Construction of the \( \alpha_3 \)-automorphism

S. Abdelalim, M. Zeriouh and M. Ziane

Abstract—Let \( \lambda \) a monomorphism from \( A \) to \( A' \) where \( A, A' \in \Gamma \), we consider \( B' \) a basic subgroup of \( A' \)
\[
B' = \bigoplus_{i \in I_k} B_k \quad \text{with} \quad B_k = \bigoplus_{i \in I_k} \langle x_{k,i} >
\]
where \( o(x_{k,i}) = p^k \quad \forall i \in I_k \), we suppose there exists \( n_0 \in \mathbb{N}^* \) such that the restriction of \( \lambda \) to \( p^{n_0} A \) is an isomorphism from \( p^{n_0} A \) to \( p^{n_0} A' \) and we pose:
\[
\lambda(A) = A_1 [6] \quad \text{and} \quad A_2 = A_1 + B_1 [2].
\]

We show that if \( \alpha \in Aut(A) \) is written in the form:
\[
\alpha = \pi A_1 + \rho \quad \text{where} \quad \pi \text{ is invertible } p\text{-adic number and } \rho \in Hom(A, A') \quad \text{with} A' \text{ is the first Ulm subgroup of } A\text{ then, there exists an automorphism } \alpha_3 \text{ of } A_3 = A_2 + B_2 \text{ such that for all } a_3 \in A_3: \alpha_3(a_3) = \pi a_3 + p^{n_0-1} a_0 \quad \text{where } a_0 \in A_1 \text{ and } \alpha_3 \lambda = \lambda \alpha.
\]

Keywords—About Abelian goups, \( p \)-group, order, direct sums of cyclic groups, basic subgroups, monomorphism group, automorphism group.

I. INTRODUCTION

In 1987, P. Schupp showed, in [4], that the extension property in the category of groups, characterizes the inner automorphisms. M. R. Pettet gives, in [5], a simpler proof of Schupp’s result and shows that the inner automorphisms of a group are also characterized by the lifting property in the category of groups. The automorphisms of abelian \( p \)-groups having the extension property in the category of abelian \( p \)-groups are characterized in [1].

II. MAIN RESULT

Proposition 2.1
(i) \( A_2 \cap B_2' \) is the direct sum of cyclic groups.
(ii) If \( A_2 \cap B_2' = B_{2,1}' \oplus B_{2,2}' \)
\[
\begin{align*}
B_{2,1}' &= \bigoplus_{i \in I_{2,1}} \langle x_{2,j} > \quad ; \quad o(x_{2,j}) = p \quad \forall i \in I_{2,1} \\
B_{2,2}' &= \bigoplus_{i \in I_{2,2}} \langle x_{2,j} > \quad ; \quad o(x_{2,j}) = p^2 \quad \forall i \in I_{2,2}
\end{align*}
\]
then there exists a subgroup \( B_{2,1} \) of \( B_2' \) such that \( B_{2,1}' \supset B_{2,1} \).

Proof
(i) Since we have \( B_2' = \bigoplus_{i \in I_2} \langle x_{2,i} > \) and \( o(x_{2,i}) = p^2 \quad \forall i \in I_2 \).
while \( p^2 B_2' = 0 \quad i.e. B_2' \) is a bounded group therefore \( A_2 \cap B_2' \) is also a bounded group then by theorem 17.2 [3]: \( A_2 \cap B_2' \) is direct sum of cyclic groups.
(ii) We have \( A_2 \cap B_2' = B_{2,1}' \oplus B_{2,2}' \) (5) with:
\[
\begin{align*}
B_{2,1}' &= \bigoplus_{i \in I_{2,1}} \langle x_{2,j} > \quad ; \quad o(x_{2,j}) = p \quad \forall i \in I_{2,1} \\
B_{2,2}' &= \bigoplus_{i \in I_{2,2}} \langle x_{2,j} > \quad ; \quad o(x_{2,j}) = p^2 \quad \forall i \in I_{2,2}
\end{align*}
\]
we have:
\[
x_{2,j} \in B_{2,1}' \subset B_2' = \bigoplus_{i \in I_2} \langle x_{2,i} > \quad ; \quad o(x_{2,i}) = p^2 \quad \forall i \in I_2
\]
then \( x_{2,j} = \sum_{j=1}^{r} m_j x_{2,j} \) where \( m_j \in \mathbb{Z} \)
therefore \( px_{2,j} = 0 = \sum_{j=1}^{r} pm_j x_{2,j} \)
hence \( \forall j = 1, \ldots , r : \quad pm_j x_{2,j} = 0 \)
then \( \forall j = 1, \ldots , r : \quad p | m_j \)
hence \( \forall j = 1, \ldots , r : \exists m_j \in \mathbb{Z} : m_j = pm_j \)
then $x_{2,i} = \sum_{j=1}^{r} p_{yi,J} x_{2,i,j} = p \sum_{j=1}^{r} m_{j} x_{2,i,j} = py_{2,i,j}$ (6)

where $y_{2,i,j} = \sum_{j=1}^{r} m_{j} x_{2,i,j} \in B_{2}^* ; o(y_{2,i,j}) = p^{2} \forall i \in I_{2,1}$ then

$B_{2,1} \subset B_{2,1}^* = \bigoplus_{i=1}^{r} x_{2,i} > ; o(y_{2,i,j}) = p^{2} \forall i \in I_{2,1}$

(iii) Since $B_{2,1}^* \oplus B_{2,2}^*$ is a subgroup of $B_2^*$ a $B_{2,1}^* \oplus B_{2,2}^*$ is the direct sum of cyclic groups of the same order $p^2$ and

$(B_{2,1}^* \oplus B_{2,2}^*) \cap p^2 B_2^* = 0$

then by proposition 27.1 [3]:

$B_{2,1}^* \oplus B_{2,2}^*$ is a direct summand of $B_2^*$.

We pose: $B_2^* = (B_{2,1}^* \oplus B_{2,2}^*) \oplus B_2$ (7) where $B_2$ is a subgroup of $B_2^*$.

**Definition 2.2**

We define the homorphism $\alpha_3$ from $B_2^*$ to $A^*$ as follows:

$\overline{\alpha_3}_{|B_2^*} = \pi id_{B_2^*}$

Under the conditions of Theorem 1.4 see [2] p. 251, we will enunciate and prove the following lemmas.

**Lemma 2.3**

$\overline{\alpha_3}_{|B_2^*} = \alpha_2$

**Proof**

By (6) and (5): $py_{2,i,j} = x_{2,i,j} \in B_{2,1}^* \subset A_2$

and by theorem 1.4 [2], we have:

$\alpha_2(x_{2,i,j}) = \pi x_{2,i,j} + p^{n_1} a_1$ (8) where $a_1 \in A_1$

We pose: $\overline{\alpha_3}(y_{2,i,j}) = \pi y_{2,i,j} + p^{n_1-1} a_1$ (9) while

\[
\overline{\alpha_3}(x_{2,i,j}) = \frac{\overline{\alpha_3}(y_{2,i,j})}{p} = \pi \overline{\alpha_3}(y_{2,i,j}) = p(\pi y_{2,i,j} + p^{n_1-1} a_1); a_1 \in A_1
\]

$= p\pi y_{2,i,j} + p^{n_1} a_1 = \pi x_{2,i,j} + p^{n_1} a_1 = \alpha_2(x_{2,i,j})$

**Lemma 2.4**

$\overline{\alpha_3}_{|B_2^* \oplus B_2^*} = \alpha_2$

**Proof**

Let $b_{2,i}^* \in B_{2,i}^*$ we have:

$\overline{\alpha_3}(b_{2,i}^*) = \alpha_3(\sum_{j=1}^{r} m_{j} x_{2,i,j})$

$= \sum_{j=1}^{r} m_{j} \overline{\alpha_3}(x_{2,i,j})$

$= \sum_{j=1}^{r} m_{j} \alpha_2(x_{2,i,j})$

$= \alpha_2(\sum_{j=1}^{r} m_{j} x_{2,i,j})$

$= \alpha_2(b_{2,i}^*)$

Let $b = b_{2,1}^* + b_{2,2}^* \in B_{2,1}^* \oplus B_{2,2}^*$ we have:

$\overline{\alpha_3}(b) = \overline{\alpha_3}(b_{2,1}^* + b_{2,2}^*)$

$= \overline{\alpha_3}(b_{2,1}^*) + \alpha_3(b_{2,2}^*)$

$= \alpha_2(b_{2,1}^*) + \alpha_2(b_{2,2})$

$= \alpha_2(b)$

**Proposition 2.5**

(i) $\forall b_{2,1}^* \in B_{2,1}^* : \overline{\alpha_3}(b_{2,1}^*) = \pi b_{2,1}^* + p^{n_1-1} a_1; a_1 \in A_1$

(ii) $\forall b_{2,2}^* \in B_{2,2}^* : \alpha_2(b_{2,2}^*) = \pi b_{2,2}^* + p^{n_1} a_1; a_1 \in A_1$

**Proof**

(i) Since we have:

$b_{2,1}^* \in B_{2,1}^* = \bigoplus_{i=1}^{r} y_{2,1,i} >$ then $b_{2,1}^* = \sum_{i=1}^{r} m_{i} y_{2,1,i}$

therefore:

$\overline{\alpha_3}(b_{2,1}^*) = \overline{\alpha_3}(\sum_{i=1}^{r} m_{i} y_{2,1,i})$

$= \sum_{i=1}^{r} m_{i} \overline{\alpha_3}(y_{2,1,i})$

$= \sum_{i=1}^{r} m_{i} (\pi y_{2,1,i} + p^{n_1-1} a_1); a_1 \in A_1$ (by (9))

$= \pi b_{2,1}^* + p^{n_1-1} \sum_{i=1}^{r} m_{i} a_1$

$= \pi b_{2,1}^* + p^{n_1-1} a_1; a_1 = \sum_{i=1}^{r} m_{i} a_1 \in A_1$

(ii) Since we have:

$b_{2,2}^* \in B_{2,2}^* = \bigoplus_{i=1}^{r} x_{2,2,j} > A_2$ by (5)

then $b_{2,2}^* = \sum_{j=1}^{r} m_{j} x_{2,2,j}$ where $x_{2,2,j} \in A_2$

therefore :
\[ \alpha_2(b^*_2) = \alpha_2 \left( \sum_{i=1}^{r} m_i x_{2,2,i} \right) \]
\[ = \sum_{i=1}^{r} m_i \alpha_2(x_{2,2,i}) \]
\[ = \sum_{i=1}^{r} m_i (\pi x_{2,2,i} + p^{\delta_i} a^*_i); a^*_i \in A_1 \]  [2]
\[ = \pi b^*_2 + p^{\delta_i} \sum_{i=1}^{r} m_i a^*_i \]
\[ = \pi b^*_2 + p^{\delta_i} a^*_i; a^*_i = \sum_{i=1}^{r} m_i a^*_i \in A_1 \]

**Definition 2.6**

We define the endomorphism \( \alpha_3 \) of \( A_3 = A_2 + B_2^* \) as follows:

\[ \begin{align*}
\alpha_{3_{z_1}} &= \alpha_2 \\
\alpha_{3_{z_2}} &= \overline{\alpha_3}
\end{align*} \]

**Remark 2.7**

\( \alpha_3 \) is well defined

because if \( a_2 + b_2 \) and \( x_2 + y_2^* \) are two elements of \( A_2 \) such that \( a_2 + b_2 = x_2 + y_2^* \)
then \( a_2 - x_2 = -b_2^* + y_2^* \)
therefore \( \overline{\alpha_3(a_2 - x_2)} = \overline{\alpha_3(-b_2^* + y_2^*)} \)
i.e. \( \alpha_3(a_2 - x_2) = \overline{\alpha_3(-b_2^* + y_2^*)} \)
i.e. \( \alpha_3(a_2 + b_2) = \alpha_3(x_2 + y_2^*) \)
i.e. \( \alpha_3(x_2 + y_2^*) = \alpha_3(a_2 + b_2') \)

**Proposition 2.8**

For all \( a_3 \in A_3 \) there exists \( n_0 \in \text{IN}^* \) and \( a_1^0 \in A_1 \)
such that \( \alpha_3(a_3) = \pi a_3 + p^{\delta_i} a_1^0 \)

**Proof**

We have: \( a_3 \in A_3 = A_2 + B_2^* \)
then \( \exists (a_2, b_2') \in (A_2 \times B_2^*) \) such that \( a_3 = a_2 + b_2' \) hence
\( \alpha_3(a_3) = \alpha_3(a_2 + b_2') \)
and by definition 2.6, we have:
\( \alpha_3(a_3) = \alpha_3(a_2) + \alpha_3(b_2') \)
and by theorem 1.4, [2] there exists \( n_0 \in \text{IN}^* \) and \( a_1^0 \in A_1 \)
such that: \( \alpha_3(a_1^0) = \pi a_1^0 + p^{\delta_i} a_1^0 + \overline{\alpha_3(b_2')} \)
and by (7) we have \( b_2' = b_{2,1}^* + b_{2,2}^* + b_2 \)
then by definition 2.2, we have:
\[ \overline{\alpha_3(b_2')} = \overline{\alpha_3(b_{2,1}^* + b_{2,2}^* + b_2)} \]
i.e. \( \overline{\alpha_3(b_2')} = \overline{\alpha_3(b_{2,1}^*)} + \overline{\alpha_3(b_{2,2}^*)} + \overline{\pi b_2} \)

The proposition 2.5 and definition 2.2 show that:
\[ \overline{\alpha_3(b_2')} = \pi b_{2,1}^* + p^{\delta_i} a_1^0 + \overline{\pi b_{2,2}^*} + p^{\delta_i} a_1^0 + \overline{\pi b_2}; a_1^0, a_1^0 \in A_1 \]
\[ = \pi(b_{2,1}^* + b_{2,2}^* + b_2) + p^{\delta_i}(a_1^0 + p a_1^0) \]
\[ = \pi b_2 + p^{\delta_i} a_1^0; a_1^0 = a_1 + p a_1^0 \in A_1 \]
therefore
\[ \alpha_3(a_3) = \pi a_3 + p^{\delta_i} a_1^0 + \pi b_2 + p^{\delta_i} a_1^0 \]
\[ = \pi(a_3 + b_2') + p^{\delta_i}(p a_1 + a_1^0) \]
\[ = \pi a_3 + p^{\delta_i} a_1^0; a_1^0 = p a_1 + a_1^0 \in A_1 \]  (10)

**Proposition 2.9**

\( \alpha_3 \) is an automorphism of \( A_3 \)

**Proof**

Let \( a_3 \in \ker \alpha_3 \) then \( \alpha_3(a_3) = 0 \)
and by (10) we have: \( \pi a_3 + p^{\delta_i} a_1^0 = 0 \)
which is equivalent to \( a_3 = -\pi^{-1} p^{\delta_i} a_1^0 \in A_1 \subset A_2 \)
and since \( \alpha_{3_{a_3}} = \alpha_2 \) then \( \alpha_3(a_3) = \alpha_2(a_3) \)

i.e. \( 0 = \alpha_3(a_3) \) then \( 0 = a_3 \)
i.e. \( \ker \alpha_3 = 0 \) then \( \alpha_3 \) is a monomorphism.

On the other hand let \( a_3 \in A_1 \)
then \( \alpha_3(a_3) = \pi a_3 + p^{\delta_i} a_1^0 \) where \( a_1^0 \in A_1 \)
hence \( a_3 = -\pi^{-1} \alpha_3(a_3) = -\pi^{-1} p^{\delta_i} a_1^0 \)
i.e. \( a_3 = \alpha_3(-\pi^{-1} a_3) = -\pi^{-1} p^{\delta_i} a_1^0 \)
and since we have \( -\pi^{-1} p^{\delta_i} a_1^0 \in A_1 \subset A_2 \)
\( \alpha_2 \in \text{Aut}(A_2) \)
then \( \exists a_2 \in A_2 \subset A_1 \) such that:
\[ -\pi^{-1} p^{\delta_i} a_1^0 = \alpha_2(a_2) = \alpha_3(a_2) \]
so \( a_3 = \alpha_3(-\pi^{-1} a_3) + \alpha_3(a_3) = \alpha_3(\pi^{-1} a_3 + a_2) \)
which means that \( \alpha_3 \) is an epimorphism and hence \( \alpha_3 \) is an automorphism of \( A_3 \).

**Proposition 2.10**

The following diagram is commutative:

\[ \begin{array}{ccc}
A & \xrightarrow{\lambda} & A_3 \\
\alpha & \downarrow & \alpha_3 \\
A & \xrightarrow{\lambda} & A_3
\end{array} \]

**Proof**
We have $\forall a \in A, \lambda(a) \in A_1 \subset A_2$
then $\alpha_2 \lambda(a) = \alpha_2 \lambda(a)$ because $a_{|A_2} = a_1$
$\alpha_\lambda(a)$ because $a_{|A_1} = a_1$
$= \lambda a \lambda^{-1}(a)$ because $\alpha_1 = \lambda a \lambda^{-1}$
$= \lambda a(a)$
therefore $\alpha_2 \lambda = \lambda a$

ACKNOWLEDGMENT

We thank Professor Abdelhakim Chillali for his helpful comments and suggestions. We also thank the referee for his suggestions and comments in the revision of this paper.

REFERENCES


S. Abdelalim: Laboratory of Mathematics, Computing and Application, Department of Mathematical and computer, Faculty of sciences University of Mohamed V Agdal, BP.1014 . Rabat, Morocco. seddikabdi@hotmail.com

M. Zeriouh: Department of Mathematical and Computer Sciences, Faculty of sciences, University of Mohamed first, BP.717 60000, Oujda, Morocco. zeriouhmostafa@gmail.com

M. Ziane: Department of Mathematical and Computer Sciences, Faculty of sciences, University of Mohamed first, BP.717 60000, Oujda, Morocco.