Credibility models for permanently updated estimates in insurance

Pacáková Viera

***Abstract*—**The article explains the possibilities of permanently updated estimates in insurance theory and practice and presents several examples of the derivation of the posterior distribution for certain estimation situations with given prior distributions. Article investigates the Bayesian estimators of the parameters of binomial, Poisson and normal distribution using quadratic loss function. The choice of the prior distribution specifies the models binomial/beta, Poisson/gamma and normal/normal. In insurance practice these models allow us permanently updated estimates of binomial probability, of the credibility premium, or credibility number of claims for short-term insurance contracts. The possibility to express the Bayesian estimators in the form of credibility formulas allows easy application of these models in insurance practice.

***Keywords*—**Bayesian estimation, Binomial/beta model, Credibility premium, Normal/normal model, Posterior distribution, Poisson/gamma model, Prior distribution.

I. Introduction

A typical feature of the insurance practice is the need to set premium at the beginning of the insurance contract. Number of occurrence of claims and the total claim amounts for insurance company in the future are the random variables. Their sufficiently, precise and reliable estimate is extremely important to determine the correct premium for next year in insurance company.

Credibility theory is a technique, or set of techniques, for calculating premiums for short term insurance contracts. The technique calculates a premium for a risk using two ingredients: past data from the risk itself and collateral data, i.e. data from other sources considered to be relevant. The essential features of a credibility premium are that it is a linear function of the past data from the risk itself and that it allows for the premium to be regularly updated as more data are collected in the future (Waters, 1994).

A credibility premium represents a compromise between the two above mentioned sources of information. The credibility formula for estimation of pure premium or claim frequency P_c in next year is:

$$P_c = Z\ P_r + (1 - Z)\ \mu \qquad (1)$$

where P_r is estimation based on own past data in insurance company, or risk, μ is estimation based on collateral data and Z is a number between zero and one, known as the credibility factor. Credibility factor Z is a measure of how much reliance the company is prepared to place on the data from the policy itself.

Credibility formula is often used in the form

$$P_c = Z\ \bar{x} + (1 - Z)\ \mu \qquad (2)$$

We will present Bayesian approach to credibility estimation by three important models for insurance practice.

Author is with the University of Pardubice, Faculty of Economics and Administration, Department of Mathematics and Quantitative Methods, Studentská 95, 532 10 Pardubice, Czech Republic; e-mail: viera.pacakova@upce.cz).

II. The Bayesian Inference

The Bayesian philosophy (1763) involves a completely different approach to statistical inference. Suppose $\mathbf{x} = (x_1, x_2, \dots x_n)$ is a random sample from a population specified by density function $f(x/\theta)$ and it is required to estimate parameter Θ.

The classical approach to point estimation treats parameters as something fixed but unknown. The essential difference in the Bayesian approach to inference is that parameters are treated as random variables and therefore they have probability distributions.

Prior information about Θ that we have before collection of any data is the prior distribution $f_\Theta(\theta)$, which is probability density function or probability mass function. The information about Θ provided by the sample data $\mathbf{x} = (x_1, x_2, \dots x_n)$ is contained in the likelihood $f(\mathbf{x}/\theta) = \prod_{i=1}^{n} f(x_i/\theta)$. Bayes theorem combines this information with the information contained in $f_\Theta(\theta)$ in the form

$$f_\Theta(\theta/\mathbf{x}) = \frac{f(\mathbf{x}/\theta)\ f(\theta)}{\int_\theta f(\mathbf{x}/\theta)\ f(\theta)\,\mathrm{d}\theta} \qquad (3)$$

that determines the posterior distribution $f_\Theta(\theta \mid \mathbf{x})$.

So after collecting appropriate data we determine the posterior distribution that is the basis of all inference concerning Θ.

Note that $f(\mathbf{x}) = \int_\theta f(\mathbf{x}/\theta)\ f(\theta)\,\mathrm{d}\theta$ does not involve Θ. It is just a constant needed to make it a proper density that integrates to unity. A useful way of expressing the posterior density is to use proportionality. We can write

$$f(\theta/\mathbf{x}) \propto f(\mathbf{x}/\theta)\, f(\theta) \tag{4}$$

or simply

posterior $\propto$ likelihood · prior.

The posterior distribution contains all available information about Θ and therefore should be used for making decisions, estimates or inferences.

The Bayesian approach to estimation states that we should always start with a prior distribution for unknown parameter, precise or vague according to the information available.

Note that we are referring to a density here implying that Θ is continuous. This concerns most applications because even when X is discrete, as in binomial or Poisson distributions, the parameters π or λ will vary in a continuous space $\langle 0; 1\rangle$ or $\langle 0, +\infty)$ respectively.

There may be some situations in which we need „non-informative" prior. For example if Θ is a binomial distributed and we have no prior information about Θ, the uniform distribution on interval $\langle 0; 1\rangle$ as a prior distribution would seem appropriate.

We often have prior information about parameters based on previous practice, respectively, estimates by experts. The values of these parameters reflect the subjective opinion of the decision maker, so Bayesian approach can be criticized as subjective.

III. The Bayesian Estimator

If we have found posterior distribution of an unknown parameter Θ, we need to answer the question how do we use the posterior distribution of Θ, given the sample data $\mathbf{x} = (x_1, x_2, \dots x_n)$, to obtain an estimator of Θ.

At first we must specify the loss function $g(\mathbf{x})$, which is a measure of the "loss" incurred when $g(\mathbf{x})$ is used as an estimator of Θ. We seek a loss function which is zero when the estimation is exactly correct, that is $g(\mathbf{x}) = \Theta$ and which increases as $g(\mathbf{x})$ gets father away from Θ.

There is one very commonly used loss function, called quadratic or squared loss. The quadratic loss is defined by

$$L(g(\mathbf{x});\theta) = [g(\mathbf{x}) - \theta]^2 \tag{5}$$

and it is related to mean square error from classical statistics.

We will show that the Bayesian estimator that arises by minimizing the expected quadratic loss is the mean of posterior distribution. So

$$E\,(L(g(\mathbf{x});\theta)) = \int [g(\mathbf{x}) - \theta]^2\ f(\theta/\mathbf{x})\,\mathrm{d}\theta$$

and

$$\frac{\partial\, E(L(g(\mathbf{x});\theta))}{\partial\, g(\mathbf{x})} = 2\int [g(\mathbf{x}) - \theta]\, f(\theta/\mathbf{x})\,\mathrm{d}\theta$$

equating to zero

$$g(\mathbf{x}) \int f(\theta/\mathbf{x})\,\mathrm{d}\theta = \int \theta\ f(\theta/\mathbf{x})\,\mathrm{d}\theta$$

Because of $\int f(\theta/\mathbf{x})\mathrm{d}\theta = 1$, we get

$$g(\mathbf{x}) = E(\theta/\mathbf{x}) \tag{6}$$

We will consider three important examples of derivation of the posterior distribution and the Bayesian estimators under the quadratic loss function for certain estimation situations with given prior distributions, important for insurance practice.

IV. The Poisson/gamma Model

Suppose we have to estimate the claim frequency for a risk and claim numbers have a Poisson distribution with parameter λ. We do not know the value of λ but before having any data from risk itself available, we assume that the prior distribution of λ is a gamma distribution G(α; β).

The claim frequency rate for a class of insurance business may lie anywhere between 0 and $+\infty$. An insurer with a large experience may quite accurately estimate the rate.

The gamma distribution may be convenient for representing uncertainty in a current estimate of the claim frequency rate. This distribution is over the whole positive range from 0 to $+\infty$, and the mean α/β can be set equal to the current best estimate. Uncertainty is represented by variance α/β^2 of the gamma distribution G(α; β).

Our objectives is to estimate the unknown parameter λ. Suppose we have n past observations $\mathbf{x} = (x_1, x_2, \dots x_n)$. The Bayesian estimate of λ, with respect to a quadratic loss function, given these data, is

$$\lambda_B = E(\lambda/\mathbf{x}) \tag{7}$$

that is the mean of the posterior density of λ.

By assumption the density function of the prior G(α; β) distribution is

$$f(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\ \lambda^{\alpha-1}\ e^{-\lambda\beta} = c_2\ e^{-\lambda\beta}\ \lambda^{\alpha-1} \tag{8}$$

The distribution of a number of claims is the Poisson with a fixed but unknown parameter λ, so the likelihood function has the expression:

$$f(\mathbf{x}/\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!}\, e^{-\lambda} = c_1\ e^{-\lambda n}\ \lambda^{\sum_{i=1}^{n} x_i} \tag{9}$$

By Bayes' theorem we get the posterior density of λ, given $\mathbf{x} = (x_1, x_2, \dots x_n)$, in the form

$$f(\lambda/\mathbf{x}) \propto e^{-\lambda n}\ \lambda^{\sum x_i} \cdot e^{-\lambda\beta}\ \lambda^{\alpha-1} = e^{-\lambda(\beta+n)}\ \lambda^{\alpha+\sum x_i - 1} \tag{10}$$

that is the gamma distribution with the new parameters

$$\alpha_1 = \alpha + \sum_{i=1}^{n} x_i \qquad \beta_1 = \beta + n \tag{11}$$

Thus the Bayesian estimator of λ using the quadratic loss is

$$\lambda_B = \frac{\alpha + \sum_{i=1}^{n} x_i}{\beta + n} = \frac{\alpha + n\bar{x}}{\beta + n} \qquad (12)$$

which can be rewritten as

$$\lambda_B = \frac{\alpha + n\bar{x}}{\beta + n} = \frac{n}{\beta + n} \cdot \bar{x} + \frac{\beta}{\beta + n} \cdot \frac{\alpha}{\beta}$$

If we put factor credibility as

$$Z = \frac{n}{\beta + n} \qquad (13)$$

then we get

$$\lambda_B = E\left(\lambda/\mathbf{x}\right) = Z\,\bar{x} + \left(1 - Z\right)\mu \qquad (14)$$

which is the credibility formula for updating claim frequency rates.

It can be seen from expression of credibility factor, since n is non-negative and β is positive, that Z is in the range zero to one and it is increasing function of n. If no past data from the risk itself are available, then $n = 0$ and Z = 0 too and the best estimate of λ is α/β, the mean of the prior gamma distribution. It can be seen that Z does not take the value one for any finite value of n.

The value of Z depends on the amount of data available for the risk n, and the collateral information through β, which reflect the variance α/β^2 of the prior distribution.

V. Application of Poisson/gamma Model

Credibility formula (14) allows easy application of Poisson/gamma model in insurance practice. We try to show it in this sample.

The annual number of claims resulting from motor third-party liability insurance in insurance company in the years 2005-2011 is given in Table I, column labeled as x_i. In Poisson/gamma model for claim numbers we have assumed our prior knowledge about the unknown parameter (annual claim rate) λ summarized by gamma distribution G$(\alpha; \beta)$ with parameters α = 8400 and β = 0,4.

The number of claims arising from the risk in each year has a Poisson distribution and we do not know the value of parameter λ. What we do know is a prior distribution of λ, which we will take to be gamma G(α; β) with parameters α = 8400 and β = 0,4. This distribution has mean $\mu = \alpha/\beta$ = 21 000. The actual numbers of claims arising each year from this risk are in the column named x_i of Table I.

Figure 1 shows the credibility factor in successive years. Calculation of credibility factors Z_i in each year i on the basis of previous year's data is an application of formula (13). We can see from this figure that the credibility factor increases with time. As time goes we collect more data from the risk itself and the higher the credibility factor should be because of the increasing reliability of the data in estimating the true but unknown expected number of claims for the risk. Mathematically the fact that Z increases with time for this particular model is simply because expression (13) is an increasing function of n for any positive value of β.

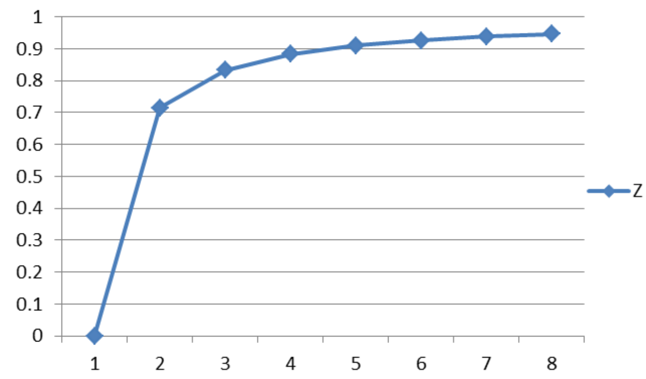

Fig. 1 Credibility factors of Poisson/gamma model in successive years

Last column denoted as λ_B contains values of Bayes estimators of annual claims numbers for each year based of past (i-1) observations by equation (14).

Table I Procedure to update Bayes estimator of λ

Year i	x_i	$\bar{x}$	Z_i	λ_B
2005	22954	-	0	21000
2006	23166	24954	0,7143	23824
2007	22402	23060	0,8333	22717
2008	19656	22841	0,8824	22624
2009	20142	22045	0,9091	21950
2010	22618	21664	0,9259	21615
2011	21544	21823	0,9375	21772
2012	-	21783	0,9459	21741

Source: Own calculation

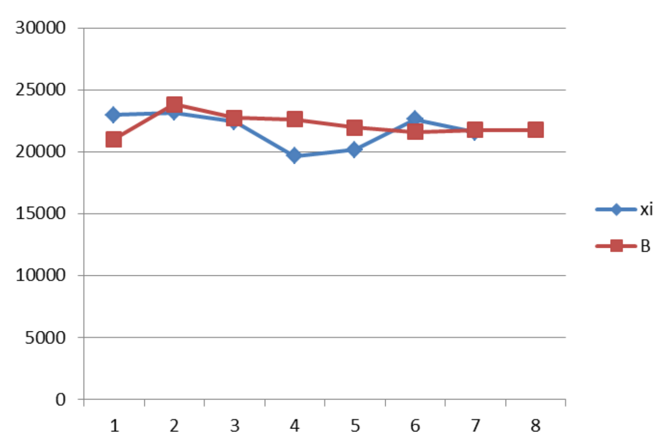

Fig. 2 Actual and estimated numbers of claims

Figure 2 shows the credibility estimate of the number of claims in successive years for the gamma G(8 400; 0,4) prior distribution for λ.

Estimated number of claims increasing with time until it reaches the level of the actual claims numbers after 7 years. This increase is due to progressively more weight, i. e. credibility, being given to the data from the risk itself and correspondingly less weight being given to the collateral data, i. e. the prior distribution of λ.

VI. The Normal/normal Model

Our problem is to estimate the pure premium, i.e. the expected aggregate claims for a risk in next year. So X is a random variable denoting total claims from a risk in a coming year and the distribution of X is normal, depends on the value of an unknown parameter θ. The conditional distribution of X/θ is normal and the unknown parameter θ is the mean of this distribution, because of

$$X/\theta \quad \sim \quad N\left(\theta;\sigma_1^2\right) \tag{15}$$

The prior distribution of θ is normal,

$$\theta \quad \sim \quad N\left(\mu;\sigma_2^2\right) \tag{16}$$

where μ, σ_1^2 σ_2^2 are known. Suppose we have n past observations of X, $\mathbf{x}=(x_1,x_2,\dots x_n)$. Our problem is to estimate $E\left(X/\theta\right)$ and we use again the Bayesian estimate with respect to the quadratic loss.

If θ was known, the pure premium would be

$$E\left(X/\theta\right)=\theta \tag{17}$$

So the problem of estimating $E\left(X/\theta\right)$ is the same as the problem of estimating parameter θ as a Bayesian estimator

$$\theta_B=E\left(\theta/\mathbf{x}\right) \tag{18}$$

i.e. the posterior mean of θ given x. We need to know the form of the posterior density function $f\left(\theta/\mathbf{x}\right)$.

Suppose we know data of n previous observations $\mathbf{x}=(x_1,x_2,\dots x_n)$ so we can express the likelihood $f\left(\theta/\mathbf{x}\right)$ as

$$f\left(\theta/\mathbf{x}\right)\propto\prod_{i=1}^{n}\mathrm{e}^{-\frac{1}{2\sigma_1^2}(x_i-\theta)^2}=e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n}(x_i-\theta)^2}\propto e^{-\frac{n}{2\sigma_1^2}\theta^2+\frac{n\bar{x}}{\sigma_1^2}\theta}$$

As we can see, the likelihood function is quadratic in θ, and can be shown to be proportional to $e^{-\frac{1}{2}\left(a_1\theta^2+a_2\theta+a_3\right)}$.

When ignoring terms not involving θ, we can express the normal prior distribution as being proportional to

$$f\left(\theta\right)=\frac{1}{\sqrt{2\pi}\,\sigma_2}e^{-\frac{(\theta-\mu)^2}{2\sigma_2^2}}\propto e^{-\frac{1}{2\sigma_2^2}\theta^2+\frac{\mu}{\sigma_2^2}\theta}$$

The posterior density $f\left(\theta/\mathbf{x}\right)$ by Bayes` theorem is proportional to

$$f\left(\theta/\mathbf{x}\right)\propto e^{-\frac{n}{2\sigma_1^2}\theta^2+\frac{n\bar{x}}{\sigma_1^2}\theta}\cdot e^{-\frac{1}{2\sigma_2^2}\theta^2+\frac{\mu}{\sigma_2^2}\theta}$$

and after adjustments

$$f\left(\theta/\mathbf{x}\right)\propto e^{-\frac{1}{2}\left(\frac{n}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)\theta^2+\left(\frac{n\bar{x}}{\sigma_1^2}+\frac{\mu}{\sigma_2^2}\right)\theta} \tag{19}$$

So the posterior distribution $f\left(\theta/\mathbf{x}\right)$ is a normal, say with parameters $\tilde{\mu},\tilde{\sigma}^2$, i. e.

$$f\left(\theta/\mathbf{x}\right)=c\,e^{-\frac{(\theta-\tilde{\mu})^2}{2\tilde{\sigma}^2}}\propto e^{-\frac{1}{2\tilde{\sigma}^2}\theta^2+\frac{\tilde{\mu}}{\tilde{\sigma}^2}\theta} \tag{20}$$

We will find the parameters $\tilde{\mu},\tilde{\sigma}^2$ by equating the power of θ^2 and θ in two different expression of $f\left(\theta/\mathbf{x}\right)$. Then we get

$$\tilde{\mu}=\frac{\mu\,\sigma_1^2+n\,\bar{x}\,\sigma_2^2}{\sigma_1^2+n\,\sigma_2^2} \tag{21}$$

$$\tilde{\sigma}^2=\frac{\sigma_1^2\,\sigma_2^2}{\sigma_1^2+n\,\sigma_2^2} \tag{22}$$

We can find the Bayesian estimation of pure premium as the mean of the posterior distribution, i. e.

$$\theta_B=\frac{\mu\,\sigma_1^2+n\,\bar{x}\,\sigma_2^2}{\sigma_1^2+n\,\sigma_2^2} \tag{23}$$

That can be rewritten as

$$E\left(\theta/\mathbf{x}\right)=Z\,\bar{x}+\left(1-Z\right)\mu \tag{24}$$

which is a credibility estimate of the pure premium $E\left(\theta/\mathbf{x}\right)$ with factor credibility

$$Z=\frac{n\sigma_2^2}{\sigma_1^2+n\sigma_2^2}=\frac{\sigma_2^2}{\frac{\sigma_1^2}{n}+\sigma_2^2}=\frac{n}{n+\frac{\sigma_1^2}{\sigma_2^2}} \tag{25}$$

VII. Application of Normal/normal Model

Total aggregate claims in a particular insurance company are modeled with a normal distribution $N\left(\theta;\sigma_1^2\right)$, where θ is unknown and $\sigma_1^2=135000^2$. Prior information about θ suggests that it is distributed by $N\left(\mu;\sigma_2^2\right)$ with known parameters $\mu=2100000$ and $\sigma_2^2=150\ 000^2$.

Table II Bayes estimations of pure premium

i	x_i	$\bar{x}$	Z	θ_B
1	2112000	0	0	2100000
2	2140000	2112000	0,55249	2106630
3	1955000	2126000	0,71174	2118505
4	2315000	2069000	0,78740	2075591
5	2280000	2130500	0,83160	2125364
6	2035000	2160400	0,86059	2151979
7	2215000	2139500	0,88106	2134802
8		2150285	0,89629	2145070

Source: Own calculation

Aggregate claims from the last seven years were not incorporated in the prior information and they are in Table II, column named x_i. We have already observed the values of x_1, x_2, …, x_n and we wish to estimate the expected aggregate claims in the coming , i. e. $(n+1)th$ year.

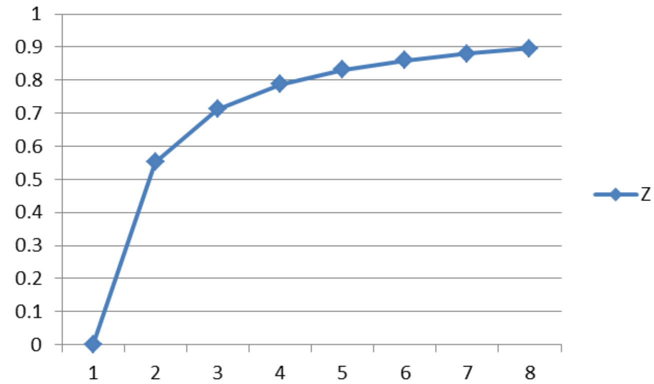

Fig. 3 Updated credibility factors of normal/normal model

Figure 3 shows the credibility factor Z in successive years. Calculation of credibility factors Z_i in each year i on the basis of previous year's data x_i of the Table II is an application of formula (25). We can see from this formula and from the figure 3 that the credibility factor Z increases with time.

The Bayes estimations of the pure premiums for each year by equation (24) with credibility factors calculated by (25) there are in the last column of Table II.

Figure 4 shows the actual and estimated aggregate claims amounts in successive years with the prior normal $N\left(\mu;\sigma_2^2\right)$ distribution for unknown parameter θ, which is the pure premium.

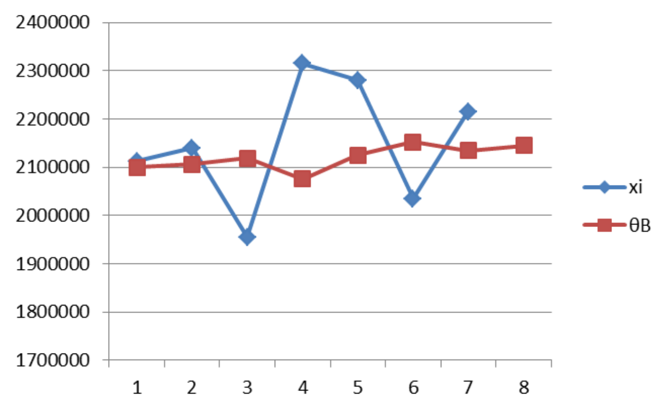

Fig. 4 Actual and estimated aggregate claims

VIII. The Binomial/Beta Model

For estimation of a binomial probability Θ from a single observation X with the prior distribution of Θ being beta with parameters α and β, we will investigate the form of the posterior distribution of Θ.

Prior beta density function by assumption is

$$f(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1},\; 0<\theta<1$$

omitting the constant $\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}$. Note that the uniform distribution on (0,1) is a special case of the beta with $\alpha = 1$ and $\beta = 1$. This corresponds to the non-informative case.

Likelihood is

$$f(x/\theta) \propto \theta^{x}(1-\theta)^{n-x},\; x=0,1,\dots,n,$$

omitting the constant $\begin{pmatrix} n \\ x \end{pmatrix}$. By (4) we get the posterior density of Θ in the form

$$f(\theta/x) \propto \theta^{x}(1-\theta)^{n-x}\theta^{\alpha-1}(1-\theta)^{\beta-1} = \theta^{\alpha+x-1}(1-\theta)^{\beta+n-x-1}$$

We can see, apart from the appropriate constant of proportionality, that it is the density of a beta random variable with parameters $\alpha'=\alpha+x$ and $\beta'=\beta+n-x$. Therefore we can conclude that the posterior distribution of Θ given X is the beta distribution with parameters

$$\begin{aligned} \alpha' &= \alpha+x \\ \beta' &= \beta+n-x \end{aligned} \tag{26}$$

Now we can determine the Bayesian estimator of Θ under quadratic loss. By (6) the Bayesian estimator is the mean of this distribution, that is

$$\theta_B = \frac{\alpha+x}{(\alpha+x)+(\beta+n-x)} = \frac{\alpha+x}{\alpha+\beta+n} \tag{27}$$

The Bayesian estimator of binomial Θ can be expressed as follows:

$$\theta_B = \frac{n}{\alpha+\beta+n}\cdot\frac{x}{n} + \frac{\alpha+\beta}{\alpha+\beta+n}\cdot\frac{\alpha}{\alpha+\beta}$$

If we put $Z=\dfrac{n}{\alpha+\beta+n}$ and $\mu=\dfrac{\alpha}{\alpha+\beta}$, which is the mean of the prior beta distribution, we get by (1) the Bayesian estimator of Θ in the form of credibility formula:

$$\theta_B = Z\cdot\frac{x}{n} + (1-Z)\cdot\mu \tag{28}$$

IX. Application of the Binomial/Beta Model

Let θ is unknown probability of getting a critical illness. To estimate this probability we have found the data about the number of insurance agreements on critical illness and the number of claims in these insurance contracts in the past seven years. Found data are shown in Table III in the columns designated as "n" and "x".

Before data collecting we have no information at all about this probability θ. In this situation if we aim the Bayes estimate of θ, we need a "non-informative" prior distribution, which is uniform distribution on <0; 1>, or beta distribution with parameters $\alpha = 1$ and $\beta = 1$. This prior distribution of the parameter θ we will use in the first year for which we have no data.

In each of subsequent year we will update the parameters α and β according to the expression (26). These updated estimates of parameters of posterior distribution include the Table III.

Table III Updated estimates of parameters of posterior distribution

year	n	x	α	β
1	524	15	1	1
2	866	35	16	510
3	2879	85	51	1341
4	4420	155	136	4175
5	5916	325	291	8400
6	8661	411	616	13991
7	9299	504	1027	22241
8			1531	31036

Source: Own calculation

In table IV we can find the Bayesian estimators θ_B of probability θ of getting a critical illness, updated in each year by new information that we have. Values of θ_B we have got in each year as a mean of posterior distribution.

Table IV Bayesian estimators of binomial probability θ

year	n	x	n/x	θ_B
1	524	15	0,0286	0,5
2	866	35	0,0404	0,0304
3	2879	85	0,0295	0,0366
4	4420	155	0,0351	0,0318
5	5916	325	0,0549	0,0335
6	8661	411	0,0475	0,0422
7	9299	504	0,0542	0,0441
8				0,0470

Source: Own calculation

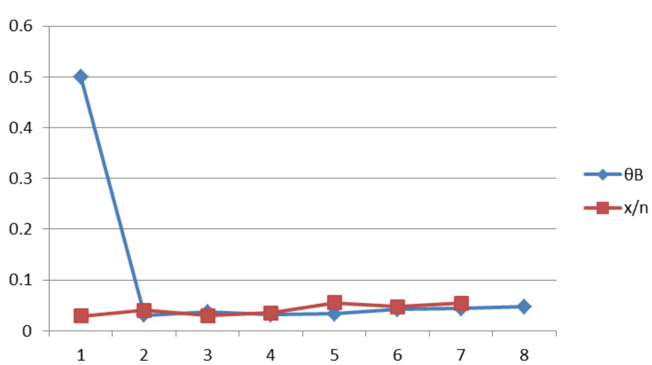

Fig. 5 Comparison of MLE and Bayesian estimator of binomial probability θ

We can see by expression (28), that Bayesian estimator θ_B is a weighted average of the maximum likelihood estimator (MLE) x/n of binomial probability θ and of the prior mean $\alpha/(\alpha+\beta)$. The prior mean can be described as "no data" estimate, and the MLE as the "all data" estimate. The Bayesian estimate is somewhere in between these two. Comparison of MLE and Bayesian estimator of probability θ provides Figure 5.

X. The Pareto/Gamma Model

The aggregate claims X from a risk are assumed to have the following Pareto density function:

$$f(x)=\frac{\theta}{(1+x)^{+(\theta+1)}}, \qquad 0<x<+\infty$$

where θ is an unknown parameter. Suppose θ has a gamma distribution G(α; β). Let $\mathbf{x}=(x_1,x_2,\dots x_n)$ be a random sample from the Pareto distribution. We will derive the posterior distribution of the parameter θ:

$$f(\theta/\mathbf{x}) \propto prior \times likelihood$$
$$\propto \theta^{\alpha-1}e^{-\beta\theta} \times \prod_{i=1}^{n}\theta(1+x_i)^{-(\theta+1)}$$
$$\propto \theta^{\alpha+n-1}e^{-\beta\theta} \times \prod_{i=1}^{n}e^{-(\theta+1)\log(1+x_i)}$$
$$\propto \theta^{\alpha+n-1}e^{-\theta(\beta+t(\mathbf{x}))}$$

where $t(\mathbf{x})=\sum_{i=1}^{n}\log(1+x_i)$.

Therefore, the posterior distribution is gamma $G(\alpha+n;\beta+t(\mathbf{x}))$ and the Bayesian estimate of θ with respect to the quadratic loss function is the mean of the posterior distribution:

$$\theta_B=\frac{\alpha+n}{\beta+\sum_{i1}^{n}\log(1+x_i)} \tag{29}$$

This form of posterior mean as Bayesian estimator of θ cannot be written in the form of the credibility formula. This is the example that the Bayesian approach to estimation may not always produce an estimator which can be rearranged to the form of credibility estimator. The Bayesian approach to credibility provides an answer to the problem of determining the credibility factor Z, at least in certain cases.

XI. Empirical Bayes Credibility Theory

The assumption that we know the prior distribution including parameters, as in previous models, is very strong and so we need a more practical tool to relax this assumption. The empirical Bayes credibility models that we will briefly discussed in this section will provide an invaluable alternative for this work.

Empirical Bayes credibility theory is the collective name for the vast literature which has developed since Bühlmann and Straub`s (1970). Although this model is a basis for other more specific models such as hierarchical, multidimensional or regression credibility models.

Our problem is to estimate the premium, or claim rate, for an individual risk. Let $X_1, X_2, ..., X_j, ..., X_n$ denote the aggregate claims, or numbers of claims, from this risk in successive periods. We want to estimate the premium but do not want to make the strong distributional assumptions of the normal/normal model. Our first assumption is that the X_j, $j=1,2,...,n$ are identically distributed. Next we assume that the distribution of X_j 's depends on a parameter θ whose is fixed but unknown. As in the Bayesian formulation we assume that θ varies over some super-population of risks, but we make no assumptions regarding the distribution of θ. We do assume that, given θ,

$$m(\theta)=E(X_j/\theta) \quad (30)$$

$$s^2(\theta)=D(X_j/\theta) \quad (31)$$

The true premium for our risk is $m(\theta)$. We will assume that we have n observed values of the X_j 's, say $x_1,x_2,...,x_n$ which we will denote $\mathbf{x}$. Our problem is to estimate $m(\theta)$ for given $\mathbf{x}$. In the empirical Bayes model we aim to found the pure premium with respect to quadratic loss, which equals $E(m(\theta)/\mathbf{x})$.

The solution put forward by Bühlmann (1967) is to find the best approximation to $E(m(\theta)/\mathbf{x})$ among those functions which are linear in the observed data. The form of empirical Bayes credibility premium has been derived as

$$E(m(\theta)/\mathbf{x})=Z\,\bar{x}+(1-Z)E(m(\theta)) \quad (32)$$

with factor credibility

$$Z=\frac{n}{n+\dfrac{E(s^2(\theta))}{D(m(\theta))}} \quad (33)$$

In empirical Bayes credibility Model 1 we use the available data (Tab. 5) to estimate the quantities $E(m(\theta))$, $D(m(\theta))$, $E(s^2(\theta))$, and hence obtain a Bühlman type credibility estimate (1967) for a particular risk. In Model 2 the approach of Bühlmann and Straub (1970) allowing for varying annual volumes in the risks, and then after standardizing annual aggregate claims by these volumes to obtain empirically based credibility estimates for pure premiums. The example of application of Model 2 we can find for example in [7], [13], [15].

Table V Baseline data for empirical Bayes credibility Model 1

Risk i	Year j 1	2	…	j	…	n
1	X_{11}	X_{12}		X_{1j}		X_{1n}
2	X_{21}	X_{22}		X_{2j}		X_{2n}
⋮	⋮	⋮		⋮		⋮
i	X_{i1}	X_{i2}		X_{ij}		X_{in}
⋮	⋮	⋮		⋮		⋮
N	X_{N1}	X_{N2}		X_{Nj}		X_{Nn}

Suppose again we want to estimate the pure premium or the average number of claims for a particular risk, which is one, say i-the risk of N similar risks. Let X_{ij} means of the total claims, respectively total number of claims for the i-th risk, $i=1,2,...,N$, in the j-th year, $j=1,2,...,n$, as in Table V.

Derived relations necessary for estimates are:

$$est\ E(m(\theta))=\bar{X} \quad (34)$$

$$estE(s^2(\theta))=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n-1}\sum_{j=1}^{n}(X_{ij}-\bar{X}_i)^2 \quad (35)$$

$$estD(m(\theta))=\frac{1}{N-1}\sum_{i'=1}^{N}(\bar{X}_i-\bar{X})^2-\frac{1}{N\,n}\sum_{i=1}^{N}\frac{1}{n-1}\sum_{j=1}^{n}(X_{ij}-\bar{X}_i)^2 \quad (36)$$

The derivation of these relations can be found in Bühlmann (1967) and Waters (1994).

XII. Application of empirical Bayes model 1

The data in Table VI show the aggregate claims in five years 2006-2010 experienced by four Czech insurance companies (1-ČP, 2-Kooperativa, 3-Allianz, 4-ČSOB) in millions of Czech crowns.

Table VI Aggregate claims in four Czech insurance companies

Insurance Company i	Year j 1	2	3	4	5
1	9331	7839	8275	8280	8190
2	10114	9399	9947	10726	11076
3	2672	2523	2510	3086	3285
4	1589	1747	2072	2095	2124

Source: www.cap,cz

The next table contains calculations for estimations $E(m(\theta))$, $D(m(\theta))$, $E(s^2(\theta))$ by expressions (34), (35), (36).

Table VII Auxiliary calculations

i	$\bar{X}_i$	$(\bar{X}_i-\bar{X})^2$	$\frac{1}{4}\sum_{j=1}^{5}(X_{ij}-\bar{X}_i)^2$
1	8383,0	6446521,00	313540,5
2	10252,4	19433990,56	435832,3
3	2815,2	9173629,44	123269,7
4	1925,4	15355425,96	58672,3
$\sum$	23376	50409566,96	931314,8

Source: Own calculation

Substituting from Table VII we get

$$est\ E(m(\theta))=\bar{X}=23376/4=5\,844$$

$$est\ E(s^2(\theta))=931314{,}8/4=232\,828{,}7$$

$$est\ D(m(\theta))=50409566{,}96/3-931314{,}8/20=$$
$$=16\,756\,623$$

Now we can calculate factor credibility Z by (33):

$$Z = \frac{n}{n + \frac{E\left(s^2(\theta)\right)}{D\left(m(\theta)\right)}} = \frac{5}{5 + \frac{232\,828{,}7}{16\,756\,623}} = 0{,}997229$$

Now we know all the necessary information for calculating the credible net premium for each insurance company using the formula (32). These estimates for the next year are shown in Table VIII.

Table VIII Estimates the credible net premiums for next year

Insurance company	$E\left(m(\theta)/\mathbf{x}\right)$
1-ČP	8375,96
2-Kooperativa	10240,18
3-Allianz	2823,59
4-ČSOB	1936,26

Source: Own calculation

XIII. Conclusion

Bayesian estimation theory provides methods for permanently updated estimates of the number of claims and of the pure premium for each coming year in insurance company. Bayesian approach combine prior information that are known before collected of any data and information provided by the sample data, which are number of claims or aggregate claim amounts in previous n years.

The approach used in the binomial/beta, Poisson/gamma and normal/normal models is essentially the same. The only one important difference is in the distributional assumptions. This approach has been very successful in these three cases. It has made the notion of collateral data very precise, by interpreting it in terms of a prior distribution and has given formulae for the calculation of the credibility factor.

The biggest advantage of the Poisson/gamma, normal/normal and binomial/beta models for insurance practice is possibility to express them in the form of credibility formulas by expressions (14), (24) and (28). These formulas allow easy application of not quite trivial theory in insurance practice, as seen from the examples in sections V, VII and IX.

However, the Bayesian approach does have a few serious drawbacks and limitations. This approach can be criticized as subjective, because we should always start with a prior distribution of estimated parameters.

Formulas (13) and (25) involve parameters, β in the former and σ_1, σ_2 in the latter, which we have assumed to be known. The values of these parameters reflect the subjective opinion of the decision maker; there is no question of estimating these parameters from data.

The second difficulty is that even if our problem fits into a Bayesian framework, the Bayesian approach may not work in the sense that it may not produce an estimate which can readily be rearranged to be in the form of a credibility estimate (1). This is the case of the model Pareto/gamma in section X.

The problem of estimation of unknown parameters when some data from related risks are available solves the so-called Empirical Bayes Credibility Theory. Its principle is briefly explained in section XI, detailed interpretation is not the subject of this article. This theory is the content of many publications, for example [1], [2], [3], [4], [5], [13], [15].

References

[1] P. J. Boland, *Statistical and Probabilistic Methods in Actuarial Science*, London: Chapman&Hall/CRC, 2007.

[2] H. Bühlmann, Experience rating and credibility, *ASTIN Bull*. 4, 119, 1967.

[3] H. Bühlmann, E. Straub, Glaubwürdigkeit für Schadensätze. Mitteilungen der Vereinigung, *Schweizerische Versicherungs-mathematiker* 70 (Credibility for loss ratios. Communications of the Association Swiss actuaries 70), pp. 111–133, 1970.

[4] H. Bühlmann, A. Gisler, *Course in Credibility Theory and its Applications*, Berlin: Springer, 2005.

[5] R. Kaas, M. Goovaerts, J. Dhaene, M. Denuit, *Modern Actuarial Risk Theory*, Boston: Kluwer Academic Publishers, 2001.

[6] E. Kotlebová, *Bayesovská štatistická indukcia v ekonomických aplikáciách* (Bayesian Statistical Inference in Economic Applications), Bratislava: Ekonóm, 2009.

[7] B. Linda, J. Kubanová, Credibility Premium Calculation in Motor Third-Party Liability Insurance, *Proceedings of the 14th WSEAS International Conference on Mathematical and Computational Methods in Science and Engineering* (MACMESE'12), Sliema, Malta, September 7-9, 2012, pp. 259-263.

[8] Meelis Kaarik, Merili Umbleja, On claim size fitting and rough estimation of risk premiums based on Estonian traffic insurance example, *International Journal of Mathematical Models and Methods in Applied Sciences*, Issue 1, Volume 5, 2011, pp. 17-24

[9] V. Pacáková, The Bayesian Inference in Actuarial Sciences, *Central European Journal for Operations Research and Economics*, Volume 5, Number 3-4, 1997, pp. 255-268.

[10] V. Pacáková, Bayesian Estimations in Insurance Theory and Practice, *Proceeding of the 14th WSEAS International Conference on Mathematical and Computational Methods in Science and Engineering* (MACMESE'12), Sliema, Malta, September 7-9, 2012, pp. 127-131.

[11] L. Přečková, Asymmetry of information during the application of the model for valuation the sum insured in case of business interruption in the Czech Republic, *International Journal of Mathematical Models and Methods in Applied Sciences*, Issue 1, Volume 5, 2011, pp. 212- 219,

[12] E. Šoltés, V. Pacáková, T. Šoltésová, Vybrané kredibilné regresné modely v havarijnom poistení (Selected credibility regression models in accident insurance), *Ekonomický časopis*, Vol. 54, No. 2, 2006, pp. 168-182.

[13] E. Šoltés, *Modely kredibility na výpočet poistného* (Models for calculating credibility premiums), Bratislava: (Publisher) Vydavateľstvo EKONÓM, 2009.

[14] Y. K. Tse, *Nonlife Actuarial Models*, Cambridge: University Press, 2009.

[15] H. R. Waters, *An Introduction to Credibility Theory*, London and Edinburgh: Institute of Actuaries and Faculty of Actuaries, 1994.

Prof. RNDr. Viera Pacáková, PhD. graduated in Econometrics (1970) at Comenius University in Bratislava, 1978 - RNDr. in Probability and Mathematical Statistics at the same university, degree PhD at University of Economics in Bratislava in 1986, assoc prof. in Quantitative Methods in Economics in 1998 and professor in Econometrics and Operation Research at University of Economics in Bratislava in 2006.

She was working at Department of Statistics, Faculty of Economic Informatics, University of Economics in Bratislava since 1970 to January 2011. At the present she has been working at Faculty of Economics and Administration in Pardubice since 2005.

Prof. Pacáková has been concentrated on actuarial science and management of financial risks since 1994 in connection with the actuarial education in the Slovak republic and she has been achieved considerable results in this area.