C^1 Positive Bernstein-Bézier Rational Quartic Interpolation

Maria Hussain, Malik Zawwar Hussain and Maryam Buttar

Abstract **–** A positivity preserving C^1 rational quartic Bernstein-Bézier interpolation scheme is developed for scattered data. The constraints are developed on the Bézier ordinates to preserve the positive shape of scattered data. The weight functions are free to improve the shape of the surface. The developed scheme has more degrees of freedom as compared to existing rational positivity preserving interpolation schemes. It is local and equally applicable to data as well as data with derivatives.

Keywords- C^1 Bernstein-Bézier rational quartic function, positivity, scattered data interpolation, weight functions, Bézier ordinates.

I. INTRODUCTION

Scattered data originates from a number of applications including earth sciences, meteorology, electronic imaging, reverse engineering, industrial design, 3D photography, ship building, aeronautics and medical modelling of human body parts. The data can be convex, monotone or positive partially or over the whole domain. However, the problem under consideration here is positivity of scattered data.

The problem of positivity preserving interpolation of scattered data has been discussed by many authors. Some of the noticeable contributions are reviewed here.

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Brodlie, Asim and Unsworth [2] discussed the problem of visualization of data by using modified quadratic Shepard method. To assure the positivity of interpolant, sufficient conditions were imposed on quadratic basis functions. To interpolate the fractional data, they inhibited the interpolant within the limits [0,1]. The generalized form of the interpolant was produced. In [6], Hussain and Hussain developed a positivity preserving interpolation scheme which was based on Coons patches. A rational function with shape parameters (Hermite form) was used to interpolate each boundary of triangle. The side-vertex interpolation was also performed by a rational function (Hermite form). The final surface patch was the convex combination of these side-vertex interpolant. The authors in [6] developed data dependent constraints on shape parameters to preserve the positive shape of data. Hussain and Hussain [7] developed a C^1 Bernstein-Bézier rational cubic scheme to preserve the positive shape of scattered data. Luo and Peng [8] presented a range restricted C^1 rational spline interpolation scheme for scattered data. The rational spline interpolant was represented as a convex combination of three cubic Bézier triangular patches. The range restricted obstacles were the polynomial surfaces of degree up to three. Mulansky and Schmidt [9] proposed a range restricted C^1 quadratic spline interpolation scheme for scattered data. The domain was triangulated by the Powell-Sabin refinement. The solution to the problem came out as a solvable system of linear inequalities for the gradients as parameters. The inequalities were solved by a global quadratic optimization problem, where the thin plate functional was the objective function and the system of inequalities were the constraints. Herrmann, Mulansky and Schmidt [4] proposed a univariate and bivariate C^1 quadratic range restricted interpolation scheme for scattered data. In [9], the range

restricting constraints were piecewise constants, whereas, in [4] the piecewise quadratic functions were the lower and upper obstacles to the value of interpolant.

In this paper, a C^1 Bernstein-Bézier rational quartic interpolation scheme is developed to preserve the positive shape of scattered data. The rational quartic Bernstein-Bézier function has 15 Bézier ordinates and 15 weight functions. Due to high number of Bézier ordinates(control points), the interpolated surface is closer to the shape of control polygon as compared to lesser degree interpolating functions. The weight functions provide degrees of freedom for surface's shape improvement if desired. The constraints are developed on the Bézier ordinates to preserve the positive shape of scattered data. If for any triangular patch, the Bézier ordinates fail to obey the developed constraints then weights functions are modified to preserve positive surface through positive scattered data.

The advantages of the proposed rational quartic interpolation scheme to the existing ones are as follows: In [4], [8] and [9], constraints were developed on the derivatives at knots to assure positivity. The proposed positivity preserving rational quartic interpolation scheme of this paper keeps the derivatives unconstrained, hence equally applicable to data as well as data with derivatives. In [7], a positivity preserving rational cubic Bernstein-Bézier interpolation scheme was proposed. The rational cubic function of [7] had ten Bézier ordinates, whereas, the rational quartic scheme of this paper has fifteen Bézier ordinates. It follows that the positive surface developed by rational quartic Bernstein-Bézier function is closer to the control polygon as compare to the rational cubic interpolant [7]. Moreover, it has fifteen weight functions but cubic interpolant [7] had ten weight functions. Thus rational quartic interpolant has more degrees of freedom as compared to [7]. Unlike [2], the proposed rational quartic interpolation scheme of this paper is local.

The remaining paper is organized as follows: The Section II presents the Bernstein-Bézier rational quartic function. The problem of positivity and C^1 continuity are discussed in Section III and Section IV respectively. Section VI concludes the paper.

II. BERNSTEIN-BÉZIER RATIONAL QUARTIC FUNCTION

The Bernstein-Bézier functions can be used to interpolate the scattered data only if the data is arranged over the triangular grid (domain is union of non-intersecting triangles). The rational Bernstein-Bézier functions possess free parameters known as weight functions. The smoothness of interpolated surface is obliged to suitable choice of weight functions. The Bernstein-Bézier rational quartic function is defined as:

$$P(u,v,w)=\frac{P_1(u,v,w)}{P_2(u,v,w)}, \qquad (1)$$

where

$$P_1(u,v,w)=\sum_{i+j+k=4} u^i v^j w^k \omega_{ijk} b_{ijk} ,$$

$$P_2(u,v,w)=\sum_{i+j+k=4} u^i v^j w^k \omega_{ijk} .$$

ω_{ijk}, $i+j+k=4$ are the weight functions. b_{400}, b_{040} and b_{004} are the Bézier ordinates at the three vertices of triangle; b_{031}, b_{022}, b_{013}, b_{301}, b_{202}, b_{103}, b_{310}, b_{220} and b_{130} are the boundary Bézier ordinates; b_{211}, b_{121}, b_{112} are the inner Bézier ordinate and u, v, w are the barycentric coordinates.

The barycentric coordinates with respect to the triangle $\Delta V_1V_2V_3$ can be computed from the following convex combination:

$$V = uV_1 + vV_2 + wV_3,$$

where $u+v+w=1$ and $u>0, v>0, w>0$. $V(x,y)$ is an arbitrary point of the triangle $\Delta V_1V_2V_3$ with vertices $V_1(x_1,y_1)$, $V_2(x_2,y_2)$ and $V_3(x_3,y_3)$.

III. POSITIVE BERNSTEIN-BÉZIER RATIONAL QUARTIC FUNCTION

In this section, the sufficient conditions are derived on the ordinates of Bernstein-Bézier rational quartic function (1) to preserve the shape of positive scattered data.

It can be easily observed that $P(u,v,w)>0$ if $P_1(u,v,w)>0$ and $P_2(u,v,w)>0$. We assume that the Bézier ordinates at the vertices are strictly positive i.e. $b_{400}>0$, $b_{040}>0$ and $b_{004}>0$. The sufficient conditions on the remaining Bézier ordinates are derived to ensure the positivity of the entire patch.

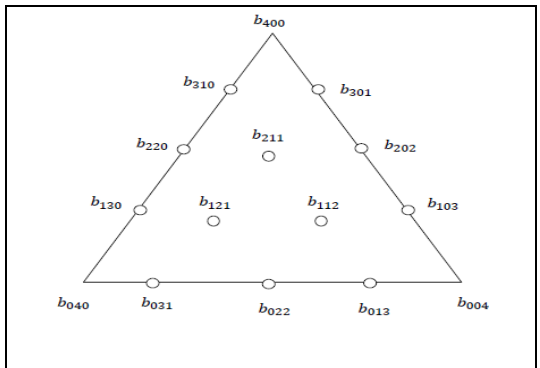

Fig. 1 Distribution of the Bézier ordinates of the Bernstein-Bézier rational quartic function.

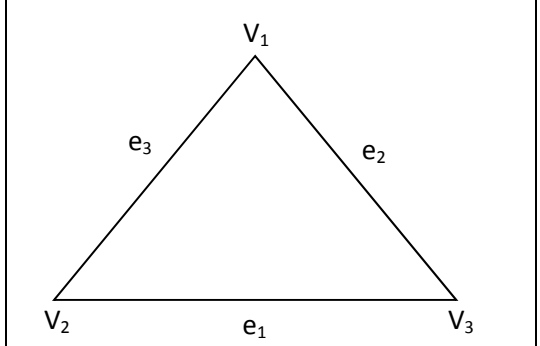

Fig. 2 Location of edges of $\Delta V_1V_2V_3$.

Let

$$b_{400}=L,\quad b_{040}=M,\quad b_{004}=N, \tag{2}$$

where $L>0$, $M>0$ and $N>0$. It is assumed that all the boundary Bézier ordinates have the same value i.e.

$$b_{310}=b_{130}=b_{301}=b_{103}=b_{031}=b_{013}=b_{220}=b_{202}$$
$$=b_{022}=b_{211}=b_{121}=b_{112}=-r, \tag{3}$$

where $r>0$. The weight functions are chosen in the following way:

$$\omega_{400}=l,\ \omega_{040}=m,\ \omega_{004}=n,\ \omega_{ijk}=t,$$
$$\text{where } i+j+k=4, i\neq 4, j\neq 4, k\neq 4, \tag{4}$$
$$l>0,\ m>0,\ n>0 \text{ and } t>0.$$

The positivity of $P_1(u,v,w)$ depends upon the Bézier ordinates b_{ijk}, $\forall i,j,k$, $i+j+k=4$. Substituting the values of Bézier ordinates and weight functions from (2)-(4) into $P_1(u,v,w)$, it is rewritten as:

$$P_1(u,v,w)=u^4lL+v^4mM+w^4nN$$
$$-rt\left(4u^3v+4uv^3+4u^3w+4uw^3+4v^3w+4vw^3\right.$$
$$+6u^2v^2+6u^2w^2+6v^2w^2+12u^2vw+12uv^2w$$
$$\left.+12uvw^2\right). \tag{5}$$

Since,

$$4u^3v+4uv^3+4u^3w+4uw^3+4v^3w+4vw^3+6u^2v^2$$
$$+6u^2w^2+6v^2w^2+12u^2vw+12uv^2w+12uvw^2$$
$$=1-u^4-v^4-w^4.$$

Utilizing the above expression in (5), $P_1(u,v,w)$ takes the form

$$P_1(u,v,w)=u^4(lL+rt)+v^4(mM+rt)$$
$$+w^4(nN+rt)-rt. \tag{6}$$

From (5) it is clear that when $r=0$, $P_1(u,v,w)>0$, when rt increases, $P_1(u,v,w)$ decreases.

Assume that l, m, n, r, and t are fixed, at the minimum value of $P_1(u,v,w)$, the following conditions hold:

$$\frac{\partial P_1}{\partial u}-\frac{\partial P_1}{\partial v}=0, \tag{7}$$

$$\frac{\partial P_1}{\partial u}-\frac{\partial P_1}{\partial w}=0. \tag{8}$$

Substituting the value of $P_1(u,v,w)$ from (6) in to (7) and (8) the following ratio is obtained

$$u:v:w=\frac{1}{(lL+rt)^{\frac{1}{3}}}:\frac{1}{(mM+rt)^{\frac{1}{3}}}:\frac{1}{(nN+rt)^{\frac{1}{3}}}. \tag{9}$$

Since $u+v+w=1$, the ratio involved in (9) provides the following values of barycentric coordinates u, v and w

$$u=\frac{1}{S\sqrt[3]{lL+rt}},\ v=\frac{1}{S\sqrt[3]{mM+rt}},$$

$$w=\frac{1}{S\sqrt[3]{nN+rt}}.$$

$$S=\frac{1}{\sqrt[3]{lL+rt}}+\frac{1}{\sqrt[3]{mM+rt}}+\frac{1}{\sqrt[3]{nN+rt}}.$$

Substituting these u, v and w into (6), we obtain the minimum value of $P_1(u,v,w)$.

$$P_1(u,v,w)=\frac{1}{\left(\frac{1}{\sqrt[3]{lL+rt}}+\frac{1}{\sqrt[3]{mM+rt}}+\frac{1}{\sqrt[3]{nN+rt}}\right)^3}-rt. \tag{10}$$

It is clear from (10), $P_1(u,v,w)>0$, when $r=0$ and $P_1(u,v,w)$ decreases as r increases. Equation (10) is rewritten as:

$$P_1(u,v,w)=rt\left(\frac{1}{R}-1\right), \tag{11}$$

where

$$R=\left(\left(\frac{lL}{rt}+1\right)^{\frac{-1}{3}}+\left(\frac{mM}{rt}+1\right)^{\frac{-1}{3}}+\left(\frac{nN}{rt}+1\right)^{\frac{-1}{3}}\right)^3.$$

Replacing $\frac{1}{r}$ by $m_1>0$ in (11), we have

$$P_1(u,v,w)=\frac{t}{m_1}\left(\frac{1}{R_1}-1\right), \tag{12}$$

where

$$R_1=\left(\left(\frac{m_1lL}{t}+1\right)^{\frac{-1}{3}}+\left(\frac{m_1mM}{t}+1\right)^{\frac{-1}{3}}+\left(\frac{m_1nN}{t}+1\right)^{\frac{-1}{3}}\right)^3.$$

To assure positivity of $P_1(u,v,w)$, we shall estimate minimum value of $r=\frac{1}{m_1}$. Solving Equation (12) for m_1 we have

$$F(m_1)=1, \tag{13}$$

where

$$F(m_1)=\frac{1}{\sqrt[3]{\frac{m_1lL}{t}+1}}+\frac{1}{\sqrt[3]{\frac{m_1mM}{t}+1}}+\frac{1}{\sqrt[3]{\frac{m_1nN}{t}+1}}.$$

It is observed that for $l>0$, $m>0$, $n>0$, $L>0$, $M>0$, $N>0$, $t>0$ and $m_1>0$, we have $F'(m_1)<0$ and $F''(m_1)>0$. Since $F''(m_1)>0$ $\forall m_1$, $F(m_1)$ is convex over the whole domain. This observation reports that $F(m_1)$ must have one minima over the entire interval. Solving the Equation (13) we shall find the required value of m_1. The Equation (13) can solved from any numerical technique of estimation of roots, however here it is solved through Newton-Raphson method. The required lower bound of Bézier ordinates b_{310}, b_{130}, b_{301}, b_{103}, b_{031}, b_{013}, b_{220}, b_{202}, b_{022}, b_{211}, b_{121} and b_{112} is $-r=-\frac{1}{m_1}$.

The above discussion can be summarized as:

Theorem 1. The Bernstein- Bézier rational quartic function (1) with $b_{400}=L$, $b_{040}=M$, $b_{004}=N$ where $L>0$, $M>0$ and $N>0$ is positive $(P(u,v,w)>0,\ \forall u,v,w)$ if the Bézier ordinates satisfy the relation b_{310}, b_{130}, b_{301}, b_{103}, b_{031}, b_{013}, b_{220}, b_{202}, b_{022}, b_{211}, b_{121}, $b_{112}\geq -r$, r is obtained by solving the Equation (13) where $r=\frac{1}{m_1}$. □

IV. C^1 POSITIVE BERNSTEIN-BÉZIER RATIONAL QUARTIC SURFACE

The construction of C^1 positivity preserving surface for the scattered data (x_i, y_i, z_i), $i = 1,2,3,...,N$, where $z_i > 0$ involves the following steps:

1. Triangulation of the domain data (x_i, y_i), $i = 1,2,3,...,N$.
2. Initialization of partial derivatives.
3. Creation of the final surface which comprises of quartic Bézier triangular patches each of which is guaranteed to remain positive.

The Bernstein-Bézier rational quartic triangular patch is C^1 at the vertices of the triangle if the boundary Bézier ordinates are:

$$\left.\begin{aligned}
b_{031} &= b_{040} + \frac{\omega_{040}}{4\omega_{031}}\overrightarrow{V_2V_3} \bullet \nabla f\big|_{V_2} \\
b_{130} &= b_{040} - \frac{\omega_{040}}{4\omega_{130}}\overrightarrow{V_1V_2} \bullet \nabla f\big|_{V_2} \\
b_{103} &= b_{004} + \frac{\omega_{004}}{4\omega_{103}}\overrightarrow{V_3V_1} \bullet \nabla f\big|_{V_3} \\
b_{013} &= b_{004} - \frac{\omega_{004}}{4\omega_{013}}\overrightarrow{V_2V_3} \bullet \nabla f\big|_{V_3} \\
b_{310} &= b_{400} + \frac{\omega_{400}}{4\omega_{310}}\overrightarrow{V_1V_2} \bullet \nabla f\big|_{V_1} \\
b_{301} &= b_{400} - \frac{\omega_{400}}{4\omega_{301}}\overrightarrow{V_3V_1} \bullet \nabla f\big|_{V_1}
\end{aligned}\right\} \qquad (14)$$

The values of boundary Bézier ordinates are calculated by adopting the approach of [7]. For an arbitrary value of weight function the boundary Bézier ordinates defined in (14) may not obey the lower bounds developed in Theorem 1. From (14), it is clear that the positivity condition proposed in Theorem 1 will be satisfied by the boundary Bézier ordinates, either by increasing the weights ω_{310}, ω_{130}, ω_{301}, ω_{103}, ω_{031}, ω_{013}, ω_{220}, ω_{202}, ω_{022} or by decreasing the weights ω_{400}, ω_{040} and ω_{004}. In such a way, the derived positivity conditions are satisfied e.g. $b_{031} > -r$ or

$$b_{040} + \frac{\omega_{040}}{4\omega_{031}}\left[(x_3 - x_2)\frac{\partial f}{\partial x}(V_2) + (y_3 - y_2)\frac{\partial f}{\partial y}(V_2)\right] > -r.$$

The weights are involved in the positivity preservation of data. ω_{310}, ω_{130}, ω_{301}, ω_{103}, ω_{031}, ω_{013}, ω_{220}, ω_{202} and ω_{022} are local to each triangle, thus the change in the values of these weights will not affect the C^1 continuity at the vertices.

A. Inner Bézier ordinates

For each triangle, now we will find the inner Bézier points. This needs to be done in order to ensure C^1 continuity across patch boundary. In this scheme each triangular surface patch is a convex combination of three quartic Bézier patches. The final surface patch is the convex combination of the triangular patches P^i, $i = 1,2,3$. The triangular patches have same vertex and boundary Bézier ordinates but different inner Bezier ordinates. The constraints are determined on the inner Bézier ordinates to ensure C^1 continuity along the edges e_i, $i = 1,2,3$ of triangle.

The final surface patch P is the convex combination of triangular patches P^i, $i = 1,2,3$. The final surface P is expressed as:

$$P = C_1P^1 + C_2P^2 + C_3P^3, \qquad (15)$$

where

$$P^1 = \frac{R + R_1}{D}, \quad P^2 = \frac{R + R_2}{D}, \quad P^3 = \frac{R + R_3}{D},$$

$$\begin{aligned}
R &= u^4\omega_{400}b_{400} + v^4\omega_{040}b_{040} + w^4\omega_{004}b_{004} \\
&\quad + 4u^3v\omega_{310}b_{310} + 4uv^3\omega_{130}b_{130} + 4u^3w\omega_{301}b_{301} \\
&\quad + 4uw^3\omega_{103}b_{103} + 4vw^3\omega_{031}b_{031} + 4vw^3\omega_{013}b_{013} \\
&\quad + 6u^2v^2\omega_{220}b_{220} + 6u^2w^2\omega_{202}b_{202} + 6v^2w^2\omega_{022}b_{022},
\end{aligned}$$

$$\begin{aligned}
R_1 &= 12u^2vw\omega_{211}b^1{}_{211} + 12uv^2w\omega_{121}b^1{}_{121} + 12uvw^2\omega_{112}b^1{}_{112}, \\
R_2 &= 12u^2vw\omega_{211}b^2{}_{211} + 12uv^2w\omega_{121}b^2{}_{121} + 12uvw^2\omega_{112}b^2{}_{112}, \\
R_3 &= 12u^2vw\omega_{211}b^3{}_{211} + 12uv^2w\omega_{121}b^3{}_{121} + 12uvw^2\omega_{112}b^3{}_{112},
\end{aligned}$$

$$D = u^4\omega_{400} + v^4\omega_{040} + w^4\omega_{004} + 4u^3v\omega_{310} + 4uv^3\omega_{130} + 4u^3w\omega_{301} + 4uw^3\omega_{103} + 4vw^3\omega_{031} + 4vw^3\omega_{013} + 6u^2v^2\omega_{220} + 6u^2w^2\omega_{202} + 6v^2w^2\omega_{022} + 12u^2vw\omega_{211} + 12uv^2w\omega_{121} + 12uvw^2\omega_{112},$$

$$C_1 = \frac{v^2w^2}{v^2w^2 + u^2w^2 + u^2v^2},$$

$$C_2 = \frac{u^2w^2}{v^2w^2 + u^2w^2 + u^2v^2},$$

$$C_3 = \frac{u^2v^2}{v^2w^2 + u^2w^2 + u^2v^2}.$$

Substituting the values of P^1, P^2 and P^3, we have:

$$P(u,v,w) = \frac{R + 12u^2vw\omega_{211}b_{211} + 12uv^2w\omega_{121}b_{121} + 12uvw^2\omega_{112}b_{112}}{D} \tag{16}$$

where

$$b_{211} = \frac{v^2w^2b^1_{211} + u^2w^2b^2_{211} + u^2v^2b^3_{211}}{v^2w^2 + u^2w^2 + u^2v^2},$$

$$b_{121} = \frac{v^2w^2b^1_{121} + u^2w^2b^2_{121} + u^2v^2b^3_{121}}{v^2w^2 + u^2w^2 + u^2v^2},$$

$$b_{112} = \frac{v^2w^2b^1_{112} + u^2w^2b^2_{112} + u^2v^2b^3_{112}}{v^2w^2 + u^2w^2 + u^2v^2}.$$

To determine the values of b^i_{211}, b^i_{121} and b^i_{112} we use the value of normal derivatives $\frac{\partial P}{\partial n_i}$, $i = 1,2,3$ along the edges e_i, $i = 1,2,3$.

Hence the inward normal derivative of P i.e. $\frac{\partial P}{\partial n_1}$ along the e_1 is

$$\frac{\partial P}{\partial n_1} = \frac{\partial P}{\partial u}(1) + \frac{\partial P}{\partial v}\left(\frac{-1}{2}\right) + \frac{\partial P}{\partial w}\left(\frac{-1}{2}\right) = \frac{\partial P}{\partial u} - \frac{1}{2}\frac{\partial P}{\partial v} - \frac{1}{2}\frac{\partial P}{\partial w}. \tag{17}$$

$n_1 = \left(1, \frac{-1}{2}, \frac{-1}{2}\right)$ is the inward normal to the edge e_1. Similarly the inward normal to the remaining edges are $n_2 = \left(\frac{-1}{2}, 1, \frac{-1}{2}\right)$ and $n_3 = \left(\frac{-1}{2}, \frac{-1}{2}, 1\right)$.

Substituting the values of $\frac{\partial P}{\partial u}$, $\frac{\partial P}{\partial v}$ and $\frac{\partial P}{\partial w}$ in (17) and after some simplification we have

$$\frac{\partial P}{\partial n_1} = \frac{A_1v^7 + B_1v^6w + C_1v^5w^2 + D_1v^4w^3 + E_1v^3w^4 + F_1v^2w^5 + G_1vw^6 + H_1w^7}{\left(a_1v^4 + b_1v^3w + c_1v^2w^2 + d_1vw^3 + e_1w^4\right)^2}, \tag{18}$$

where

$$A_1 = 4\omega_{040}\omega_{130}b_{130} - 2\omega_{040}\omega_{031}b_{031} + \left(2\omega_{040}\omega_{040} + 2\omega_{040}\omega_{400} - 4\omega_{040}\omega_{130} - 2\omega_{040}\omega_{031}\right)b_{040},$$

$$B_1 = 16\omega_{031}\omega_{130}b_{130} + 12\omega_{040}\omega_{121}b_{121} - 6\omega_{040}\omega_{022}b_{022} + \left(6\omega_{031}\omega_{040} - 12\omega_{121}\omega_{040} - 8\omega_{031}\omega_{040} + 6\omega_{022}\omega_{040}\right)b_{040} + \left(8\omega_{040}\omega_{031} - 6\omega_{400}\omega_{031} - 16\omega_{130}\omega_{031}\right)b_{031},$$

$$C_1 = 12\omega_{040}\omega_{112}b_{112} + 48\omega_{031}\omega_{121}b_{121} + 24\omega_{022}\omega_{130}b_{130} + \left(2\omega_{013}\omega_{040} - 6\omega_{022}\omega_{040} - 12\omega_{112}\omega_{040}\right)b_{040} - \left(24\omega_{130}\omega_{022} + 6\omega_{400}\omega_{022} - 12\omega_{040}\omega_{022} + 12\omega_{031}\omega_{022}\right)b_{022} + 12\omega_{022}\omega_{031}b_{031},$$

$$D_1 = 4\omega_{040}\omega_{103}b_{103} + 48\omega_{031}\omega_{112}b_{112} + 16\omega_{013}\omega_{130}b_{130} + 72\omega_{022}\omega_{121}b_{121} - 2\omega_{040}\omega_{004}b_{004} + \left(2\omega_{013}\omega_{040} + 2\omega_{004}\omega_{040} - 12\omega_{103}\omega_{040}\right)b_{040} - \left(48\omega_{112}\omega_{031} + 12\omega_{022}\omega_{031}\right)b_{031} + \left(8\omega_{040}\omega_{013} - 16\omega_{031}\omega_{013} - 16\omega_{130}\omega_{013} - 2\omega_{400}\omega_{013}\right)b_{013} + \left(12\omega_{031}\omega_{022} - 72\omega_{121}\omega_{022}\right)b_{022},$$

$$E_1 = 4\omega_{004}\omega_{130}b_{130} + 16\omega_{031}\omega_{103}b_{103} + 48\omega_{013}\omega_{121}b_{121} + 72\omega_{022}\omega_{112}b_{112} - 2\omega_{004}\omega_{040}b_{040} + \left(2\omega_{040}\omega_{004} - 4\omega_{130}\omega_{004} - 6\omega_{031}\omega_{004}\right)b_{004} + \left(8\omega_{013}\omega_{031} + 6\omega_{004}\omega_{031} - 40\omega_{103}\omega_{031}\right)b_{031} + \left(16\omega_{031}\omega_{013} - 48\omega_{121}\omega_{013} - 12\omega_{022}\omega_{013}\right)b_{013} - \left(72\omega_{112}\omega_{022} + 12\omega_{012}\omega_{022}\right)b_{022},$$

$$F_1 = 12\omega_{004}\omega_{121}b_{121} + 48\omega_{013}\omega_{112}b_{112} + 24\omega_{022}\omega_{103}b_{103} - 6\omega_{004}\omega_{031}b_{031} + \left(6\omega_{031}\omega_{004} - 12\omega_{121}\omega_{004} - 6\omega_{022}\omega_{004}\right)b_{004} + \left(12\omega_{022}\omega_{013} - 48\omega_{112}\omega_{013} - 16\omega_{013}\omega_{013}\right)b_{013} + \left(6\omega_{004}\omega_{022} - 48\omega_{103}\omega_{022} + 12\omega_{013}\omega_{022}\right)b_{022},$$

$$G_1 = 12\omega_{004}\omega_{112}b_{112} + 16\omega_{013}\omega_{103}b_{103} - 6\omega_{004}\omega_{022}b_{022}$$
$$+\left(6\omega_{022}\omega_{004} - 6\omega_{013}\omega_{004} - 12\omega_{112}\omega_{004}\right)b_{004}$$
$$+\left(2\omega_{004}\omega_{013} + 8\omega_{013}\omega_{013} - 24\omega_{103}\omega_{013}\right)b_{013},$$
$$H_1 = 4\omega_{004}\omega_{103}b_{103} - 2\omega_{004}\omega_{013}b_{013}$$
$$+\left(2\omega_{013}\omega_{004} - 4\omega_{103}\omega_{004}\right)b_{004}.$$

Since we are interested in the linear value of $\frac{\partial P}{\partial n_1}$ (both numerator and denominator of (18) should be linear). Numerator of (18) is made linear by the following substitutions:

$$B_1 = 6A_1 + H_1,\ C_1 = 15A_1 + 6H_1,$$
$$D_1 = 20A_1 + 15H_1,\ E_1 = 15A_1 + 20H_1,$$
$$F_1 = 6A_1 + 15H_1,$$
$$G_1 = A_1 + 6H_1.$$

Similarly for the denominator, where $a_1 = \omega_{040}$, $b_1 = 4\omega_{031}$, $c_1 = 6\omega_{022}$, $d_1 = 4\omega_{013}$, $e_1 = \omega_{004}$.
The expression
$\left(a_1v^4 + b_1v^3w + c_1v^2w^2 + d_1vw^3 + e_1w^4\right)^2$ is linear if

$c_1 = a_1e_1$, where $c_1 = \frac{a_1^{\ 2} + e_1^{\ 2}}{2}$,

$a_1 = e_1$ i.e. $\omega_{040} = \omega_{004}$.

The above computation leads to the following system of equations:

$$\omega_{031} = \frac{3\omega_{040} + \omega_{004}}{4},\ \omega_{022} = \frac{\omega_{040} + \omega_{004}}{2},$$
$$\omega_{013} = \frac{\omega_{040} + 3\omega_{004}}{4},\ \omega_{301} = \frac{3\omega_{400} + \omega_{004}}{4},$$
$$\omega_{202} = \frac{\omega_{400} + \omega_{004}}{2},\ \omega_{103} = \frac{\omega_{400} + 3\omega_{004}}{4},$$
$$\omega_{310} = \frac{3\omega_{400} + \omega_{040}}{4},\ \omega_{220} = \frac{\omega_{400} + \omega_{040}}{2},$$
$$\omega_{130} = \frac{\omega_{400} + 3\omega_{040}}{4}.$$

From the above discussion it is clear that only three parameters $\left(\omega_{400}, \omega_{040}, \omega_{004}\right)$ are free while the values of remaining are dependent on these parameters.

Let $\omega_{400} = \omega_{040} = \omega_{004} = \omega_{013} = \omega_{103} = \omega_{130} = \omega_{031} = \omega_{301} = \omega_{310} = \omega_{022} = \omega_{202} = \omega_{220} = k$, where k, is any positive real number.

Similarly the normal derivatives along the other edges i.e. $\frac{\partial P}{\partial n_2}$ and $\frac{\partial P}{\partial n_3}$ can be computed. Linearity of normal derivatives along the edges provides the following set of restrictions on the Bézier ordinates and weight functions.

$$12k\omega_{121}b_{121}^1 - 12k\omega_{121}b_{040} - 6k^2b_{022} = A_1^{'}, \tag{19}$$

$$12k\omega_{112}b_{112}^1 + 48k\omega_{121}b_{121}^1 - 12k\omega_{112}b_{040}$$
$$-30k^2b_{022} = A_2^{'}, \tag{20}$$
$$48k\omega_{112}b_{112}^1 + 72k\omega_{121}b_{121}^1 + 72k\omega_{121}b_{022}$$
$$-48k\omega_{112}b_{031} + 12k^2b_{022} = A_3^{'}, \tag{21}$$
$$48k\omega_{121}b_{121}^1 - 48k\omega_{121}b_{013} + 72k\omega_{112}b_{112}^1$$
$$-72k\omega_{112}b_{022} - 12k^2b_{022} = A_4^{'}, \tag{22}$$
$$12k\omega_{121}b_{121}^1 + 48k\omega_{112}b_{112}^1 - 30k^2b_{022} - 48k\omega_{112}b_{013}$$
$$-12k\omega_{121}b_{004} = A_5^{'}, \tag{23}$$
$$12k\omega_{112}b_{112}^1 - 12k\omega_{112}b_{004} - 6k^2b_{022} = A_6^{'}, \tag{24}$$
$$12k\omega_{211}b_{211}^2 - 12k\omega_{211}b_{400} - 6k^2b_{202} = B_1^{'}, \tag{25}$$
$$48k\omega_{211}b_{211}^2 + 12k\omega_{112}b_{112}^2 - 12k\omega_{112}b_{400}$$
$$-48k\omega_{211}b_{301} - 30k^2b_{202} = B_2^{'}, \tag{26}$$
$$12k\omega_{211}b_{211}^2 + 48k\omega_{112}b_{112}^2 - 72k\omega_{112}b_{202}$$
$$-72k\omega_{211}b_{202} - 12k^2b_{202} = B_3^{'}, \tag{27}$$
$$48k\omega_{211}b_{211}^2 + 72k\omega_{112}b_{112}^2 - 48k\omega_{211}b_{103}$$
$$+12k^2b_{202} = B_4^{'}, \tag{28}$$
$$48k\omega_{112}b_{112}^2 - 12k\omega_{211}b_{004} - 48k\omega_{112}b_{103}$$
$$-30k^2b_{202} = B_5^{'}, \tag{29}$$
$$12k\omega_{112}b_{004} = B_6^{'}, \tag{30}$$
$$12k\omega_{211}b_{211}^3 + 12k\omega_{211}b_{400} - 6k^2b_{220} = C_1^{'}, \tag{31}$$
$$12k\omega_{121}b_{121}^3 + 48k\omega_{211}b_{211}^3 - 48k\omega_{211}b_{310}$$
$$-12k_{121}b_{400} - 30k^2b_{220} = C_2^{'}, \tag{32}$$
$$72k\omega_{211}b_{211}^3 + 48k\omega_{121}b_{121}^3 - 48k\omega_{121}b_{310}$$
$$-72k\omega_{211}b_{220} + 34k^2b_{220} = C_3^{'}, \tag{33}$$
$$48k\omega_{211}b_{211}^3 + 72k\omega_{121}b_{121}^3 - 48k\omega_{211}b_{130} + 12k^2b_{220}$$

$$-72k\omega_{121}b_{220} = C_4^{'}, \tag{34}$$

$$12k\omega_{211}b_{211}^3 + 48k\omega_{121}b_{121}^3 - 48k\omega_{121}b_{130}$$
$$-12k\omega_{211}b_{040} - 30k^2 b_{220} = C_5^{'}, \tag{35}$$

$$12k\omega_{121}b_{121}^3 - 12k\omega_{121}b_{040} - 6k^2 b_{220} = C_6^{'}, \tag{36}$$

where $A_i^{'}$, $B_i^{'}$ and $C_i^{'}$, $i = 1,2,...,6$ are known(expressions of known Bézier ordinates b_{130}, b_{310}, b_{301}, b_{103}, b_{031}, b_{013}, b_{400}, b_{040} and b_{004}).

The set of equations (19)-(36) has lesser unknowns than number of equations. It provides a flexible choice of unknowns. The optimal choice of boundary Bézier ordinates and weights are those which establish first derivative continuity at vertices and edges and inherits positive shape of data as positive surface.

V. NUMERICAL EXAMPLES

In this section, the results developed in Section II-Section IV are implemented on two positive scattered sets taken in Table 1 and Table 2. Firstly the data sets are arranged by Delaunay triangulation method. Fig. 3 and Fig. 6 are the Delaunay triangulation of the data sets of Table 1 and Table 2 respectively. The linear interpolation of these positive data sets are shown in Fig. 4 and Fig. 7 which clearly indicate that shape of surface is positive over the whole domain. The interpolation of these data sets by the scattered data interpolation scheme developed in Sections II-IV shown in Fig. 5(Table 1) and Fig. 8(Table 2). It is clear from Fig. 5 and Fig. 8 that positive shape of the surface is preserved.

Table 1. A positive scattered data set.

x	-3	-3	-3	-3	-3
y	-3	-2	0	1	2
F	1.0497	0.9687	0.7689	0.8338	0.9687
x	-3	-2	-2	-2	-2
y	3	-3	-2	0	3
F	1.0497	1.8102	1.2884	0.0007	1.8102
x	-1	-1	-1	-1	-1
y	-3	-2	-1	1	3
F	1.7430	1.2602	0.4554	0.4554	1.7430
x	0	0	0	0	0
y	-3	-2	0	1	2
F	0.9100	0.9100	0.9100	0.9100	0.9100
x	0	1	1	1	1
y	3	-3	-2	-1	0
F	0.9100	0.0770	0.5598	1.3646	1.7515
x	1	1	2	2	2
y	2	3	-3	-1	1
F	0.5598	0.0770	0.0098	1.4013	1.4013
x	2	3	3	3	3
y	3	-3	-2	-1	0
F	0.0098	0.7703	0.8513	0.9862	1.0511
x	3	3	3		
y	1	2	3		
F	0.9862	0.8513	0.7703		

Table 2. A positive scattered data set.

x	0	0	0	0	0
y	0	0.1250	0.2500	0.3750	0.6250
F	0.7664	0.8220	0.8026	0.6464	0.3892
x	0	0	0	0.1250	0.1250
y	0.7500	0.8750	1.0000	0	0.3750
F	0.3395	0.3026	0.2703	0.8467	0.8278
x	0.1250	0.1250	0.1250	0.2500	0.2500
y	0.5000	0.8750	1.0000	0	0.2500
F	0.5245	0.2817	0.2516	0.8189	1.1653
x	0.2500	0.2500	0.2500	0.2500	0.2500
y	0.3750	0.5000	0.6250	0.8750	1.0000
F	0.8523	0.5046	0.3382	0.2447	0.2222
x	0.3750	0.3750	0.3750	0.3750	0.3750
y	0	0.1250	0.6250	0.7500	1.0000
F	0.6331	0.8042	0.2620	0.1089	0.1842
x	0.5000	0.5000	0.5000	0.5000	0.5000
y	0	0.2500	0.3750	0.6250	0.7500
F	0.4349	0.5381	0.4604	0.2096	0.0437
x	0.5000	0.5000	0.6250	0.6250	0.6250
y	0.8750	1.0000	0	0.1250	0.5000
F	0.0944	0.1460	0.3203	0.4002	0.3601
x	0.6250	0.6250	0.6250	0.7500	0.7500
y	0.6250	0.8750	1.0000	0	0.1250
F	0.2124	0.1203	0.1124	0.2521	0.3846
x	0.7500	0.7500	0.7500	0.8750	0.8750
y	0.3750	0.6250	1.0000	0	0.2500
F	0.6190	0.2105	0.0810	0.1796	0.4675
x	0.8750	0.8750	0.8750	0.8750	1.0000
y	0.3750	0.6250	0.8750	1.0000	0
F	0.4958	0.1512	0.0630	0.0553	0.1076
x	1.0000	1.0000	1.0000	1.0000	1.0000
y	0.1250	0.2500	0.3750	0.5000	0.6250
F	0.1552	0.2302	0.2405	0.1610	0.0831
x	1.0000	1.0000	1.0000		
y	0.7500	0.8750	1.0000		
F	0.0504	0.0406	0.0359		

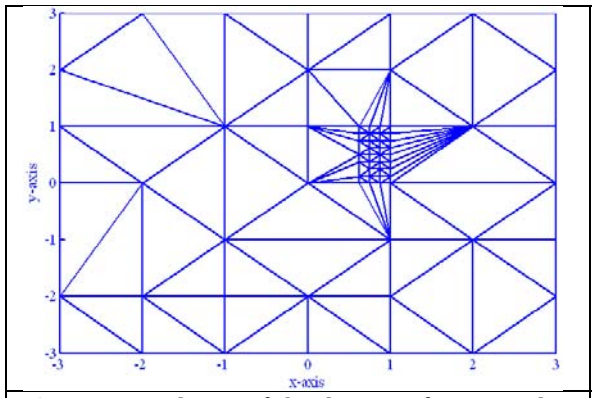

Fig. 3 Triangulation of the domain of positive data set in Table 1.

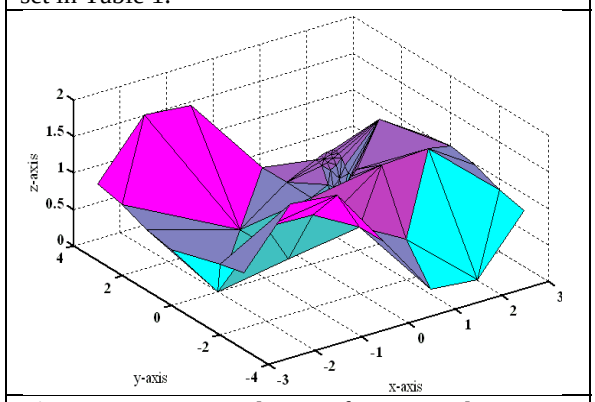

Fig. 4 Linear interpolation of positive data set in Table 1.

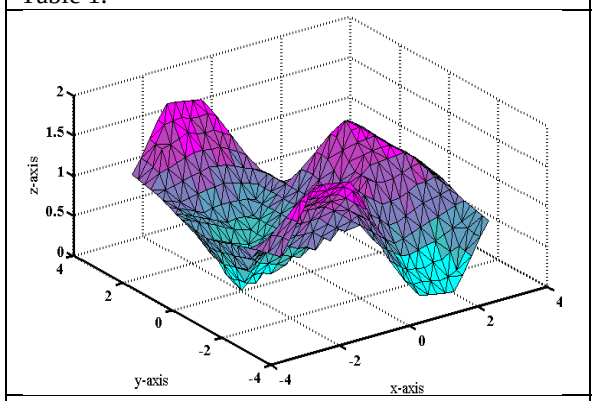

Fig. 5 Positive rational quartic surface.

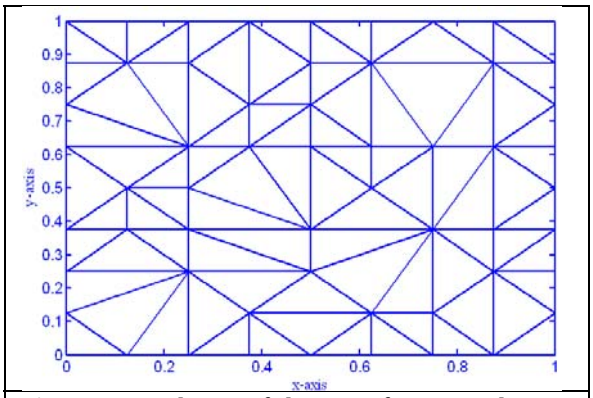

Fig. 6 Triangulation of domain of positive data set in Table 2.

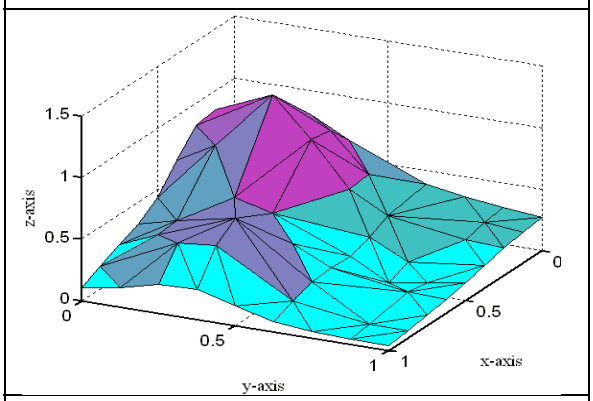

Fig. 7 Linear interpolation of positive data set in Table 2.

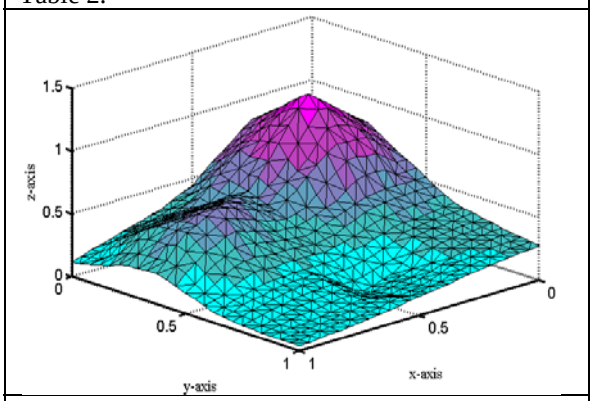

Fig. 8 Positive rational quartic surface.

VI. CONCLUSION

In this study an interpolation scheme is developed to preserve the shape of positive scattered data. The Bernstein-Bézier rational quartic function is used as interpolation tool. The Bernstein Bézier rational quartic function is the combination of Bézier ordinates and weight functions. These weight functions behave like

shape refinement parameters (free parameters). The used interpolation technique requires pre-triangulation of data. The data is triangulated such that each data point is lying at the vertex of triangle. Data points provide the values of vertex Bézier ordinates. The values of boundary Bézier are computed by C^1 continuity at the vertices of triangles. The C^1 continuity along the edges of triangles provides the value of remaining Bézier ordinates and weight functions. The lower bounds of inner and boundary Bézier ordinates are determined to assure positivity of triangular patch. If these Bézier ordinates do not obey the developed constraints of positivity then these are modified by the values of weight functions to assure positivity.

The developed interpolation is local, C^1, applicable to both data and data with derivatives and provides opportunity of shape refinement by suitable choice of weights. The rational structure of interpolant incorporates data having singularities.

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Table 3. Numerical results of Fig. 5.

$\frac{\partial F}{\partial x}(V_1)$	$\frac{\partial F}{\partial y}(V_1)$	$\frac{\partial F}{\partial x}(V_2)$	$\frac{\partial F}{\partial y}(V_2)$	$\frac{\partial F}{\partial x}(V_3)$	$\frac{\partial F}{\partial y}(V_3)$
0.3385	-0.0662	0.2030	0.2395	0.6481	0.0967
0.3541	0.4770	0.7605	0.0810	0.1673	0.2395
0.3541	0.4770	-0.1462	0.6496	-0.4168	0.2682
0.2039	0.4767	-0.1462	0.6496	-0.1429	0.3795
-0.4151	0.0999	-0.4365	-0.0313	-0.1429	0.3795
-0.4151	0.0999	-0.1429	0.3795	0.1673	0.2395
-0.4368	0	-0.1462	0.6496	0.2039	0.4767
0.4164	0.3529	0.7605	0.0810	0.3831	0.1447
0.4164	0.3529	0.3831	0.1447	0.2030	0.2395
0.3420	-0.4822	0.1673	-0.2255	0.7605	-0.0810
0.3420	-0.4822	-0.4168	-0.4023	-0.0998	-0.4243
-0.3648	-0.0293	0.2030	0.2395	0.3385	-0.0662
-0.3648	-0.0293	0.3385	-0.0662	-0.0998	-0.4243
0.0970	-0.4292	0.0523	-0.5950	-0.1892	-0.6438
0.0970	-0.4292	-0.2726	-0.0877	0.0523	-0.5950
0.0970	-0.4292	-0.1892	-0.6438	-0.4368	0
-0.5086	0.1674	-0.2726	-0.0877	0.2030	0.2395
-0.5086	0.1674	0.2030	0.2395	0.3831	0.1447
-0.5086	0.1674	-1.0782	-0.0280	-0.2726	-0.0877
-0.4151	-0.0999	0.1673	-0.2255	-0.1842	-0.2984
-0.4151	-0.0999	-0.1842	-0.2984	-0.4365	-0.0313
0.3230	0.0356	0.2039	0.4767	-0.1429	0.3795
0.3230	0.0356	0.6481	0.0967	0.2039	0.4767
0.3230	0.0356	0.3385	-0.0662	0.6481	0.0967
-0.6056	0.4204	-0.3648	-0.0293	-0.8330	0
-0.6056	0.4204	0.4164	0.3529	0.2030	0.2395
-0.6056	0.4204	0.2030	0.2395	-0.3648	-0.0293
0.3277	-0.3432	0.7605	-0.0810	0.3377	-0.3749
0.3277	-0.3432	0.3377	-0.3749	0.0523	-0.5950
0.3541	0.4700	0.1673	0.2395	-0.1429	0.3795
0.3541	0.4700	-0.1429	0.3795	-0.1462	0.6496
-0.4368	0	-0.8330	0	-0.4168	0.2682
-0.4368	0	-0.4168	0.2682	-0.1462	0.6496
-0.3680	-0.2914	-0.8330	0	-0.4368	0
-0.3680	-0.2914	-0.4368	0	-0.1892	-0.6438
-0.3680	-0.2914	0.0523	-0.5950	0.3377	-0.3749
-0.3680	-0.2914	-0.1892	-0.6438	0.0523	-0.5950
-0.3680	-0.0877	0.3277	-0.3432	0.0523	-0.5950
-0.2726	-0.0877	-1.0782	-0.0280	0.3277	-0.3432
0.3420	-0.4822	-0.1842	-0.2984	0.1673	-0.2255
0.3420	-0.4822	-0.0998	-0.4243	-0.1842	-0.2984
-0.3648	-0.0293	-0.4168	-0.4023	-0.8330	0
-0.3648	-0.0293	-0.0998	-0.4243	-0.4168	-0.4023
0.0970	-0.4292	0.2030	0.2395	-0.2726	-0.0877
0.0970	-0.4292	0.6481	0.0967	0.2030	0.2395
0.0970	-0.4292	0.2039	0.4767	0.6481	0.0967
0.0970	-0.4292	-0.4368	0	0.2039	0.4767
0.3230	0.0356	-0.4365	-0.0313	-0.1842	-0.2984
0.3230	0.0356	-0.1429	0.3795	-0.4365	-0.0313
0.3230	0.0356	-0.1842	-0.2984	-0.0998	-0.4243
0.3230	0.0356	-0.0998	-0.4243	0.3385	-0.0662

Table 4. Numerical results of Fig. 8.

$\frac{\partial F}{\partial x}(V_1)$	$\frac{\partial F}{\partial y}(V_1)$	$\frac{\partial F}{\partial x}(V_2)$	$\frac{\partial F}{\partial y}(V_2)$	$\frac{\partial F}{\partial x}(V_3)$	$\frac{\partial F}{\partial y}(V_3)$
-1.6735	1.0083	-0.3225	-0.6049	-1.1498	1.3772
-0.1281	-0.8943	1.1717	-1.6762	0.2507	-2.0246
0.8933	-2.0513	0.2507	-2.0246	1.1717	-1.6762
-1.3532	-0.5728	-0.3225	-0.6049	-1.6735	1.0083
-1.5332	1.3660	-1.3242	1.0709	-1.6735	1.0083
-1.5332	1.3660	-1.6735	1.0083	-1.1498	1.3772
-0.5430	-1.2417	-0.5131	-1.8853	-0.3892	-1.2378
-0.6477	0.9433	-0.3282	1.0381	-0.3549	0.9333
0.2924	0.5204	1.0558	0.2771	0.6424	0.4448
0.2924	0.5204	-1.1498	1.3772	-0.3225	-0.6049
0.2924	0.5204	-0.3225	-0.6049	1.0558	0.2771
-0.5754	-2.3701	0.8933	-2.0513	-0.3225	-0.6049
-0.5754	-2.3701	-0.3892	-1.2378	-0.5131	-1.8853
1.7071	-0.8607	0.8933	-2.0513	1.1717	-1.6762
1.7071	-0.8607	1.0558	0.2771	-0.3225	-0.6049
1.7071	-0.8607	-0.3225	-0.6049	0.8933	-2.0513
-0.3357	0.2227	-0.1381	0.3972	-0.1224	0.2553
-0.3357	0.2227	-0.2784	-0.1787	-0.5345	-0.1667
-0.6519	-0.7629	-1.5533	-0.3904	-1.2949	-0.7794
-0.2526	-0.1507	-0.1034	-1.0139	-0.6519	-0.7629
-0.2526	-0.1507	-0.6519	-0.7629	-0.3782	-0.1342
-0.1209	-1.2098	-0.3892	-1.2378	0.1838	-1.2572
-0.1209	-1.2098	-0.5430	-1.2417	-0.3892	-1.2378
-0.6861	0.7318	-1.3242	1.0709	-0.7312	0.6392
-1.3450	0.6226	-0.6477	0.9433	-0.5760	0.3808
-1.4327	0.5010	-1.3450	0.6226	-1.8624	0.3052
-2.2530	-0.1128	-1.2949	-0.7794	-1.5333	-0.3904
-0.4341	-0.3013	-0.6519	-0.7629	-1.2949	-0.7794
-0.4341	-0.3013	-0.3782	-0.1342	-0.6519	-0.7629
-0.1800	-0.0508	-0.2286	-0.0409	-0.2526	-0.1507
-0.1363	-0.0287	-0.2526	-0.1507	-0.3782	-0.1342
0.0357	0.0282	-0.2286	-0.0409	-0.2600	-0.0632
0.0357	0.0282	-0.1034	-1.0139	-0.2526	-0.1507
0.0357	0.0282	-0.2526	-0.1507	-0.2286	-0.0409
-0.1349	-0.5913	-0.1381	0.3972	-0.6632	-0.4288
-0.1349	-0.5913	0.0357	0.0282	-0.1381	0.3972
-0.2084	-0.2578	-0.1672	-0.2768	-0.1604	-0.1961
-0.2302	-0.3872	0.1838	-1.2572	-0.3892	-1.2378
-0.2302	-0.3872	-0.3892	-1.2378	-1.3532	-0.5728
-0.2302	-0.3872	-1.3532	-0.5728	-0.6861	0.7318
-0.2302	-0.3872	-0.6861	0.7318	-0.3282	1.0381
-0.2302	-0.3872	-0.3282	1.0381	-1.4327	0.5010
-0.2302	-0.3872	-1.4327	0.5010	-1.5533	-0.3904
-0.2302	-0.3872	-0.1034	-1.0139	0.1838	-1.2572
0.1281	-1.0147	-0.1349	-0.5913	-0.1209	-1.2098
0.1281	-1.0147	0.1838	-1.2572	-0.1034	-1.0139
0.1281	-1.0147	-0.1209	-1.2098	0.1838	-1.2572
0.1281	-1.0147	-0.1034	-1.0139	0.0357	0.0282
0.1281	-1.0147	0.0357	0.0282	-0.1349	-0.5913
-0.4712	-0.9759	-0.6632	-0.4288	-0.5345	-0.1667
-0.4712	-0.9759	0.2507	-2.0246	-0.5131	-1.8853
-0.4712	-0.9759	-0.1281	-0.8943	0.2507	-2.0246
-0.4712	-0.9759	-0.5131	-1.8853	-0.5430	-1.2417
-0.4712	-0.9759	-0.5430	-1.2417	-0.6632	-0.4288

-0.4712	-0.9759	-0.1604	-0.1961	-0.1281	-0.8943
-0.4712	-0.9759	-0.5345	-0.1667	-0.2084	-0.2578
-0.4712	-0.9759	-0.2084	-0.2578	-0.1604	-0.1961
-0.5754	-2.3701	-0.5131	-1.8853	0.2507	-2.0246
-0.5754	-2.3071	0.2507	-2.0246	0.8933	-2.0513
-0.5754	-2.3071	-1.3532	-0.5728	-0.3892	-1.2378
-0.5754	-2.3071	-0.3225	-0.6049	-1.3532	-0.5728
-0.6632	-0.4288	-0.1381	0.3972	-0.3357	0.2227
-0.6632	-0.4288	-0.3357	0.2227	-0.5345	-0.1667
-0.6861	0.7318	-0.7312	0.6392	-0.3549	0.9333
-0.6861	0.7318	-0.3549	0.9333	-0.3282	1.0381
-0.6861	0.7318	-1.6735	1.0083	-1.3242	1.0709
-0.6861	0.7318	-1.3532	-0.5728	-1.6735	1.0083
-1.4327	0.5010	-0.3282	1.0381	-0.6477	0.9433
-1.4327	0.5010	-0.6477	0.9433	-1.3450	0.6226
-2.2530	-0.1128	-1.5533	-0.3904	-1.4327	0.5010
-2.2530	-0.1128	-1.4327	0.5010	-1.8624	0.3052
-0.1363	-0.0287	-0.1552	-0.0376	-0.1800	-0.0508
-0.1363	-0.0287	-0.1800	-0.0508	-0.2526	-0.1507
0.0357	0.0282	-0.2600	-0.0632	-0.1224	0.2553
0.0357	0.0282	-0.1224	0.2553	-0.1381	0.3972
-0.1349	-0.5913	-0.6632	-0.4288	-0.5430	-1.2417
-0.1349	-0.5913	-0.5430	-1.2417	-0.1209	-1.2098
-0.2084	-0.2578	-0.2784	-0.1787	-0.1880	-0.2364
-0.2084	-0.2578	-0.5345	-0.1667	-0.2784	-0.1787
-0.2084	-0.2578	-0.1370	-0.2404	-0.1672	-0.2768
-0.2084	-0.2578	-0.1880	-0.2364	-0.1370	-0.2404
-0.2302	-0.3872	-1.5533	-0.3904	-0.6519	-0.7629
-0.2302	-0.3872	-0.6519	-0.7629	-0.1034	-1.0139