpose, we introduce new dependent variables $p(\eta)$, $q(\eta)$, $g(\eta)=h(\eta)$, $n(\eta)$, $s(\eta)=\theta(\eta)$ and $t(\eta)$ so that Eqs. (9) – (11) can be written as

$$f'=p, \tag{17}$$

$$p'=q, \tag{18}$$

$$g'=n, \tag{19}$$

$$s'=t, \tag{20}$$

$$(1+K)q'+fq-p^2+Kn-S\left(p+\frac{1}{2}\eta q\right)=0, \tag{21}$$

$$\left(1+\frac{K}{2}\right)n'+fn-pg-K(2g+q)-S\left(\frac{3}{2}g+\frac{1}{2}\eta n\right)=0, \tag{22}$$

$$\frac{1}{\Pr}t'+ft-ps-S\left(s+\frac{1}{2}\eta t\right)=0. \tag{23}$$

In terms of the new dependent variables, the boundary conditions (12) are given by

$$\begin{aligned} &f(0)=0,\quad p(0)=1,\quad g(0)=-mq(0),\quad t(0)=-1,\\ &p(\eta)\to 0,\quad g(\eta)\to 0,\quad s(\eta)\to 0\quad \text{as}\quad \eta\to\infty. \end{aligned} \tag{24}$$

We now consider the segment $\eta_{j-1}\eta_j$, with $\eta_{j-1/2}$ as the midpoint, which is defined as below:

$$\eta_0=0,\quad \eta_j=\eta_{j-1}+h_j,\quad \eta_J=\eta_\infty, \tag{25}$$

where h_j is the $\Delta\eta$ - spacing and $j=1,2,...,J$ is a sequence number that indicates the coordinate location. The finite-difference approximation equations (17)-(23) are written for the midpoint $\eta_{j-1/2}$ of the segment $\eta_{j-1}\eta_j$. This procedure gives

$$\frac{f_j-f_{j-1}}{h_j}=\frac{p_j+p_{j-1}}{2}=p_{j-1/2}, \tag{26}$$

$$\frac{p_j-p_{j-1}}{h_j}=\frac{q_j+q_{j-1}}{2}=q_{j-1/2}, \tag{27}$$

$$\frac{g_j-g_{j-1}}{h_j}=\frac{n_j+n_{j-1}}{2}=n_{j-1/2}, \tag{26}$$

$$\frac{s_j-s_{j-1}}{h_j}=\frac{t_j+t_{j-1}}{2}=t_{j-1/2}, \tag{29}$$

$$\begin{aligned} &(1+K)\frac{q_j-q_{j-1}}{h_j}+(fq)_{j-1/2}-\left(p^2\right)_{j-1/2}+K(n)_{j-1/2}-\\ &S\left[(p)_{j-1/2}+\frac{1}{2}\eta(q)_{j-1/2}\right]=0 \end{aligned} \tag{30}$$

$$\begin{aligned} &\left(1+\frac{K}{2}\right)\frac{n_j-n_{j-1}}{h_j}+(fn)_{j-1/2}-(pg)_{j-1/2}-\\ &K\left[2(g)_{j-1/2}+(q)_{j-1/2}\right]-S\left[\frac{3}{2}(g)_{j-1/2}+\frac{1}{2}\eta(n)_{j-1/2}\right]=0 \end{aligned} \tag{31}$$

$$\begin{aligned} &\frac{1}{\Pr}\frac{t_j-t_{j-1}}{h_j}t'+(ft)_{j-1/2}-(ps)_{j-1/2}-\\ &S\left[(s)_{j-1/2}+\frac{1}{2}\eta(t)_{j-1/2}\right]=0. \end{aligned} \tag{32}$$

Rearranging of expressions (26)-(32) gives

$$f_j-f_{j-1}-\frac{1}{2}h_j\left(p_j+p_{j-1}\right)=0, \tag{33}$$

$$p_j-p_{j-1}-\frac{1}{2}h_j\left(q_j+q_{j-1}\right)=0, \tag{34}$$

$$g_j-g_{j-1}-\frac{1}{2}h_j\left(n_j+n_{j-1}\right)=0, \tag{35}$$

$$s_j-s_{j-1}-\frac{1}{2}h_j\left(t_j+t_{j-1}\right)=0, \tag{36}$$

$$\begin{aligned} &(1+K)q_j-q_{j-1}+\frac{1}{4}h_j\left(f_j+f_{j-1}\right)\left(q_j+q_{j-1}\right)-\\ &\frac{1}{4}h_j\left(p_j+p_{j-1}\right)^2+\frac{1}{2}h_jK\left(n_j+n_{j-1}\right)-\\ &\frac{1}{2}h_j\,S\left[\left(p_j+p_{j-1}\right)+\frac{1}{2}\eta\left(q_j+q_{j-1}\right)\right]=0 \end{aligned} \tag{37}$$

$$\begin{aligned} &\left(1+\frac{K}{2}\right)n_j-n_{j-1}+\frac{1}{4}h_j\left(f_j+f_{j-1}\right)\left(n_j+n_{j-1}\right)-\\ &\frac{1}{4}h_j\left(p_j+p_{j-1}\right)\left(g_j+g_{j-1}\right)-\\ &K\frac{1}{2}h_j\left[2\left(g_j+g_{j-1}\right)+\left(q_j+q_{j-1}\right)\right]-\\ &S\frac{1}{2}h_j\left[\frac{3}{2}\left(g_j+g_{j-1}\right)+\frac{1}{2}\eta\left(n_j+n_{j-1}\right)\right]=0 \end{aligned} \tag{38}$$

$$\begin{aligned} &\frac{1}{\Pr}t_j-t_{j-1}+\frac{1}{4}h_j\left(f_j+f_{j-1}\right)\left(t_j+t_{j-1}\right)-\\ &\frac{1}{4}h_j\left(p_j+p_{j-1}\right)\left(s_j+s_{j-1}\right)-\\ &S\frac{1}{2}h_j\left[\left(s_j+s_{j-1}\right)+\frac{1}{2}\eta\left(t_j+t_{j-1}\right)\right]=0. \end{aligned} \tag{39}$$

Equations (33)-(39) are imposed for $j=1,2,3,...,J$, and the transformed boundary layer thickness η_J is to be sufficiently large so that it is beyond the edge of the boundary layer. The boundary conditions are

$$\begin{aligned} &f_0=0,\quad p_0=1,\quad g_0=-mq_0,\quad t_0=-1,\\ &p_J=0,\quad g_J=0,\quad s_J=0. \end{aligned} \tag{40}$$

(ii) Newton's method

To linearize the nonlinear system (33)-(39), we use Newton's method, by introducing the following expression:

$$\begin{aligned} &f_j^{(k+1)}=f_j^{(k)}+\delta f_j^{(k)},\ p_j^{(k+1)}=p_j^{(k)}+\delta p_j^{(k)},\\ &q_j^{(k+1)}=q_j^{(k)}+\delta q_j^{(k)},\ g_j^{(k+1)}=g_j^{(k)}+\delta g_j^{(k)},\\ &n_j^{(k+1)}=n_j^{(k)}+\delta n_j^{(k)},s_j^{(k+1)}=s_j^{(k)}+\delta s_j^{(k)},\\ &t_j^{(k+1)}=t_j^{(k)}+\delta t_j^{(k)} \end{aligned} \tag{41}$$

where $k = 0,1,2,\ldots$. We then insert the left-hand side expressions in place of $\delta f^{(k)}$, $\delta p^{(k)}$, $\delta q^{(k)}$, $\delta g^{(k)}$, $\delta n^{(k)}$, $\delta s^{(k)}$ and $\delta t^{(k)}$. This procedure yields the following linear system (the superscript k is dropped for simplicity):

$$\delta f_j - \delta f_{j-1} - \frac{h_j}{2}\left(\delta p_j + \delta p_{j-1}\right) = \left(r_1\right)_{j-1/2}, \tag{42}$$

$$\delta p_j - \delta p_{j-1} - \frac{h_j}{2}\left(\delta q_j + \delta q_{j-1}\right) = \left(r_2\right)_{j-1/2}, \tag{43}$$

$$\delta g_j - \delta g_{j-1} - \frac{h_j}{2}\left(\delta n_j + \delta n_{j-1}\right) = \left(r_3\right)_{j-1/2}, \tag{44}$$

$$\delta s_j - \delta s_{j-1} - \frac{h_j}{2}\left(\delta t_j + \delta t_{j-1}\right) = \left(r_4\right)_{j-1/2}, \tag{45}$$

$$\begin{aligned}&\left(a_1\right)\delta q_j + \left(a_2\right)\delta q_{j-1} + \left(a_3\right)\delta f_j + \\ &\quad \left(a_4\right)\delta f_{j-1} + \left(a_5\right)\delta p_j + \left(a_6\right)\delta p_{j-1} + \\ &\quad \left(a_7\right)\delta n_j + \left(a_8\right)\delta n_{j-1} = \left(r_5\right)_{j-1/2},\end{aligned} \tag{46}$$

$$\begin{aligned}&\left(b_1\right)\delta n_j + \left(b_2\right)\delta n_{j-1} + \left(b_3\right)\delta f_j + \\ &\quad \left(b_4\right)\delta f_{j-1} + \left(b_5\right)\delta p_j + \left(b_6\right)\delta p_{j-1} + \\ &\quad \left(b_7\right)\delta g_j + \left(b_8\right)\delta g_{j-1} + \left(b_9\right)\delta q_j + \\ &\quad \left(b_{10}\right)\delta q_{j-1} = \left(r_6\right)_{j-1/2},\end{aligned} \tag{47}$$

$$\begin{aligned}&\left(c_1\right)\delta t_j + \left(c_2\right)\delta t_{j-1} + \left(c_3\right)\delta f_j + \\ &\quad \left(c_4\right)\delta f_{j-1} + \left(c_5\right)\delta p_j + \left(c_6\right)\delta p_{j-1} + \\ &\quad \left(c_7\right)\delta s_j + \left(c_8\right)\delta s_{j-1} = \left(r_7\right)_{j-1/2},\end{aligned} \tag{48}$$

where

$$\begin{aligned}
&\left(a_1\right)_j = 1 + K + \frac{1}{2}h_j f_{j-1/2} - \frac{S}{4}h_j\eta, \\
&\left(a_2\right)_j = \left(a_1\right)_j - 2\left(1+K\right),\ \left(a_3\right)_j = \frac{1}{2}h_j q_{j-1/2}, \\
&\left(a_4\right)_j = \left(a_3\right)_j,\ \left(a_5\right)_j = -h_j p_{j-1/2} - \frac{S}{2}h_j, \\
&\left(a_6\right)_j = \left(a_5\right)_j,\ \left(a_7\right)_j = \frac{1}{2}h_j K,\ \left(a_8\right)_j = \left(a_7\right)_j, \\
&\left(b_1\right)_j = 1 + \frac{K}{2} + \frac{1}{2}h_j f_{j-1/2} - \frac{S}{4}h_j\eta, \\
&\left(b_2\right)_j = \left(b_1\right)_j - 2\left(1+\frac{K}{2}\right),\ \left(b_3\right)_j = \frac{1}{2}h_j n_{j-1/2}, \\
&\left(b_4\right)_j = \left(b_3\right)_j,\ \left(b_5\right)_j = -\frac{1}{2}h_j g_{j-1/2}, \\
&\left(b_6\right)_j = \left(b_5\right)_j, \\
&\left(b_7\right)_j = -\frac{1}{2}h_j p_{j-1/2} - Kh_j - \frac{3}{4}Sh_j, \\
&\left(b_8\right)_j = \left(b_7\right)_j,\ \left(b_9\right)_j = -\frac{K}{2}h_j,\ \left(b_{10}\right)_j = \left(b_9\right)_j, \\
&\left(c_1\right)_j = \frac{1}{\Pr} + \frac{1}{2}h_j f_{j-1/2} - \frac{S}{4}h_j\eta, \\
&\left(c_2\right)_j = \left(c_1\right)_j - \frac{2}{\Pr},\ \left(c_3\right)_j = \frac{1}{2}h_j t_{j-1/2}, \\
&\left(c_4\right)_j = \left(c_3\right)_j,\ \left(c_5\right)_j = -\frac{1}{2}h_j s_{j-1/2}, \\
&\left(c_6\right)_j = \left(c_5\right)_j,\ \left(c_7\right)_j = -\frac{1}{2}h_j p_{j-1/2} - \frac{S}{2}h_j, \\
&\left(c_8\right)_j = \left(c_7\right)_j,
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
&\left(r_1\right)_{j-1/2} = -f_j + f_{j-1} + h_j p_{j-1/2}, \\
&\left(r_2\right)_{j-1/2} = -p_j + p_{j-1} + h_j q_{j-1/2}, \\
&\left(r_3\right)_{j-1/2} = -g_j + g_{j-1} + h_j n_{j-1/2}, \\
&\left(r_4\right)_{j-1/2} = -s_j + s_{j-1} + h_j t_{j-1/2}, \\
&\left(r_5\right)_{j-1/2} = \left(1+K\right)\left(q_j - q_{j-1}\right) - h_j\left(fq\right)_{j-1/2} + \\
&\qquad h_j\left(p^2\right)_{j-1/2} - Kh_j\left(n\right)_{j-1/2} + \\
&\qquad Sh_j\left[\left(p\right)_{j-1/2} + \frac{\eta}{2}\left(q\right)_{j-1/2}\right], \\
&\left(r_6\right)_{j-1/2} = \left(1+\frac{K}{2}\right)\left(n_j - n_{j-1}\right) - h_j\left(fn\right)_{j-1/2} + \\
&\qquad h_j\left(gp\right)_{j-1/2} + Kh_j\left[2\left(g\right)_{j-1/2} + \left(q\right)_{j-1/2}\right] + \\
&\qquad \frac{S}{2}h_j\left[3\left(g\right)_{j-1/2} + \eta\left(n\right)_{j-1/2}\right], \\
&\left(r_7\right)_{j-1/2} = \frac{1}{\Pr}\left(t_j - t_{j-1}\right) - h_j\left(ft\right)_{j-1/2} + \\
&\qquad h_j\left(ps\right)_{j-1/2} + Sh_j\left[\left(s\right)_{j-1/2} + \frac{\eta}{2}\left(t\right)_{j-1/2}\right].
\end{aligned} \tag{50}$$

The boundary conditions (40) become

$$\begin{aligned}&\delta f_0 = 0,\quad \delta p_0 = 0,\quad \delta g_0 = 0,\quad \delta t_0 = 0, \\ &\delta p_J = 0,\quad \delta g_J = 0,\quad \delta s_J = 0,\end{aligned} \tag{51}$$

which just express the requirement for the boundary conditions to remain constant during the iteration process.

(iii) Block-elimination method

The linearized difference equations (42)-(48) can be solved by the block-elimination method as outlined by Cebeci and Bradshaw [22], since the system has block-tridiagonal structure. Commonly, the block-tridiagonal structure consists of variables or constants, but here an interesting feature can be observed that it consists of block matrices. In a matrix-vector form, Eqs. (42)-(48) can be written as

$$\boldsymbol{A}\boldsymbol{\delta} = \boldsymbol{r} \tag{52}$$

where

$$\boldsymbol{A} = \begin{bmatrix} [A_1] & [C_1] & & & & & \\ [B_2] & [A_2] & [C_2] & & & & \\ & & \ddots & & & & \\ & & \ddots & & & & \\ & & \ddots & & & & \\ & & & [B_{J-1}] & [A_{J-1}] & [C_{J-1}] & \\ & & & & [B_J] & [A_J] & \end{bmatrix}$$

$$\boldsymbol{\delta} = \begin{bmatrix} [\delta_1] \\ [\delta_2] \\ \vdots \\ [\delta_{J-1}] \\ [\delta_J] \end{bmatrix} \text{ and } \boldsymbol{r} = \begin{bmatrix} [r_1] \\ [r_2] \\ \vdots \\ [r_{J-1}] \\ [r_J] \end{bmatrix}.$$

The elements of the matrices are as follows:

$$[A_1] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2}h_1 & 0 & 0 & 0 & -\frac{1}{2}h_1 & 0 & 0 \\ 0 & -\frac{1}{2}h_1 & 0 & 0 & 0 & -\frac{1}{2}h_1 & 0 \\ 0 & 0 & -\frac{1}{2}h_1 & 0 & 0 & 0 & -\frac{1}{2}h_1 \\ (a_2)_1 & (a_8)_1 & 0 & (a_3)_1 & (a_1)_1 & (a_7)_1 & 0 \\ (b_{10})_1 & (b_2)_1 & 0 & (b_3)_1 & (b_9)_1 & (b_1)_1 & 0 \\ 0 & 0 & (c_2)_1 & (c_3)_1 & 0 & 0 & (c_1)_1 \end{bmatrix} \tag{53}$$

$2 \le j \le J$,

$$[A_J] = \begin{bmatrix} -\frac{1}{2}h_J & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -\frac{1}{2}h_J & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -\frac{1}{2}h_J & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -\frac{1}{2}h_J \\ (a_6)_J & 0 & (a_{10})_J & (a_3)_J & (a_1)_J & (a_7)_J & 0 \\ (b_6)_J & (b_8)_J & 0 & (b_3)_J & (b_9)_J & (b_1)_J & 0 \\ (c_6)_J & 0 & (c_8)_J & (c_3)_J & 0 & 0 & (c_1)_J \end{bmatrix} \tag{54}$$

$$[B_J] = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}h_J & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h_J & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h_J \\ 0 & 0 & 0 & (a_4)_J & (a_2)_J & (a_8)_J & 0 \\ 0 & 0 & 0 & (b_4)_J & (b_{10})_J & (b_2)_J & 0 \\ 0 & 0 & 0 & (c_4)_J & 0 & 0 & (c_2)_J \end{bmatrix} \tag{55}$$

$1 \le j \le J-1$,

$$[C_J] = \begin{bmatrix} -\frac{1}{2}h_J & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ (a_5)_J & 0 & (a_9)_J & 0 & 0 & 0 & 0 \\ (b_5)_J & (b_7)_J & 0 & 0 & 0 & 0 & 0 \\ (c_5)_J & 0 & (c_7)_J & 0 & 0 & 0 & 0 \end{bmatrix} \tag{56}$$

$2 \le j \le J$,

$$[\delta_1] = \begin{bmatrix} \delta q_0 \\ \delta n_0 \\ \delta s_0 \\ \delta f_1 \\ \delta q_1 \\ \delta n_1 \\ \delta t_1 \end{bmatrix}, \qquad [\delta_j] = \begin{bmatrix} \delta q_{j-1} \\ \delta n_{j-1} \\ \delta s_{j-1} \\ \delta f_1 \\ \delta q_j \\ \delta n_j \\ \delta t_j \end{bmatrix} \tag{57}$$

and $1 \le j \le J$,

$$[r_j] = \begin{bmatrix} (r_1)_{j-1/2} \\ (r_2)_{j-1/2} \\ (r_3)_{j-1/2} \\ (r_4)_{j-1/2} \\ (r_5)_{j-1/2} \\ (r_6)_{j-1/2} \\ (r_7)_{j-1/2} \end{bmatrix}. \tag{58}$$

To solve Eq. (52), we assume that $\boldsymbol{A}$ is nonsingular and it can be factorized as

$$\boldsymbol{A} = \boldsymbol{LU}, \tag{59}$$

where

$$\boldsymbol{L} = \begin{bmatrix} [\alpha_1] & & & & \\ [\beta_2] & [\alpha_2] & & & \\ & & \ddots & & \\ & & \ddots & [\alpha_{J-1}] & \\ & & & [\beta_J] & [\alpha_J] \end{bmatrix}$$

and

$$\boldsymbol{U} = \begin{bmatrix} [I_1] & [\Gamma_1] & & & \\ & [I_1] & & & \\ & & \ddots & & \\ & & \ddots & [I_1] & [\Gamma_{J-1}] \\ & & & & [I_1] \end{bmatrix}$$

where $[I]$ is a 7×7 identity matrix, while $[\alpha_i]$ and $[\Gamma_i]$ are 7×7 matrices in which elements are determined by the following equations:

$$[\alpha_i] = [A_1], \tag{60}$$

$$[A_1][\Gamma_1] = [C_1], \tag{61}$$

$$[\alpha_i] = [A_1] - [B_j][\Gamma_{J-1}], \quad j = 2,3,...,J, \tag{62}$$

$$[\alpha_i][\Gamma_j] = [C_j], \quad j = 2,3,...,J-1. \tag{63}$$

Substituting Eq. (59) into Eq. (52), we obtain

$$\boldsymbol{LU\delta = r}\,. \tag{64}$$

If we define

$$\boldsymbol{U\delta = W}\,, \tag{65}$$

Eq. (64) becomes

$$\boldsymbol{LW = r}\,, \tag{66}$$

where

$$\boldsymbol{W} = \begin{bmatrix} [W_1] \\ [W_2] \\ \vdots \\ [W_{J-1}] \\ [W_J] \end{bmatrix},$$

and $[W_j]$ are 7 × 1 column matrices. The elements of $\boldsymbol{W}$ can be determined from Eq. (65) by the following relations:

$$[\alpha_1][W_1] = [r_1], \tag{67}$$

$$[\alpha_i][W_j] = [r_j] - [B_j][W_{j-1}], \quad 2 \le j \le J. \tag{68}$$

When the elements of $\boldsymbol{W}$ have been found, Eq. (65) gives the solution for $\boldsymbol{\delta}$ in which the elements are found from the following relations:

$$[\delta_J] = [W_J], \tag{69}$$

$$[\delta_i] = [W_j] - [\Gamma_j][\delta_{j+1}], \quad 1 \le j \le J-1. \tag{70}$$

Once the elements of $\boldsymbol{\delta}$ are found, Eqs. (42)-(48) can be used to find the $(k+1)$th interation in Eq. (41). These calculations are repeated until the convergence criterion is satisfied. In laminar boundary layer calculation, the wall shear stress parameter $q(0)$ is commonly used as the convergence criterion (Cebeci and Bradshaw [22]). This is probably because in boundary layer calculations, it is found that the greatest error usually appears in the wall shear stress parameter. Thus, this convergence criterion is used in the present study. Calculations are stopped when

$$\left|\delta q_0^{(k)}\right| < \in_1, \tag{71}$$

where $\in_1$ is a small prescribed value. In this study, $\in_1 = 0.00001$ is used, which gives about four decimal places accuracy for most of the predicted quantities as suggested by Bachok and Ishak [25,27] and Ali et al. [28,29].

The present method has a second-order accuracy, unconditionally stable and is easy to be programmed, thus making it highly attractive for production use. The only disadvantage is the large amount of once-and-for-all algebra needed to write the difference equations and to set up their solutions.

IV. RESULTS AND DISCUSSION

The step size $\Delta\eta$ in η, and the position of the edge of the boundary-layer η_∞ have to be adjusted for difference values of the parameters to maintain the necessary accuracy. In this study, the values of $\Delta\eta$ between 0.001 and 0.1 were used, depending on the values of the parameters considered, in order that the numerical values obtained are mesh independent, at least to four decimal places. However, a uniform grid of $\Delta\eta = 0.01$ was found to be satisfactory for a convergence criterion of 10^{-5} which gives accuracy to four decimal places, in nearly all cases. On the other hand, the boundary-layer thickness η_∞ between 5 and 50 was chosen where the infinity boundary condition is achieved. The results are given to carry out a parametric study showing the influences of the unsteadiness parameter S, material parameter K and Prandtl number Pr. For validation of the numerical method used in this study, the case when $S = 0$ (steady-state flow) has also been considered and the results are compared with those of Elbashbeshy [11], as well as the series solution given by Eq.

(16). The quantitative comparison is shown in Table 1 and it is found to be in a very good agreement.

The velocity profiles for various values of S and K are presented in Figs. 1 and 2. Figure 2 shows that the velocity gradient at the surface is larger for larger values of S, which produces larger skin friction coefficient $\left|f''(0)\right|$. We note that the parameter Pr have no influence on the flow field, which is clear from Eqs. (9)-(11). It is evident from Figure 2 that the boundary layer thickness increases with K. The velocity gradient at the surface $\left|f''(0)\right|$ decreases as K increases. Thus, micropolar fluids show drag reduction compared to viscous fluids.

Figures 3-6 show the temperature profiles for selected values of parameters. The temperature profiles are found to subside monotonously to zero as η increases. These curves represent the physically realistic case. As can been seen from Figs. 3-6, the surface temperature $\theta(0)$ decreases with increasing S, K and Pr. Thus, the local Nusselt number $1/\theta(0)$, which represent the heat transfer rate at the surface increases when S, K or Pr increases. Figure 7 shows that the surface temperature $\theta(0)$ decreases with increasing values of K. Thus, the heat transfer rate at the surface $1/\theta(0)$ is higher for a micropolar fluid ($K > 0$) compared to a Newtonian fluid ($K = 0$). On the other hand, for a fixed value of K, the surface temperature $\theta(0)$ decreases when Pr is increased, i.e. the heat transfer rate at the surface $1/\theta(0)$ increases with Pr. This is because the higher Prandtl number fluid has a lower thermal conductivity (or a higher viscosity) which results in thinner thermal boundary layer and hence, higher heat transfer rate at the surface (see Fig. 5).

The effect of m on the angular velocity, when the other parameters are fixed to unity is presented in Fig. 8. As expected, the microrotation at the surface $h(0)$ is more dominant for larger values of m. Finally, Figs. 1-6 show that the far field boundary conditions (12) are satisfied asymptotically, thus supporting the numerical results obtained.

TABLE 1 Variations of $\theta(0)$ for different values of S and Pr

S	Pr	Elbashbeshy [11]	Eq. (16)	Numerical results
0	0.72	1.2253	1.236657472	1.2367
	1	1.0000	1.000000000	1.0000
1	0.72			0.9116
	1			0.8591

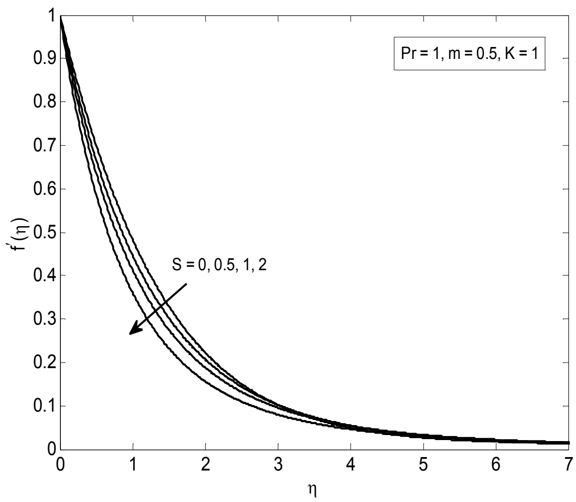

FIG. 1 Velocity profiles $f'(\eta)$ for some values of S when $Pr=1,\ m=0.5$ and $K=1$

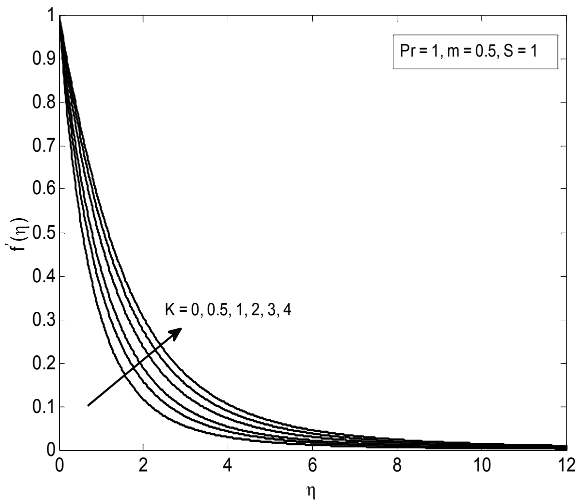

FIG. 2 Velocity profiles $f'(\eta)$ for some values of K when $Pr=1,\ m=0.5$ and $S=1$

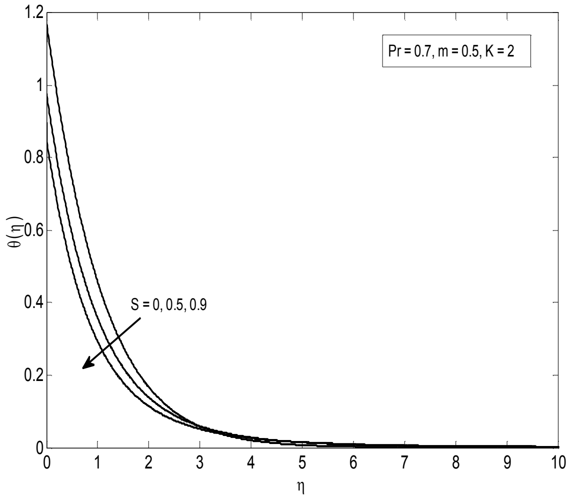

FIG. 3 Temperature profiles $\theta(\eta)$ for some values of S when $Pr = 0.7$, $m = 0.5$ and $K = 2$

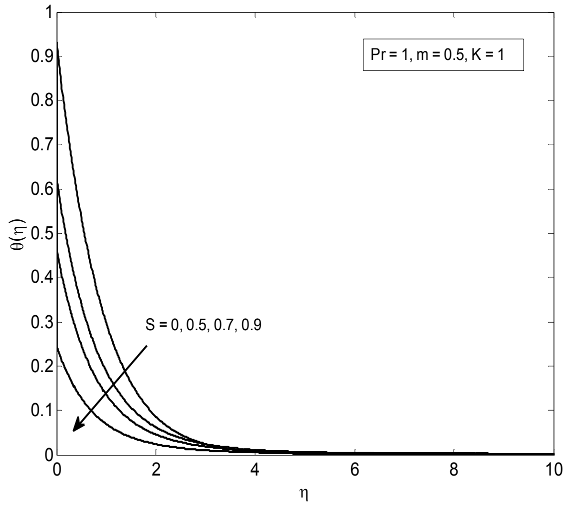

FIG. 4 Temperature profiles $\theta(\eta)$ for some values of S when $\Pr = 1$, $m = 0.5$ and $K = 1$

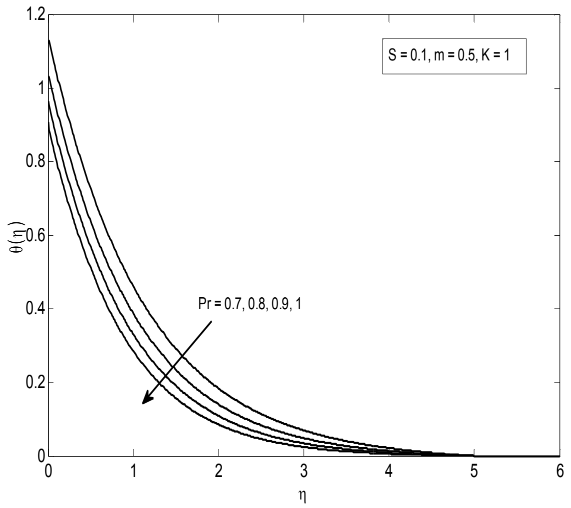

FIG. 5 Temperature profiles $\theta(\eta)$ for some values of Pr when $S = 0.1$, $m = 0.5$ and $K = 1$

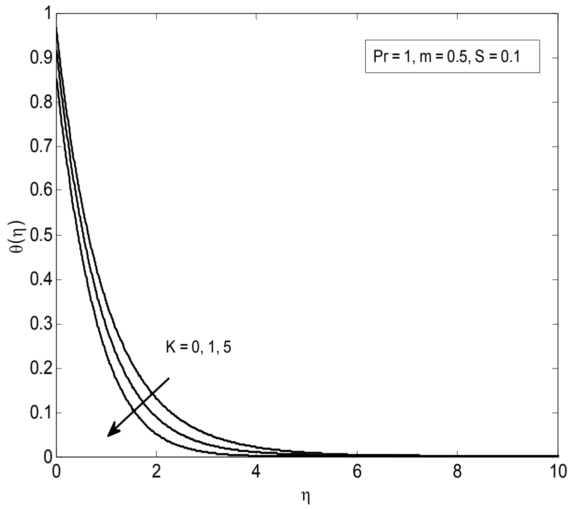

FIG. 6 Temperature profiles $\theta(\eta)$ for some values of K when $\Pr = 1$, $m = 0.5$ and $S = 0.1$

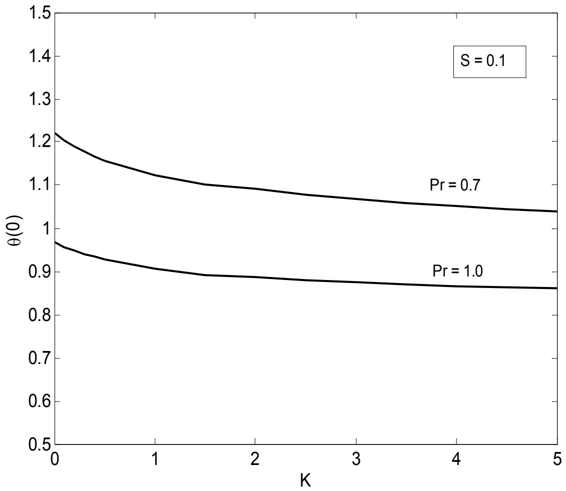

FIG. 7 Variations of $\theta(0)$ with K for $Pr = 0.7$ and $Pr = 1.0$ when $S = 0.1$

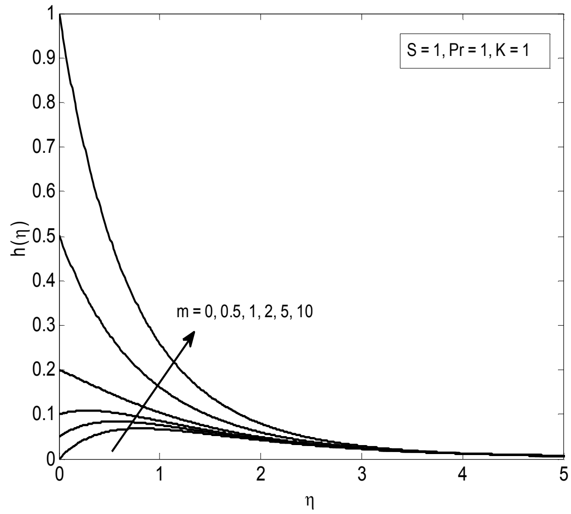

FIG. 8 Angular velocity profiles $h(\eta)$ for some values of m when $S = 1$, $\Pr = 1$ and $K = 1$

V. CONCLUSIONS

We have theoretically studied the similarity solutions of the unsteady boundary layer flow and heat transfer due to a stretching surface. A new similarity solution has been devised, which transform the time-dependent governing equations to ordinary differential equations. We discussed the effects of the governing parameters *S*, *K* and Pr on the fluid flow and heat transfer characteristics. The numerical results compared very well with previously reported cases, as well as the series solution for the steady-state flow. We found that the heat transfer rate at the surface $1/\theta(0)$ increases with *S*, *K* and Pr. Further, the heat transfer rate at the surface is higher for a micropolar fluid compared to a Newtonian fluid.

REFERENCES

[1] K. Abdell and F. Magpantay, "Approximate analytic solutions for mixed and forced convection heat transfer from an unsteady no-uniform flow past a rotating cylinder," *WSEAS Transactions on Heat and Mass Transfer,* vol. 2, pp. 6 – 16, 2007.

[2] T. Al-Azab, " Unsteady mixed convection heat and mass transfer past an infinite porous plate which thermophoresis effect, " *WSEAS Transactions on Heat and Mass Transfer*, vol. 4, pp. 23-33, 2009.

[3] L.J. Crane, "Flow past a stretching plate," *J. Appl. Math. Phys. (ZAMP),* vol. 21, pp. 645 – 647, 1970.

[4] P. S. Gupta and A. S. Gupta, " Heat and mass transfer on a stretching sheet with suction or blowing, " *Can. J. Chem. Eng*, vol. 55, pp. 744-746, 1977.

[5] L. J. Grubka and K. M. Bobba, "Heat transfer characteristics of a continuous stretching surface with variable temperature," *ASME J. Heat Transfer*, vol. 107, pp. 248-250, 1985.

[6] C. Y. Wang, " The three-dimensional flow due to a stretching surface," *Phys. Fluids*, vol. 27, pp. 1915-1917, 1984.

[7] C. H. Chen, "Laminar mixed convection adjacent to vertical, continuously stretching sheets," *Heat and Mass Transfer*, vol. 33, pp. 471-476, 1998.

[8] E. Magyari and B. Keller, " Heat and mass transfer in the boundary layers on an exponentially stretching continuous surface," *J. Phys. D: Appl. Phys,* vol. 32, pp. 577-585, 1999.

[9] M. I. Char and C. K. Chen, " Temperature field in non-Newtonian flow over a stretching plate with variable heat flux," *Int. J. Heat Mass Transfer,* vol. 31, pp. 917-921, 1988.

[10] C. R. Lin and C. K. Chen, "Exact solution of heat transfer from stretching surface with variable heat flux," *Heat Mass Transfer*, vol. 33, pp. 477-480, 1998.

[11] E. M. A. Elbashbeshy, "Heat transfer over a stretching surface with variable surface heat flux," J. Phys. D: Appl. Phys. Vol. 31, pp. 1951-1954, 1998.

[12] A. Ishak, R. Nazar and I. Pop, " Heat transfer over a stretching surface with variable heat flux in micropolar fluids," *Phys. Lett. A*, vol. 372, pp. 559-561, 2008.

[13] C. D. S. Devi, H. S. Takhar and G. Nath, "Unsteady mixed convection flow in stagnation region adjacent to a vertical surface," *Heat Mass Transfer*, vol. 26, pp. 71-79, 1991.

[14] H. I. Andersson, J. B. Aarseth and B. S. Dandapat, "Heat transfer in a liquid film on an unsteady stretching surface," *Int. J. Heat Mass Transfer*, vol. 43, pp. 69-74, 2000.

[15] R. Nazar, N. Amin and I. Pop, " Unsteady boundary layer flow due to stretching surface in a rotating fluid," *Mech. Res. Commun*., vol. 31, pp. 121-128, 2004.

[16] A. Ishak, R. Nazar an, I. Pop, " Heat transfer over an unsteady stretching permeable surface with prescribed wall temperature," *Nonlinear Anal. RWA*, vol. 10, pp. 2909-2913, 2009.

[17] J.C. Williams III and T.B. Rhyne, " Boundary layer development on a wedge impulsively set into motion," *SIAM J. Appl. Math*., vol. 38, pp. 215-224, 1980.

[18] A. Ishak, R. Nazar and I. Pop," Mixed convection stagnation point flow of a micropolar fluid towards a stretching sheet," *Meccanica*, vol. 43, pp. 411-418, 2008.

[19] G. Ahmadi, "Self-similar solution of incompressible micropolar boundary layer flow over a semi-infinite plate," *Int. J. Eng. Sci.,* vol. 14, pp. 639-646, 1976.

[20] A. Ishak, R. Nazar and I. Pop, "Moving wedge and flat plate in a micropolar fluid," *Int. J. Eng. Sci*., vol. 44, pp. 1225-1236, 2006.

[21] M. Abramowitz and I.A. Stegun, "Handbook of Mathematical Functions", Dover, New York, 1965.

[22] T. Cebeci and P. Bradshaw, " Physical and Computational Aspects of Convective Heat Transfer", Springer, New York, 1988.

[23] N. Bachok, A. Ishak and I. Pop," Mixed convection boundary layer flow near the stagnation point on a vertical surface embedded in a porous medium with anisotropy effect," *Trans. Porous Med*, vol. 82, pp. 363-373, 2010.

[24] N. Bachok, A. Ishak and I. Pop," Boundary-layer flow of nanofluids over a moving surface in a flowing fluid," *Int. J. of Thermal Sciences*, vol. 49, pp. 1663-1668, 2010.

[25] N. Bachok and A. Ishak," The effects of suction and injection on a moving flat plate in a parallel stream with prescribed surface heat flux," *WSEAS Transactions on Heat and Mass Transfer*, vol. 5, pp. 73-82, 2010.

[26] N. Bachok and A. Ishak," Flow and heat transfer over a stretching cylinder with prescribed surface heat flux," *Malaysian Journal of Mathematical Sciences*, vol. 4, pp. 159-169, 2010.

[27] N. Bachok and A. Ishak," Flow and heat transfer characteristic on a moving flat plate in a parallel stream with prescribed surface heat flux," *Proceedings of the 9th WSEAS Int. Conf. on Appl of Comp Eng*, pp. 115-118, 2010.

[28] F. M. Ali, R. Nazar and N. M. Arifin," MHD viscous flow and heat transfer due to a permeable shrinking sheet with prescribed surface heat flux," *Proceedings of the 9th WSEAS Int. Conf. on Appl of Comp Eng*, pp. 260-263, 2010.

[29] F. M. Ali, R. Nazar and N. M. Arifin," MHD viscous flow and heat transfer induced by a permeable shrinking sheet with prescribed surface heat flux," *WSEAS Transactions on Mathematics*, pp. 365-375, 2010.