Robust Stability Analysis of Discrete-Time Systems with Parametric Uncertainty: A Graphical Approach

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***Abstract*—**The principal aim of this paper is to describe an elegant, efficient and simple to use approach to investigation of robust stability for discrete-time polynomials with various uncertainty structures. The key idea is based on generalization of the value set concept and the zero exclusion condition under robust D-stability framework. The set of examples for polynomials with single parameter uncertainty, interval polynomials, affine linear uncertainty structure and also more complicated uncertainty structures given in the paper provides not only the robust stability analyses but also the simple Matlab codes as an inspiration for practical implementation of performed tests.

***Keywords*—**Discrete-Time Systems, Parametric Uncertainty, Robust Stability, Value Set Concept, Zero Exclusion Condition.

I. Introduction

Without a shadow of a doubt, stability represents the most important property of the control loops and its ensurance is the critical task for all applications. When one considers an uncertain system (see e.g. [1] – [6] for related robustness problems), the attention is aimed to so-called robust stability, i.e. the stability must be guaranteed for all possible systems from a priori assumed family.

Robust stability of interval polynomials has been deeply studied topic during the last decades [7] – [11]. The situation is much easier for the continuous-time cases as the famous Kharitonov theorem can be directly applied [12]. Since the Kharitonov-like extremal results are not generally available for discrete-time systems, an array of useful alternative approaches have been proposed by various researchers – see e.g. [7] – [11]. Nevertheless, many of the developed methods require some restrictions and pre-conditions and so they suffer from the lack of generality. Moreover, the majority of more general results are, in author's opinion, relatively complicated to use. The situation is even more difficult for polynomials with not only interval (independent) but more complex uncertainty structures.

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This paper presents very universal graphical approach to robust stability analysis of systems with parametric uncertainty adopted mainly from [13]. As it can be applied for general stability regions, its utilization for discrete-time interval polynomials is advantageous and very easily performable. The principal idea is based on the value set concept and general version of the zero exclusion condition considered under robust *D*-stability framework.

This work extends the contribution [14] where robust stability of discrete-time interval polynomials was investigated with the help of the Polynomial Toolbox for Matlab. This paper deals with the topic more generally and furthermore, it presents the simple Matlab "mini-programs" for robust stability analysis of discrete-time families of polynomials with various uncertainty structures. Interested reader can find the similar programs but mainly for continuous-time systems in [15], [16]. The paper is the extended version of the conference contribution [17].

The paper is organized as follows. In Section 2, the solved problem is formulated. The Section 3 then briefly describes the embellishment of the value set concept and the zero exclusion condition for robust D-stability framework. Further, a number of simulation examples including the visualizations of value sets and the simple Matlab codes for their plotting are presented in the extensive Section 4. And finally, Section 5 offers some conclusion remarks.

II. Problem Formulation

The issue of stability of systems can be considered as the problem of stability of their characteristic polynomials. The discrete-time uncertain polynomial can be written in the form:

$$p(z,q)=\sum_{i=0}^{n}\rho_i(q)z^i \tag{1}$$

where z is the complex variable, q is the vector of uncertainty and ρ_i are coefficient functions.

Then, the family of polynomials is defined by [13]:

$$P=\{p(\cdot,q):q\in Q\} \tag{2}$$

where Q is the uncertainty bounding set (represented by a multidimensional box in this contribution).

Generally, the family of polynomials (2) is robustly stable if and only if $p(z,q)$ is stable for all $q \in Q$, i.e. all roots of $p(z,q)$ must lie within the unit circle. Since the direct calculation of roots can be impractical due to potentially enormously long computation times, the more efficient techniques had to be studied [18].

Among an array of existing methods which have been developed and described e.g. in [7] – [11], the graphical approach based on combination of the value set concept and the zero exclusion condition [13] seems to be very elegant, universal and powerful tool.

III. Zero Exclusion Condition for Robust *D*-Stability

The basic continuous-time version of the value set concept and the zero exclusion condition can be found e.g. in [13], [19]. This work employs its improved version, described hereafter, which is extended and generalized to so-called robust *D*-stability framework [13]. The main idea remains the same, but it allows investigating robust stability for an arbitrary stability region *D*, e.g. for a unit circle in case of discrete-time polynomials with parametric uncertainty.

Suppose a family of polynomials (2). The value set at any evaluation point $x \in \mathbf{C}$ is determined by:

$$p(x,Q) = \{p(x,q) : q \in Q\} \tag{3}$$

In other words, $p(x,Q)$ is the image of Q under $p(x,\cdot)$. For instance, in discrete-time case substitute z for a point at the unit circle in a family $P = \{p(z,q) : q \in Q\}$ and let the vector of uncertain parameters q range over the set Q.

The zero exclusion condition formulated in [13] says: Let D be an open subset of the complex plane and assume that (2) is a family of polynomials with invariant degree, uncertainty bounding set Q which is pathwise connected. Moreover, suppose that the coefficient functions $\rho_i(q)$ are continuous and that (2) has at least one D-stable member $p(\cdot, q_0)$. Then (2) is robustly D-stable if and only if:

$$0 \notin p(x,Q) \tag{4}$$

for all $x \in \partial D$, where ∂D denotes the boundary of D.

IV. Illustrative Examples

This section is intended to demonstrate the graphical tests of robust stability by means of simple Matlab codes which practically implement the value set concept and the zero exclusion condition for families of discrete-time polynomials with various uncertainty structures.

A. Single Parameter Uncertainty

First, consider the family of discrete-time polynomials with single parameter uncertainty:

$$p(z,q) = 3z^3 + (1+2q)z^2 + (1+q)z + q; \quad q \in \langle 0.8; 1.2 \rangle \tag{5}$$

In order to be robustly stable, the necessary condition is that the polynomial must have a stable member. For example, $q = 1$ leads to the polynomial:

$$p(z,1) = 3z^3 + 3z^2 + 2z + 1 \tag{6}$$

which is stable (it has the roots $r_1 = -0.7181$; $r_{2,3} = -0.141 \pm 0.\overline{6}j$).

The value sets of the family (5) can be plotted by using the Matlab code:

```
%discrete-time polynomial with single parameter uncertainty
%robustly stable case
clear all
hold all
for c=0:0.01:1 %generalized frequency range
  count=1; %auxiliary counter
  for q=0.8:0.01:1.2 %sampling of uncertain coefficient
    z=exp(j*c*2*pi); %unit circle
    p(count)=3*z^3+(1+2*q)*z^2+(1+q)*z+q;…
    % the polynomial
    count=count+1; %counter increment
  end
  x=real(p); %real part
  y=imag(p); %imaginary part
  plot(x,y,'.')
end
hold off
```

The obtained straight line value sets are shown in fig. 1, where the value set for each frequency contains 41 points.

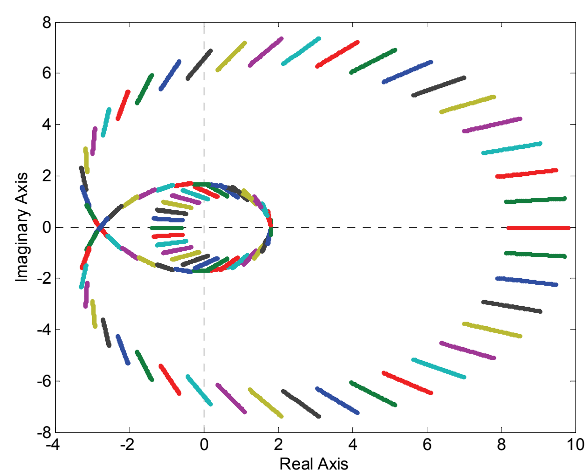

Fig. 1 value sets for the family (5) with single parameter uncertainty – robustly stable case

From the robust stability point of view, the most important fact following from the fig. 1 is that the origin of the complex plane (zero point) is excluded from the value sets. Besides, the family (5) has a stable member (as it has been verified already). Thus, in concordance with the condition given in Section 3, the family (5) can be concluded as (Schur) robustly stable one.

Now, assume the similar family as in the previous case but with wider range of possible variation of uncertain parameter q.

$$p(z,q) = 3z^3 + (1+2q)z^2 + (1+q)z + q; \quad q \in \langle 0.4; 1.6 \rangle \tag{7}$$

It has been already shown that the family has a stable member e.g. for $q = 1$. The value sets are visualized in fig. 2. They can be obtained using the same code as fig. 1 with just appropriate modification of uncertain coefficient sampling (q=0.4:0.01:1.6). It means that each plotted value set has 121 points.

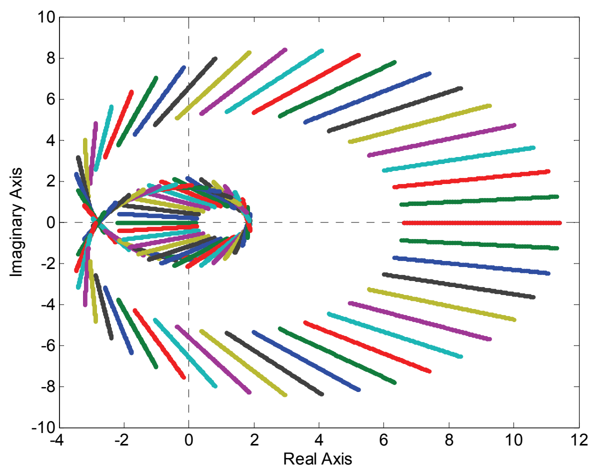

Fig. 2 value sets for the family (7) with single parameter uncertainty – robustly unstable case

The value sets from fig. 2 include the complex plane origin which means that the family (7) is not robustly stable, i.e. stability of the family is not guaranteed for all possible values of q.

The last example from this section will demonstrate the importance of testing the existence of a stable member. Suppose the single parameter family:

$$p(z,q) = z^3 + (3+2q)z^2 + (2+q)z + 2 + q; \quad q \in \langle 1; 2 \rangle \tag{8}$$

Using the analogical code as in the previous cases, one can obtain the value sets depicted in fig. 3.

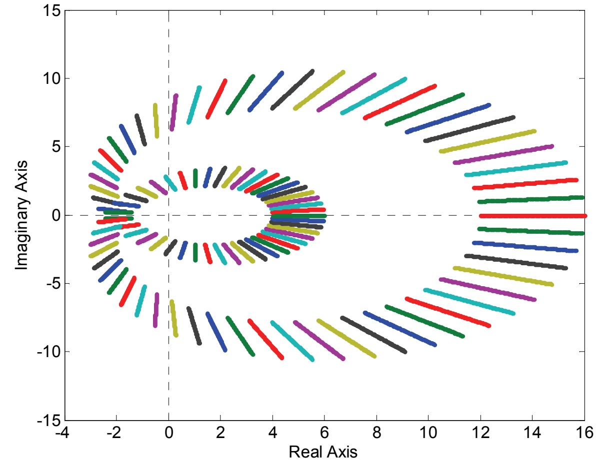

Fig. 3 value sets for the family (8) with single parameter uncertainty – robustly UNSTABLE case

At first glance, the family seems to be robustly stable since the zero point is excluded from the value sets. However, simple stability test of arbitrary family member, e.g. for $q = 1.5$:

$$p(z,1.5) = z^3 + 6z^2 + 3.5z + 3.5; \tag{9}$$

reveals that it is not stable (roots $r_1 = -5.4777$; $r_{2,3} = -0.2612 \pm 0.7555j$). The problem is that the family has all of its members unstable, so even if the stability boundary is not crossed during the zero exclusion test, the family is definitely not robustly stable.

B. Interval Uncertainty

Now, more parameters can be uncertain, however the polynomial coefficients must vary independently. For instance, consider the fifth order discrete-time interval polynomial taken from [14] and subsequently from [15], [16]:

$$p_1(z,q) = [1,2] + [3,4]z + [5,6]z^2 + [7,8]z^3 + [9,10]z^4 + \cdots \cdots [11,12]z^5 \tag{10}$$

A possible Matlab code for plotting the value sets of this polynomial based on the brute-force parameter gridding can look like [15], [16]:

```
%discrete-time interval polynomial
%parameter gridding
%stable case
clear all
hold all
for c=0:0.01:1 %generalized frequency range
 count=1; %auxiliary counter
 for q0=1:0.5:2 %sampling of uncertain coefficients
  for q1=3:0.5:4
   for q2=5:0.5:6
```

```
   for q3=7:0.5:8
    for q4=9:0.5:10
     for q5=11:0.5:12
      z=exp(j*c*2*pi); %unit circle
      p(count)=q0+q1*z+q2*z^2+q3*z^3+ q4*z^4+…
      q5*z^5; % the polynomial
      count=count+1; %counter increment
     end
    end
   end
  end
 end
end
x=real(p); %real part
y=imag(p); %imaginary part
plot(x,y,'.')
end
hold off
```

The resulting plot is shown in fig. 4.

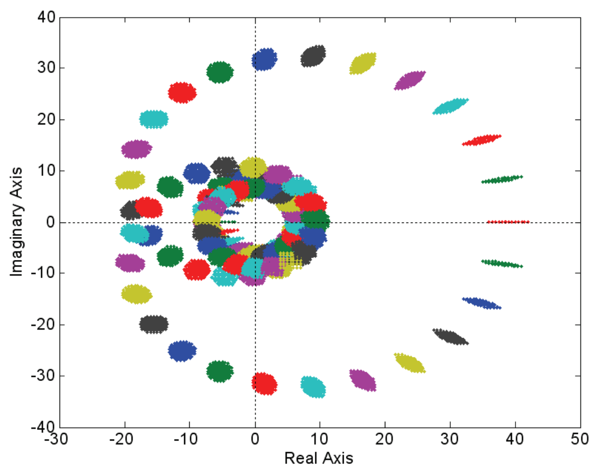

Fig. 4 value sets for the family (10) – “parameter gridding” approach – robustly stable case

Note, that the value set for each frequency consists of $3^6 = 729$ points (3 “sampled” values for each of 6 uncertain coefficients).

As can be seen, the origin of the complex plane is excluded from the value sets. Moreover, the family (10) definitely has a stable member which can be easily verified by choosing any fixed polynomial from the family and checking its stability. So, the family is robustly stable.

Alternatively, the same graphical test can be performed by the Matlab routine based on construction of generators and plotting their convex hull for each generalized frequency, i.e.:

```
%discrete-time interval polynomial
%convex hull of 64 generators
%stable case
clear all
hold all
for c=0.000001:0.01:1 %generalized frequency range
 count=1; %auxiliary counter
 for q1=0:1 %uncertain parameters for 2^6=64…
 %generators
  for q2=0:1
   for q3=0:1
    for q4=0:1
     for q5=0:1
      for q6=0:1
       z=exp(j*c*2*pi); %unit circle
       p0=1+3*z+5*z^2+7*z^3+9*z^4+11*z^5;…
       %nominal polynomial
       p1=1; %auxiliary polynomials
       p2=z;
       p3=z^2;
       p4=z^3;
       p5=z^4;
       p6=z^5;
       p(count)=p0+q1*p1+q2*p2+q3*p3+ q4*p4+…
       q5*p5+q6*p6;…
       %uncertain polynomial structure
       count=count+1; %counter increment
      end
     end
    end
   end
  end
 end
 x=real(p); %real part
 y=imag(p); %imaginary part
 k=convhull(x,y); %convex hull
 plot(x(k),y(k))
end
hold off
```

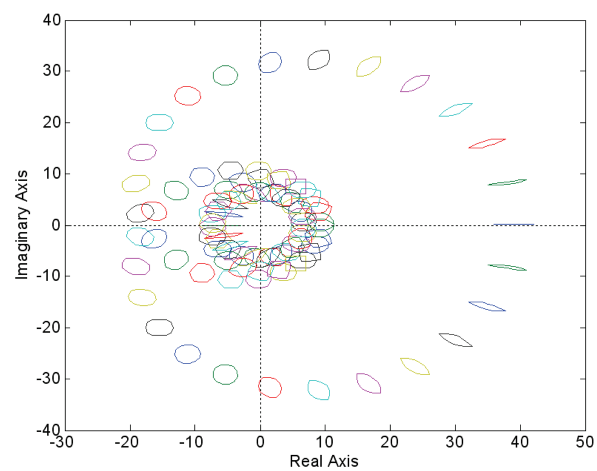

Fig. 5 value sets for the family (10) – “convex hull” approach – robustly stable case

The fig. 5 shows the graphical output of this mini-program. Compared to fig. 4, the second plot is simpler and it generally requires less calculation of the value set points. Nevertheless, the figs. 4 and 5 represent the same robust stability analysis obviously with the same result. That is the family (10) is robustly stable.

Now, assume that the upper bound of the interval coefficient by z^4 from the polynomial (10) is markedly extended while the rest of the polynomial remains the same, i.e. [14]:

$$p_2(z,q) = [1,2] + [3,4]z + [5,6]z^2 + [7,8]z^3 + [9,19]z^4 + \cdots \\ \cdots [11,12]z^5 \quad (11)$$

The trivial modifications of the both Matlab codes from the previous example (change of "q4=9:0.5:10" to "q4=9:0.5:19" in "parameter gridding" approach and change of "q5=0:1" to "q5=0:10" in "convex hull" approach) lead to the value sets depicted in figs. 6 and 7.

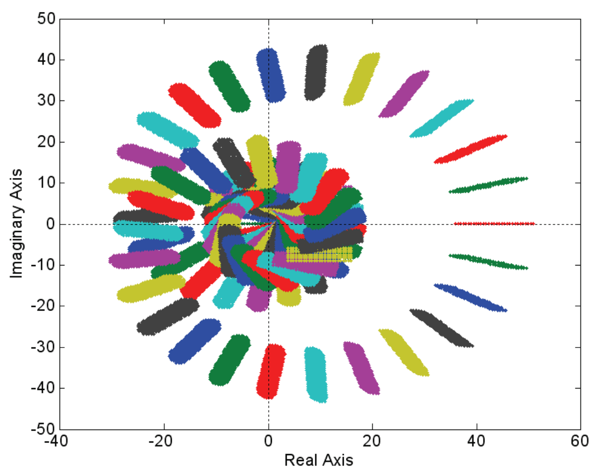

Fig. 6 value sets for the family (11) – "parameter gridding" approach – robustly unstable case

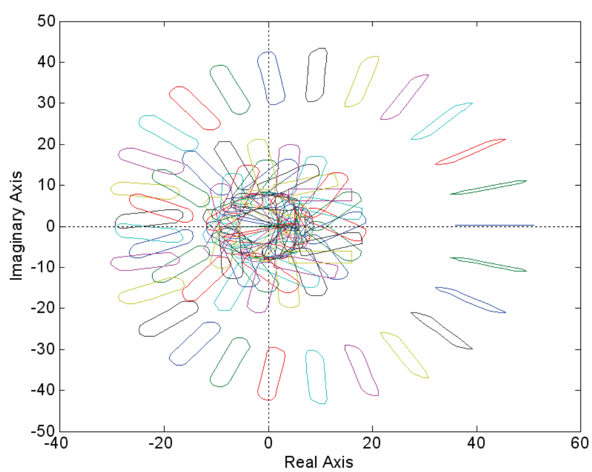

Fig. 7 value sets for the family (11) – "convex hull" approach – robustly unstable case

As can be seen from both fig. 6 and fig. 7, the complex plane origin is included in the value sets. Thus, the family (11) is not robustly stable.

C. Affine Linear Uncertainty Structure

Next, the family with so-called affine linear uncertainty structure [13], [19] will be investigated. Mutually dependent coefficients in this structure are affine linear functions, i.e. the uncertain parameters can be only in the first power and must not be multiplied mutually. For example, the family with affine linear uncertainty structure is considered in the form:

$$p(z,q) = (3q_1 + 2q_2 - q_3 + 5)z^3 + (2q_1 - q_2 + 4)z^2 + \cdots \\ \cdots (q_1 + 3q_3 + 3)z + (2q_1 - 2q_2 + 3q_3 + 1); \\ |q_i| \le 0.3 \text{ for } i = 1, 2, 3 \quad (12)$$

The simple Matlab programs for plotting the corresponding value sets are similar to the codes presented in the interval uncertainty section. First, the depiction based on the brute-force parameter gridding can by obtained by using:

```
%discrete-time polynomial - affine linear uncertainty structure
%parameter gridding
%stable case
clear all
hold all
for c=0:0.01:1 %generalized frequency range
  count=1; %auxiliary counter
  for q1=-0.3:0.05:0.3 %sampling of uncertain parameters
    for q2=-0.3:0.05:0.3
      for q3=-0.3:0.05:0.3
        z=exp(j*c*2*pi); %unit circle
        p(count)=(3*q1+2*q2-q3+5)*z^3+(2*q1-q2+4)*z^2+...
        (q1+3*q3+3)*z+(2*q1-2*q2+3*q3+1); % the polynomial
        count=count+1; %counter increment
      end
    end
  end
  x=real(p); %real part
  y=imag(p); %imaginary part
  plot(x,y,'.')
end
hold off
```

The resulting plot is shown in fig. 8. Now, the visualized value set for each frequency consists of $13^3 = 2197$ points (13 "sampled" values for each of 3 uncertain coefficients).

The origin of the complex plane is not included in the value sets and the family (12) contains a stable member (e.g. for $q_1 = q_2 = q_3 = 0$ which means $p(z,0) = 5z^3 + 4z^2 + 3z + 1$). Thus, the family is robustly stable.

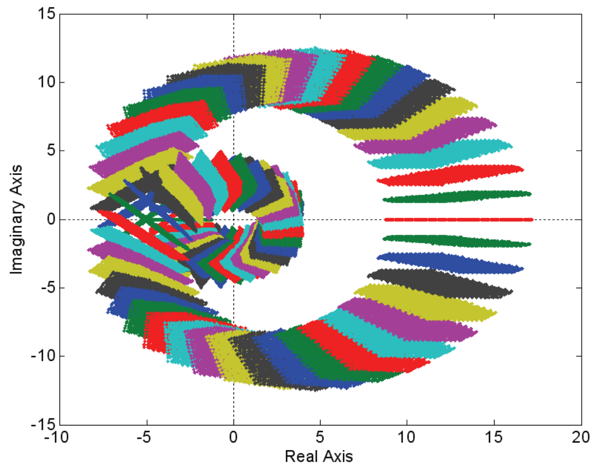

Fig. 8 value sets for the family (12) – "parameter gridding" approach – robustly stable case

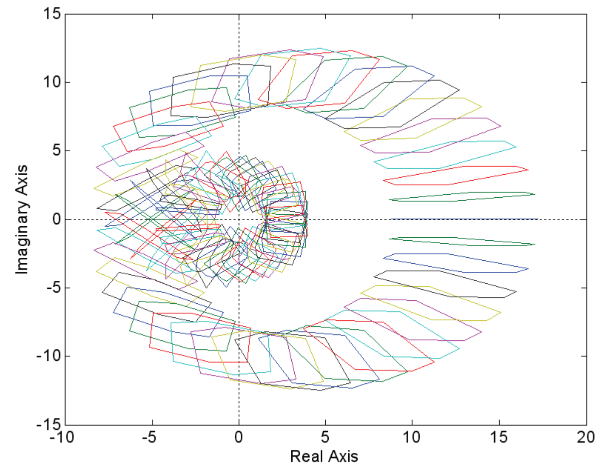

Fig. 9 value sets for the family (12) – "convex hull" approach – robustly stable case

Then, the modification of the Matlab code based on construction of generators and their convex hulls can look like:

```
%discrete-time polynomial - affine linear uncertainty structure
%convex hull of 8 generators
%stable case
clear all
hold all
for c=0.00001:0.01:1 %generalized frequency range
  count=1; %auxiliary counter
  for q1=-0.3:0.6:0.3 %uncertain parameters for 2^3=8...
  %generators
    for q2=-0.3:0.6:0.3
      for q3=-0.3:0.6:0.3
        z=exp(j*c*2*pi); %unit circle
        p(count)=(3*q1+2*q2-q3+5)*z^3+(2*q1-q2+4)*z^2+...
        (q1+3*q3+3)*z+(2*q1-2*q2+3*q3+1); % the polynomial
        count=count+1; %counter increment
      end
    end
  end
  x=real(p); %real part
  y=imag(p); %imaginary part
  k=convhull(x,y); %convex hull
  plot(x(k),y(k))
end
hold off
```

The fig. 9 shows the final value sets (which are the same as in fig. 8).

The wider bounds in the same polynomial with affine linear uncertainty structure (12):

$$\begin{aligned} & p(z,q) = (3q_1 + 2q_2 - q_3 + 5)z^3 + (2q_1 - q_2 + 4)z^2 + \cdots \\ & \cdots (q_1 + 3q_3 + 3)z + (2q_1 - 2q_2 + 3q_3 + 1); \\ & |q_i| \leq 0.5 \text{ for } i = 1, 2, 3 \end{aligned} \tag{13}$$

and appropriate modifications of presented Matlab mini-programs result in the value sets which are shown in figs. 10 and 11.

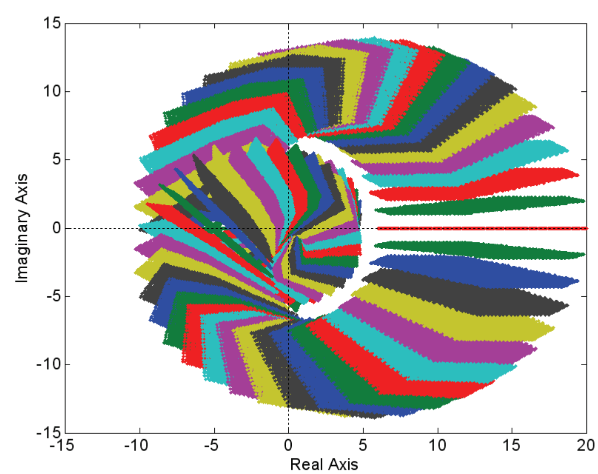

Fig. 10 value sets for the family (13) – "parameter gridding" approach – robustly unstable case

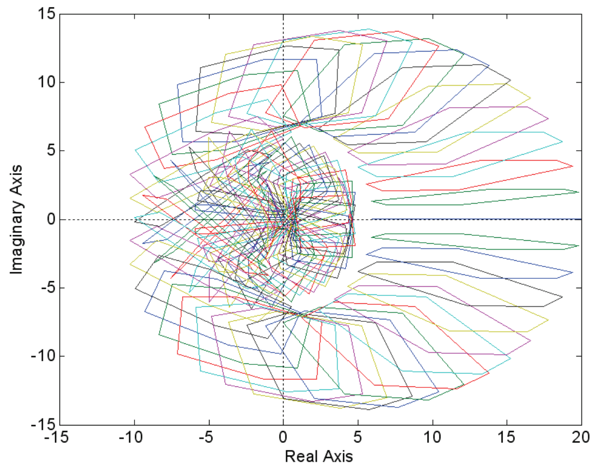

Fig. 11 value sets for the family (13) – "convex hull" approach – robustly unstable case

Obviously, the zero point is included in the value sets and thus the family (13) is robustly unstable.

D. More Complicated Uncertainty Structures

The more complicated relation among uncertain coefficients the more complex shape of the value sets [13], [19]. The multilinear and more general uncertainty structures have the value sets which are not convex anymore, but the investigation itself does not change technically. This part presents the example of family with such complicated uncertainty structure. More specifically, the family with polynomial uncertainty structure is supposed:

$$\begin{aligned} p(z,q) &= 5z^4 + \left(q_1 - q_2^2 + 4\right)z^3 + \left(q_1^2 + q_1 q_2^2 + 3\right)z^2 + \cdots \\ &\cdots \left(2q_1^3 + 3q_2^3 + 2\right)z + \left(3q_1^2 q_2^2 + 1\right); \\ &|q_i| \le 0.7 \;\text{ for }\; i = 1,2 \end{aligned} \tag{14}$$

In this case of non-convex value sets, only the brute-force parameter gridding approach is utilized. The relevant Matlab code can look like:

```
%discrete-time polynomial - polynomial uncertainty structure
%parameter gridding
%stable case
clear all
hold all
for c=0:0.01:1 %generalized frequency range
  count=1; %auxiliary counter
  for q1=-0.7:0.01:0.7 %sampling of uncertain parameters
    for q2=-0.7:0.01:0.7
      z=exp(j*c*2*pi); %unit circle
      p(count)= 5*z^4+(q1-q2^2+4)*z^3+...
      (q1^2+q1*q2^2+3)*z^2+(2*q1^3+3*q2^3+2)*z+...
      (3*q1^2*q2^2+1); % the polynomial
      count=count+1; %counter increment
    end
  end
  x=real(p); %real part
  y=imag(p); %imaginary part
  plot(x,y,'.')
end
hold off
```

The obtained non-convex value sets are depicted in fig. 12.

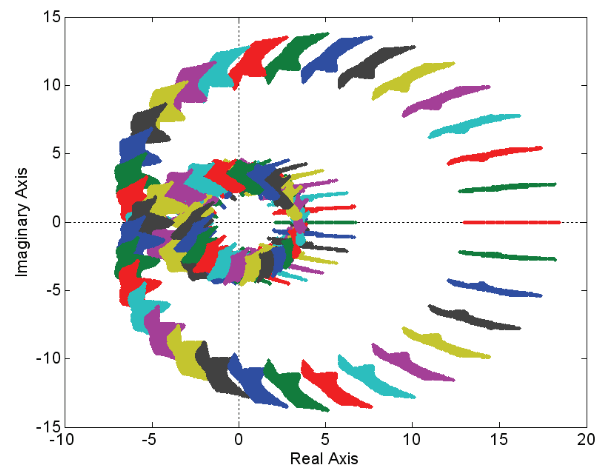

Fig. 12 value sets for the family (14) – robustly unstable case

Thanks to the position of the value sets and the complex plane origin and thanks to the fact that the family has a stable member one can say the family with polynomial uncertainty structure (14) is robustly stable.

V. Conclusion

This paper has been focused on relatively simple but powerful and very general graphical approach to investigation of robust stability for discrete-time polynomials with various uncertainty structures. The short programs, presented within the scope of the illustrative examples, could serve as an inspiration tool for practical execution of the stability tests in Matlab environment.

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