Solution of Iterative Ordinary Differential Equation by Numerical Integration Method

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Abstract—In [1], A. Pelczar introduced and proved the existence and uniqueness of the second order iterative ordinary differential equations. The proof of the existence and uniqueness theorem of the general equation of iterative ordinary differential equation was given by M. Podisuk in [2]. In [3], M. Podisuk introduced and proved the existence and uniqueness of the simple iterative ordinary differential equations. In [4], M. Podisuk and W. Sanprasert introduced the integration method for finding the numerical solution of the initial value problem of ordinary differential equation with the help of Taylor series expansion. This integration method gives the way of solving for the numerical solution of the iterative ordinary differential equation. However the method of finding the analytical solution of the iterative ordinary differential equation is not known.

Keywords—Iterative-ordinary-differential-equation Integration-method Initial-value-problem Taylor-series-expansion Gauss-Legendre-Quadrature-formula.

I. INTRODUCTION

The general form of the iterative ordinary differential equation of order m is in the form

$$y'(x) = f(x, y(x), y^2(x), \dots, y^m(x)), \quad x \in [0,a] \tag{1}$$

with the initial condition

$$y(0) = c \tag{2}$$

where

$$y^2(x) = y(y(x)),$$
$$y^3(x) = y(y^2(x)),$$
$$\dots$$
$$y^m(x) = y(y^{m-1}(x)).$$

The simple iterative ordinary differential equation of order m is in the form

$$y'(x) = y^m(x), \quad x \in [0,\infty) \tag{3}$$

with the initial condition

$$y(0) = c \neq 0. \tag{4}$$

In this paper, we will use the integration method to find the numerical solution of the equation of the type of equations (1)-(2) and (3)-(4).

II. FORMULATION

The numerical formula for finding the numerical solution of the equations

$$y'(x) = f(x, y), \quad x \in [a,b] \tag{5}$$

with the initial condition

$$y(a) = c \tag{6}$$

which is given in [4] is in the form

$$y(x_{m+1}) = y_{m+1} = y(x_m) + \int_{x_m}^{x_{m+1}} f(x)dx \tag{7}$$

and

$$y_{m+1} = y(x_m) + h\int_0^1 f(x_m + hs)ds \qquad (8)$$

and

$$y_{m+1} \cong y_m + \frac{h}{2}\sum_{k=1}^{n} A_k f(x_m + hs_k, y(x_m + hs_k)) \qquad (9)$$

where the values A_k and s_k are the values of weights and points of the Gauss-Legendre Quadrature formula.

The value of $y(x_m + hs_k)$ is obtained by the Taylor series expansion that is

$$y(x_m + hs_k) \cong y(x_m) + hs_k y'(x_m) + \frac{1}{2}h^2 s_k^2 y''(x_m) + ... + \frac{1}{n!}h^n s_k^n y^n(x_m) \qquad (10)$$

In this paper, we will use n=3 and we will solve for the value of $y(y(x_m))$ from the formula

$$y(x_m + h) = y(x_m) + hy'(x_m), \qquad (11)$$

We will solve for the value of $y(y(y(x_m)))$ from the formula

$$y(x_m + h) = y(x_m) + hsy'(x_m) + \frac{1}{2}h^2 s^2 y''(x_m), \qquad (12)$$

and we solve for the value of $y(y(y(y(x_m))))$ from the formula

$$y(x_m + h) = y(x_m) + hsy'(x_m) + \frac{1}{2}h^2 s^2 y''(x_m) + \frac{1}{6}h^3 s^3 y'''(x_m). \qquad (13)$$

III. EXAMPLE

The following examples will illustrate our idea of finding numerical solution of the iterative ordinary differential equations. All examples use Gauss-Legendre Quadrature formula of 1 point, 2 points, 3 points, 4 points, 5 points and 6 points.

Example1.

Given $y'(x) = \frac{x}{9} - y(y(x)),$

$x \in [0,1]$

and

$$y(0) = \frac{1}{2}.$$

Find the numerical value of $y(x)$ at x = 0.0001, x = 0.001, x = 0.01, x = 0.1 and x = 1. The analytical solution of the above equations is $y(x) = \frac{1}{2} - \frac{x}{3}$.

We have $$y'(x) = \frac{x}{9} - y(y(x)) \qquad (14)$$

thus

$$y''(x) = \frac{1}{9} - y'(x)\left(\frac{1}{9}y(x) - y(y(y(x)))\right) \qquad (15)$$

and

$$y'''(x) = \left(y(y(y(x))) - \frac{1}{9}y(x)\right)y''(x) - y'(x) \left(\frac{x}{81} - \frac{1}{9}y(y(x)) - y'(x)y''(x) \left(\frac{1}{9}y(y(x)) - y(y(y(y(x))))\right)\right). \qquad (16)$$

We may approximate the value of $y(y(x_m))$ by letting $p = y(x_m) - x_m$ then $y(x_m) = x_m + p$ and by the formula (14) we obtain

$$y(x_m + p) \cong y(x_m) + py'(x_m) = y_m + \frac{1}{9}px_m - py(y(x_m))$$

then

$$y(y(x_m)) \cong y(x_m) + \frac{1}{9}px_m - py(y(x_m))$$

thus

$$y(y(x_m)) \cong \frac{y(x_m) + \frac{1}{9}px_m}{1+p}.$$

We may approximate the value of $y(y(y(x_m)))$ by letting

$$q = y(y(x_m)) - x_m$$

then

$y(y(x_m)) = x_m + q$ and by the formula (15), we obtain

$$y(x_m + q) \cong y(x_m) + qy'(x_m) + 0.5q^2 y''(x_m)$$

then

$$y(y(y(x_m))) \cong y(x_m) + qy'(x_m) + 0.5q^2 \left(\frac{1}{9} - \frac{1}{9}y(x_m)y'(x_m - y(y(y(x_m)))\right)$$

thus

$$y(y(y(x_m))) \equiv \frac{y(x_m) + qy'(x_m) + \frac{0.5}{9} q^2 \left(1 - y(x_m) y'(x_m)\right)}{1 + 0.5q^2}.$$

We may approximate the value of $y(y(y(y(x_m))))$ by letting

$$r = y(y(y(x_m))) - x_m$$

then

$$y(y(y(x_m))) = x_m + r$$

and by the formula (16) we obtain

$$y(x_m + r) \cong y(x_m) + ry'(x_m) + \frac{1}{2} r^2 y''(x_m) + \frac{1}{6} r^3 y'''(x_m)$$

then

$$\begin{aligned} y(y(y(y(x_m)))) &\cong y(x_m) + ry'(x_m) \\ &+ 0.5r^2 y''(x_m) + \\ &+ \frac{1}{6} r^3 \left(y(y(y(x_m) - \frac{1}{9} y(x_m) \right) y''(x_m) \\ &- y'(x_m) \begin{pmatrix} \frac{x}{81} - \frac{y(y(x_m))}{9} - y'(x_m) y''(x_m) \\ \left(\frac{y(y(x_m))}{9} - y(y(y(y(x)))) \right) \end{pmatrix}. \end{aligned}$$

Thus

$$y(y(y(y(x_m)))) \equiv \frac{A}{1 + \frac{1}{6} r^3 \left(y'(x_m)\right)^2 y''(x_m)}$$

where

$$\begin{aligned} A &= y(x_m) + ry'(x_m) + \frac{1}{2} r^2 y''(x_m) + \\ &\frac{1}{6} r^3 y(y(y(x_m))) - \frac{1}{9} y(x_m) y''(x_m) \quad . \\ &- \frac{1}{9} y'(x_m) \begin{pmatrix} \frac{x}{9} - y(y(x_m)) - \\ y'(x_m) y''(x_m) y(y(x_m)) \end{pmatrix} \end{aligned}$$

We will do the same manner for the second example.

Now we have the value $y(x_m)$ and the approximated values of

$y(y(x_m))$, $y(y(y(x_m)))$ and $y(y(y(y(x_m))))$

then we may find the values of

$$y\left(\frac{h}{2} s_k + \frac{x_m + x_{m+1}}{2}\right), \; k = 1,2,3,...,n$$

and put in the formula (9) to obtain the value of y_{m+1}.

We will do the same manner for the second example.

The results of this example are as follow;

One point formula

$h = 0.1$

$y(0.1) = 0.46687626822$

with the absolute error

0.0002096015528

$y(1.0) = 0.16444052220$

with the absolute error

0.0022261444647

$h = 0.01$

$y(0.01) = 0.49667111896$

with the absolute error

0.0000044522967

$y(1.0) = 0.16628426153$

with the absolute error

0.0003824051323

$h = 0.001$

$y(0.001) = 0.49966694271$

with the absolute error

0.0000002760457

$y(1.0) = 0.16650678158$

with the absolute error

0.0001598850336

$h = 0.0001$

$y(0.0001) = 0.49996669258$

with the absolute error 0.0000000259088

$y(1.0) = 0.9999878011$

with the absolute error 0.0001372636846.

Two point formula

$h = 0.1$

$y(0.1) = 0.46687490587$

with the absolute error

0.000208039201

$y(1.0) = 0.16437453598$

with the absolute error

0.002292130689

$h = 0.01$

$y(0.01) = 0.49667111744$

with the absolute error
0.000004450771
$y(1.0) = 0.16628346569$
with the absolute error
0.000383200972
$h = 0.001$
$y(0.001) = 0.49966694271$
with the absolute error
0.000000276044
$y(1.0) = 0.16650677348$
with the absolute error 0.000159893137
$h = 0.0001$
$y(0.0001) = 0.4999666926$
with the absolute error
0.000000025909
$y(1.0) = 0.16652940241$
with the absolute error
0.000137263765.

Three point formula

$h = 0.1$
$y(0.1) = 0.46687470584$
with the absolute error
0.000208039141
$y(1.0) = 0.16437448831$
with the absolute error
0.002292178352
$h = 0.01$
$y(0.01) = 0.49667111744$
with the absolute error
0.0000044507719
$y(1.0) = 0.16628346568$
with the absolute error
0.0003832009809
$h = 0.001$
$y(0.001) = 0.49966694271$
with the absolute error
0.0000002760439
$y(1.0) = 0.16650677348$
with the absolute error
0.0001598931367
$h = 0.0001$
$y(0.0001) = 0.4999666926$
with the absolute error
0.0000000259088
$y(1.0) = 0.16652940221$
with the absolute error
0.0001372637657.

Four point formula

$h = 0.1$
$y(0.1) = 0.46687470581$
with the absolute error
0.0002080391414
$y(1.0) = 0.16437448826$
with the absolute error
0.0022921784021
$h = 0.01$
$y(0.01) = 0.49667111744$
with the absolute error
0.0000044507719
$y(1.0) = 0.16628346568$
with the absolute error
0.0003832009809
$h = 0.001$
$y(0.001) = 0.49966694271$
with the absolute error
0.0000002760439
$y(1.0) = 0.16650677348$
with the absolute error
0.0001598931367
$h = 0.0001$
$y(0.0001) = 0.49996669258$
with the absolute error
0.0000000259088
$y(1.0) = 0.16652940241$
with the absolute error
0.0001372637655.

Five point formula

$h = 0.1$
$y(0.1) = 0.46687470581$
with the absolute error
0.0002080391414
$y(1.0) = 0.16437448826$
with the absolute error
0.0022921784021
$h = 0.01$
$y(0.01) = 0.49667111744$
with the absolute error
0.0000044507719
$y(1.0) = 0.16628346568$

with the absolute error
0.0003832000809
$h = 0.001$
$y(0.001) = 0.49966694271$
with the absolute error
0.0000002760439
$y(1.0) = 0.16650677348$
with the absolute error
0.0001598931367
$h = 0.0001$
$y(0.0001) = 0.4999666926$
with the absolute error
0.0000000259088
$y(1.0) = 0.16652940241$
with the absolute error
0.0001372637657.

Six point formula
$h = 0.1$
$y(0.1) = 0.46687470581$
with the absolute error 0.0002080391414
$y(1.0) = 0.16437448826$
with the absolute error 0.0022921784018
$h = 0.01$
$y(0.01) = 0.49667111744$
with the absolute error 0.0000044507719
$y(1.0) = 0.16628346568$
with the absolute error 0.0003832009809
$h = 0.001$
$y(0.001) = 0.49966694271$
with the absolute error 0.0000002760439
$y(1.0) = 0.16650677348$
with the absolute error
0.00012192912
$h = 0.0001$
$y(0.0001) = 0.4999666926$
with the absolute error
0.0000000259088
$y(1.0) = 0.16652940241$
with the absolute error
0.0001372637657.

Since we know the values of $y(x_m)$, $y(y(x_m))$, $y(y(y(x_m)))$ and $y(y(y(y(x_m))))$ then we may compute the approximate value of $y_{m+1} \approx y(x_{m+1})$ by the Taylor series expansion and the results are as follow;
$h = 0.1$
$y(0.1) = 0.55517343868$
with the absolute error
0.0025879796458
$y(1.0) = 0.13662924835$
with the absolute error
0.0071515695763
$h = 0.01$
$y(0.01) = 0.50549845487$
with the absolute error
0.0004733713267
$y(1.0) = 0.14057690216$
with the absolute error
0.0466281073700
$h = 0.001$
$y(0.001) = 0.50054960146$
with the absolute error
0.0000493513771
$y(1.0) = 0.14089747107$
with the absolute error
0.0498338022350
$h = 0.0001$
$y(0.0001) = 0.5000549577$
with the absolute error
0.0000049551536
$y(1.0) = 0.14092836962$
with the absolute error
0.0501427860100.

We will do the same manner for the second example.

We can see that the Integral method give the better results than the Taylor series expansion.

Example 2.

Given
$$y'(x) = y(y(x)),$$
$$x \in [0, \infty)$$
and
$$y(0) = 0.25.$$
Find the values of
$y(0.0001), y(0.001), y(0.01), y(0.1)$ and $y(1)$.

The solution of the given problem is not known.

We have

$$y'(x) = y(y(x))$$

thus

$$y''(x) = y(y(x)) \cdot yy(y(x))$$

and

$$y'''(x) = y(y(x)) \cdot yy(y(x)) \cdot y(y(y(y(x)))).$$

Now let

$$p = y(x_m) - x_m$$

and

$$x_m + p = y(x_m)$$

thus

$$y(x_m + p) = y(y(x_m))$$

then

$$\begin{aligned} y(y(x_m)) &= y(x_m + p) \\ &= y(x_m) + py'(x_m) \\ &= y(x_m) + pyy((x_m)) \end{aligned}$$

thus

$$y(y(x_m)) = \frac{y(x_m)}{1-p}.$$

Now let

$$q = y(y(x_m)) - x_m$$

and

$$x_m + q = y(y(x_m))$$

then

$$y(x_m + q) + y(y(y(x_m)))$$

thus

$$y(y(y(x_m))) \cong y(x_m) + qy'(x_m) + \frac{q^2}{2} y''(x_m)$$

and

$$y(y(y(x_m))) \cong y(x_m) + qyy(x_m) + \frac{q^2}{2} y(y(x_m)) + \frac{q^3}{6} y(y(y(x_m)))$$

and

$$y(y(y(x_m))) \cong \frac{y(x_m) + qy(y(x_m))}{1 - \frac{1}{2} q^2 y(y(x_m))}.$$

Now let

$$r = y(y(y(x_m))) - x_m$$

and

$$x_m + r = y(y(y(x_m)))$$

then

$$y(x_m + r) = y(y(y(y(x_m))))$$

thus

$$y(y(y(y(x_m)))) \cong y(x_m) + ry'(x_m) + \frac{1}{2} r^2 y''(x_m) + \frac{1}{6} r^3 y'''(x_m)$$

and

$$\begin{aligned} y(y(y(y(x_m)))) &\cong y(x_m) + ry''(x_m) \\ &+ \frac{1}{2} r^2 y(y(x_m)) y(y(y(x_m))) \\ &+ \frac{1}{6} r^3 y(y(x_m)) y(y(y(x_m))) y(y(y(y(x_m)))) \end{aligned}$$

then

$$(y(y(y(y(x_m)))) - \frac{1}{6} r^3 y(y(x_m)) y(y(y(x_m))) y(y(y(y(x_m))))) \cong y(x_m) + ry''(x_m) + \frac{1}{2} r^2 y(y(x_m)) y(y(y(x_m)))$$

thus

$$(y(y(y(y(x_m)))) = \frac{y(x_m) + ry(y(x_m)) + \frac{1}{2} r^2 y(y(x_m)) y(y(y(x_m)))}{1 - \frac{1}{6} r^3 y(y(x_m)) y(y(y(x_m)))}.$$

The results of this example are as follow;

One point formula

$h = 0.1$
 $y(0.1) = 0.28457205201$
 $y(1.0) = 0.66510476199$
$h = 0.01$
 $y(0.01) = 0.25338137005$
 $y(1.0) = 0.66483699992$
$h = 0.001$
 $y(0.001) = 0.2503373321$
 $y(1.0) = 0.66487250234$
$h = 0.0001$
 $y(0.0001) = 0.250033773321$
 $y(1.0) = 0.66487670555$.

Two point formula

$h = 0.1$
 $y(0.1) = 0.28458261382$
 $y(1.0) = 0.66529248209$
$h = 0.01$
 $y(0.01) = 0.25338137965$
 $y(1.0) = 0.66483880910$
$h = 0.001$
 $y(0.001) = 0.25033740505$
 $y(1.0) = 0.66487252037$

$h = 0.0001$

$y(0.0001) = 0.25003321$

$y(1.0) = 0.66487670575$.

Three point formula

$h = 0.1$

$y(0.1) = 0.28458261955$

$y(1.0) = 0.66529253887$

$h = 0.01$

$y(0.01) = 0.25338137965$

$y(1.0) = 0.66483880910$

$h = 0.001$

$y(0.001) = 0.25033740505$

$y(1.0) = 0.66487252037$

$h = 0.0001$

$y(0.0001) = 0.25003373321$

$y(1.0) = 0.66487670575$.

Four point formula

$h = 0.1$

$y(0.1) = 0.28458261758$

$y(1.0) = 0.66529253888$

$h = 0.01$

$y(0.01) = 0.25338137965$

$y(1.0) = 0.66483880910$

$h = 0.001$

$y(0.001) = 0.250338137965$

$y(1.0) = 0.66487252037$

$h = 0.0001$

$y(0.0001) = 0.25033740505$

$y(1.0) = 0.66487670575$.

Five point formula

$h = 0.1$

$y(0.1) = 0.28458261758$

$y(1.0) = 0.66529253888$

$h = 0.01$

$y(0.01) = 0.25338137965$

$y(1.0) = 0.66483880010$

$h = 0.001$

$y(0.001) = 0.25033740505$

$y(1.0) = 0.66487670575$

$h = 0.0001$

$y(0.0001) = 0.25003373321$

$y(1.0) = 0.66487670575$.

Six point formula

$h = 0.1$

$y(0.1) = 0.28458261756$

$y(1.0) = 0.66529253888$

$h = 0.01$

$y(0.01) = 0.25338137965$

$y(1.0) = 0.66483880910$

$h = 0.001$

$y(0.001) = 0.25033740505$

$y(1.0) = 0.66487252037$

$h = 0.0001$

$y(0.0001) = 0.25003373321$

$y(1.0) = 0.66487670575$.

M. Podisuk, in [3], obtained the power series solution of this problem as follws;

$$
\begin{aligned}
y(x) &= 0.25 + 0.3358287413915x \\
&+ 0.05891459621489x^2 \\
&+ 0.00814643051611x^3 \\
&+ 0.00113355377363x^4 \\
&+ 1.13379302067348\times 10^{-4}x^5 \\
&+ 1.336315502857493\times 10^{-5}x^6 \\
&+ 1.440157500266146\times 10^{-6}x^7 \\
&+ 1.4868707169291\times 10^{-7}x^8 \\
&+ 2.175934559677333\times 10^{-8}x^9 \\
&+ 5.1010115114809001\times 10^{-9}x^{10} \\
&+ 1.280868547464546\times 10^{-9}x^{11} \\
&+ 2.8475642215\times 10^{-11}x^{12} \\
&+ 5.233923447461539\times 10^{-11}x^{13} \\
&+ 7.940553943571429\times 10^{-12}x^{14} \\
&+ 7.405461107333333\times 10^{-12}x^{14} \\
&+ 9.398457818750001\times 10^{-13}x^{15} \\
&+ 9.950508176470589\times 10^{-14}x^{16} \\
&+ 8.835303333333333\times 10^{-15}x^{17} \\
&+ 6.577757894736842\times 10^{-16}x^{18} \\
&+ 4.12905\times 10^{-17}x^{19} \\
&+ 2.185714285714286\times 10^{-18}x^{20} \\
&+ 1.0\times 10^{-19}x^{21} \\
&+ 2.608695652173914\times 10^{-21}x^{22} .
\end{aligned}
$$

When we use the above power series then we obtain the following results;

$$y(0.0001) = 0.25003358346$$
$$y(0.001) = 0.25033588766$$
$$y(0.01) = 0.25336418703$$
$$y(0.1) = 0.28418028103$$
$$y(1.0) = 0.65415168144\,.$$

We can see that the value of y(x) from the power series is less than the value of y(x) from the numerical computation. The reason is that the power series that we used is only of degree twenty two.

IV. CONCLUSION

The numerical results of integration method are satisfied. However if we improve the formulas (10), (11) and (12) of the integration method then we would obtain the better results and we may use other Gauss Quadrature formulas instead of the Gauss Legendre Quadrature formulas.. We strongly recommend the integration method with the help of the Taylor series expansion to approximate the numerical solution of the iterative ordinary differential equation.

V. REFERENCES

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