Continuous-Time and Discrete Multivariable 1DOF Controllers

Marek Kubalcik, Vladimír Bobal

***Abstract*—** The paper is focused on a design and implementation of a 1DOF (one degree of freedom) multivariable controller. The controller was designed in both discrete and continuous-time versions. The control algorithm is based on polynomial theory and pole – placement. The controller integrates an on – line identification of an ARX model of a controlled system and a control synthesis on the basis of the identified parameters. The model parameters are recursively estimated using the recursive least squares method.

***Keywords*—** multivariable control, control algorithms, adaptive control, polynomial methods, pole assignment, recursive identification.

I. Introduction

TYPICAL technological processes require the simultaneous control of several variables related to one system. Each input may influence all system outputs. The design of a controller for such a system must be quite sophisticated if the system is to be controlled adequately. There are many different methods of controlling MIMO (multi input – multi output) systems [1]. Several of these use decentralized PID controllers [2], others apply single input-single-output (SISO) methods extended to cover multiple inputs [3]. The classical approach to the control of multi-input–multi-output (MIMO) systems is based on the design of a matrix controller to control all system outputs at one time. The basic advantage of this approach is its ability to achieve optimal control performance because the controller can use all the available information about the controlled system. Controllers are based on various approaches and various mathematical models of controlled processes. A standard technique for MIMO control systems uses polynomial methods [4], [5], [6], [7], [8] and is also used in this paper. Controller synthesis is reduced to the solution of linear Diophantine equations [9].

One controller, which enables control of TITO (two input-two output) systems, is presented. The proposed control algorithm is based on the 1DOF (one degree of freedom) configuration [10]. The controller was realized both in discrete and continuous-time versions. Both versions of the controller were realized both with fixed parameters and as self-tuning controllers [11], [12] with recursive identification of a model of the controlled system. The recursive least squares method is used in the identification part.

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II. Mathematical model of the controlled process

A general transfer matrix of a two-input–two-output system with significant cross-coupling between the control loops is expressed as (for continuous-time systems $q = s$ as the derivative operator and for discrete systems $q = z^{-1}$ as the delay operator)

$$\boldsymbol{G}(q)=\begin{bmatrix} G_{11}(q) & G_{12}(q) \\ G_{21}(q) & G_{22}(q) \end{bmatrix} \tag{1}$$

$$\boldsymbol{Y}(q)=\boldsymbol{G}(q)\boldsymbol{U}(q) \tag{2}$$

where $\boldsymbol{U}(q)$ and $\boldsymbol{Y}(q)$ are vectors of the manipulated variables) and the controlled variables, respectively.

$$\boldsymbol{Y}(q)=[y_1(q), y_2(q)]^T \quad \boldsymbol{U}(q)=[u_1(q), u_2(q)]^T \tag{3}$$

It may be assumed that the transfer matrix can be transcribed to the following form of the matrix fraction:

$$\boldsymbol{G}(q)=\boldsymbol{A}^{-1}(q)\boldsymbol{B}(q)=\boldsymbol{B}_1(q)\boldsymbol{A}_1^{-1}(q) \tag{4}$$

where the polynomial matrices $\boldsymbol{A}\in R_{22}[q]$, $\boldsymbol{B}\in R_{22}[q]$ represent the left coprime factorization of matrix $\boldsymbol{G}(q)$ and the matrices $\boldsymbol{A}_1\in R_{22}[q]$, $\boldsymbol{B}_1\in R_{22}[q]$ represent the right coprime factorization of $\boldsymbol{G}(q)$. The further described algorithms are based on a model with polynomials of second order. This model proved to be effective for control of several TITO laboratory processes [13], where controllers based on a model with polynomials of the first order failed.

A. Discrete Model

Polynomial matrices of the discrete model are given by following exressions

$$\boldsymbol{A}(z^{-1})=\begin{bmatrix} 1+a_1z^{-1}+a_2z^{-2} & a_3z^{-1}+a_4z^{-2} \\ a_5z^{-1}+a_6z^{-2} & 1+a_7z^{-1}+a_8z^{-2} \end{bmatrix} \tag{5}$$

$$\boldsymbol{B}(z^{-1})=\begin{bmatrix} b_1z^{-1}+b_2z^{-2} & b_3z^{-1}+b_4z^{-2} \\ b_5z^{-1}+b_6z^{-2} & b_7z^{-1}+b_8z^{-2} \end{bmatrix} \tag{6}$$

The matrices can be converted to difference equations

$$y_1(k) = -a_1 y_1(k-1) - a_2 y_1(k-2) - a_3 y_2(k-1) - a_4 y_2(k-2) + \\ + b_1 u_1(k-1) + b_2 u_1(k-2) + b_3 u_2(k-1) + b_4 u_2(k-2) \tag{7}$$

$$y_2(k) = -a_5 y_1(k-1) - a_6 y_1(k-2) - a_7 y_2(k-1) - a_8 y_2(k-2) + \\ + b_5 u_1(k-1) + b_6 u_1(k-2) + b_7 u_2(k-1) + b_8 u_2(k-2) \tag{8}$$

B. *Continuous-Time Model*

Polynomial matrices of the continuous-time model are defined as follows

$$\boldsymbol{A}(s) = \begin{bmatrix} s^2 + a_1 s + a_2 & a_3 s + a_4 \\ a_5 s + a_6 & s^2 + a_7 s + a_8 \end{bmatrix} \tag{9}$$

$$\boldsymbol{B}(s) = \begin{bmatrix} b_1 s + b_2 & b_3 s + b_4 \\ b_5 s + b_6 & b_7 s + b_8 \end{bmatrix} \tag{10}$$

Differential equations describing dynamical behavior of the system are

$$y_1'' + a_1 y_1' + a_2 y_1 + a_3 y_2' + a_4 y_2 = b_1 u_1' + b_2 u_1 + b_3 u_2' + b_4 u_2 \tag{11}$$

$$y_2'' + a_5 y_1' + a_6 y_1 + a_7 y_2' + a_8 y_2 = b_5 u_1' + b_6 u_1 + b_7 u_2' + b_8 u_2 \tag{12}$$

III. Design of 1DOF Controllers

The 1DOF configuration of the closed loop system is depicted in Fig. 1.

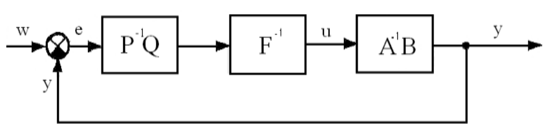

Fig. 1 Block diagram of 1DOF configuration

The controller can be described both by left and right matrix fractions as well as the controlled system

$$\boldsymbol{G}_R(q) = \boldsymbol{P}^{-1}(q)\boldsymbol{Q}(q) = \boldsymbol{Q}_1(q)\boldsymbol{P}_1^{-1}(q) \tag{13}$$

Where $\boldsymbol{P} \in R_{22}[q]$, $\boldsymbol{Q} \in R_{22}[q]$, $\boldsymbol{Q}_1 \in R_{22}[q]$, $\boldsymbol{P}_1 \in R_{22}[q]$ are polynomial matrices.

The vector of input reference signals is defined as

$$\boldsymbol{W}(q) = \boldsymbol{F}_w^{-1}(q)\boldsymbol{h}(q) \tag{14}$$

Further, the reference signals are considered as step functions. In this case $\boldsymbol{h}(q)$ is a vector of constants and $\boldsymbol{F}_w(q)$ is in the case of the discrete system expressed as

$$\boldsymbol{F}_w(z^{-1}) = \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix} \tag{15}$$

and in the case of the continuous-time system as

$$\boldsymbol{F}_w(s) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \tag{16}$$

The compensator $\boldsymbol{F}(q)$ is a component formally separated from the controller. It has to be included in the controller to fulfil the requirement on the asymptotic tracking. If the reference signals are step functions, then $\boldsymbol{F}(q)$ is an integrator.

The control law (operator q will be omitted from some operations for the purpose of simplification) is defined as

$$\boldsymbol{U} = \boldsymbol{F}^{-1}\boldsymbol{Q}_1\boldsymbol{P}_1^{-1}\boldsymbol{E} \tag{17}$$

where $\boldsymbol{E}$ is a vector of control errors. Using matrix operations it is possible to modify this vector to the form

$$\boldsymbol{E} = \boldsymbol{W} - \boldsymbol{Y} = \boldsymbol{F}\boldsymbol{P}_1(\boldsymbol{A}\boldsymbol{F}\boldsymbol{P}_1 + \boldsymbol{B}\boldsymbol{Q}_1)^{-1}\boldsymbol{A}\boldsymbol{W} \tag{18}$$

Asymptotic tracking of the reference signals is then fulfilled if $\boldsymbol{F}\boldsymbol{P}_1$ is divisible by $\boldsymbol{F}_w$.

It is possible to derive the following equation for the system output

$$\boldsymbol{Y} = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{F}^{-1}\boldsymbol{P}^{-1}\boldsymbol{Q}\boldsymbol{E} = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{F}^{-1}\boldsymbol{P}^{-1}\boldsymbol{Q}(\boldsymbol{W} - \boldsymbol{Y}) \tag{19}$$

which can be modified to

$$\boldsymbol{Y} = \boldsymbol{P}_1(\boldsymbol{A}\boldsymbol{F}\boldsymbol{P}_1 + \boldsymbol{B}\boldsymbol{Q}_1)^{-1}\boldsymbol{B}\boldsymbol{Q}_1\boldsymbol{P}_1^{-1}\boldsymbol{W} \tag{20}$$

The determinant of the matrix in the denominator ($\boldsymbol{A}\boldsymbol{F}\boldsymbol{P}_1 + \boldsymbol{B}\boldsymbol{Q}_1$) is the characteristic polynomial of the MIMO system. The roots of this polynomial matrix determine the behaviour of the closed loop system. They must be inside the unit circle (of the Gauss complex plane) in case of the discrete system and on the left side of the Gauss complex plain in case of the continuous-time system for the system to be stable. Conditions of BIBO stability can be defined by the following Diophantine matrix equation:

$$\boldsymbol{A}\boldsymbol{F}\boldsymbol{P}_1 + \boldsymbol{B}\boldsymbol{Q}_1 = \boldsymbol{M} \tag{21}$$

where $\boldsymbol{M} \in R_{22}[q]$ is a stable diagonal polynomial matrix. If the system has the same number of inputs and outputs, matrix $\boldsymbol{M}$ can be chosen as diagonal, which allows easier computation of the controller parameters. Correct pole placement of the matrix $\boldsymbol{M}$ is very important for good control performance.

For the continuous-time case the matrix $\boldsymbol{M}$ takes the following form

$$\boldsymbol{M}(s) = \begin{bmatrix} s^4 + m_1 s^3 + \\ + m_2 s^2 + m_3 s + m_4 & 0 \\ 0 & s^4 + m_5 s^3 + m_6 s^2 + \\ & + m_7 s + m_8 \end{bmatrix} \tag{22}$$

and for the discrete system it takes the form

$$\boldsymbol{M}(z^{-1}) = \begin{bmatrix} 1 + m_1 z^{-1} + m_2 z^{-2} + \\ + m_3 z^{-3} + m_4 z^{-4} & 0 \\ 0 & 1 + m_1 z^{-1} + m_2 z^{-2} + \\ & + m_3 z^{-3} + m_4 z^{-4} \end{bmatrix} \tag{23}$$

A. Design of Discrete Controller

The degree of the controller polynomial matrices depends on the internal properness of the closed loop. The structures of matrices $\boldsymbol{P}_1$ and $\boldsymbol{Q}_1$ were chosen so that the number of unknown controller parameters equals the number of algebraic equations resulting from the solution of the Diophantine equation (21) using the method of uncertain coefficients:

$$\boldsymbol{P}_1\left(z^{-1}\right)=\begin{bmatrix} 1+p_1z^{-1} & p_2z^{-1} \\ p_3z^{-1} & 1+p_4z^{-1} \end{bmatrix} \tag{24}$$

$$\boldsymbol{Q}_1\left(z^{-1}\right)=\begin{bmatrix} q_1+q_2z^{-1}+q_3z^{-2} & q_4+q_5z^{-1}+q_6z^{-2} \\ q_7+q_8z^{-1}+q_9z^{-2} & q_{10}+q_{11}z^{-1}+q_{12}z^{-2} \end{bmatrix} \tag{25}$$

The solution of the Diophantine equation results in a set of algebraic equations with unknown controller parameters.

$$\begin{bmatrix} -a_2 & -a_4 & 0 & 0 & b_2 & 0 & 0 & b_4 \\ a_2-a_1 & a_4-a_3 & 0 & b_2 & b_1 & 0 & b_4 & b_3 \\ a_1-1 & a_3 & b_2 & b_1 & 0 & b_4 & b_3 & 0 \\ 1 & 0 & b_1 & 0 & 0 & b_3 & 0 & 0 \\ -a_6 & -a_8 & 0 & 0 & b_6 & 0 & 0 & b_8 \\ a_6-a_5 & a_8-a_7 & 0 & b_6 & b_5 & 0 & b_8 & b_7 \\ a_5 & a_7-1 & b_6 & b_5 & 0 & b_8 & b_7 & 0 \\ 0 & 1 & b_5 & 0 & 0 & b_7 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{bmatrix}=\begin{bmatrix} m_4 \\ m_3+a_2 \\ m_2-a_2+a_1 \\ m_1-a_1+1 \\ 0 \\ a_6 \\ a_5-a_6 \\ -a_5 \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} -a_2 & -a_4 & 0 & 0 & b_2 & 0 & 0 & b_4 \\ a_2-a_1 & a_4-a_3 & 0 & b_2 & b_1 & 0 & b_4 & b_3 \\ a_1-1 & a_3 & b_2 & b_1 & 0 & b_4 & b_3 & 0 \\ 1 & 0 & b_1 & 0 & 0 & b_3 & 0 & 0 \\ -a_6 & -a_8 & 0 & 0 & b_6 & 0 & 0 & b_8 \\ a_6-a_5 & a_8-a_7 & 0 & b_6 & b_5 & 0 & b_8 & b_7 \\ a_5 & a_7-1 & b_6 & b_5 & 0 & b_8 & b_7 & 0 \\ 0 & 1 & b_5 & 0 & 0 & b_7 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_2 \\ p_4 \\ q_5 \\ q_6 \\ q_7 \\ q_{10} \\ q_{11} \\ q_{12} \end{bmatrix}=\begin{bmatrix} 0 \\ a_4 \\ a_3-a_4 \\ -a_3 \\ m_4 \\ m_3+m_4 \\ m_2-a_4+a_7 \\ m_1-a_3+1 \end{bmatrix} \tag{27}$$

The controller parameters are obtained by solving these equations. The parameters are then used for computation of the control law. The control law is defined by the following difference equations:

$$\begin{aligned} u_1(k)&=(1\text{-}p_1\text{-}p_4)u_1(k-1)+(p_1+p_4+p_2p_3\text{-}p_1p_4)u_1(k-2)+ \\ &+(p_1p_4\text{-}p_2p_3)u_1(k-3)+q_1e_1(k)+(q_2+p_4q_1\text{-}p_3q_4)e_1(k-1)+ \\ &+(q_3+p_4q_2\text{-}p_3q_5)e_1(k-2)+(p_4q_3\text{-}p_3q_6)e_1(k-3)+q_4e_2(k)+ \\ &+(q_5+p_1q_4\text{-}p_2q_1)e_2(k-1)+(q_6+p_1q_5\text{-}p_2q_2)e_2(k-2) \\ &+(p_1q_6\text{-}p_2q_3)e_2(k-3) \end{aligned} \tag{28}$$

$$\begin{aligned} u_2(k)&=(1\text{-}p_1\text{-}p_4)u_2(k-1)+(p_1+p_4+p_2p_3\text{-}p_1p_4)u_2(k-2)+ \\ &+(p_1p_4\text{-}p_2p_3)u_2(k-3)+q_7e_1(k)+(q_8+p_4q_7\text{-}p_3q_{10})e_1(k-1)+ \\ &+(q_9+p_4q_8\text{-}p_3q_{11})e_1(k-2)+(p_4q_9\text{-}p_3q_{12})e_1(k-3)+q_{10}e_2(k)+ \\ &+(q_{11}+p_1q_{10}\text{-}p_2q_7)e_2(k-1)+(q_{12}+p_1q_{11}\text{-}p_2q_8)e_2(k-2) \\ &+(p_1q_{12}\text{-}p_2q_9)e_2(k-3) \end{aligned} \tag{29}$$

B. Design of Continuous-Time Controller

Polynomial matrices of the continuous-time controller are as follows:

$$\boldsymbol{P}_1(s)=\begin{bmatrix} s+p_1 & p_2 \\ p_3 & s+p_4 \end{bmatrix} \tag{30}$$

$$\boldsymbol{Q}_1(s)=\begin{bmatrix} q_1s^2+q_2s+q_3 & q_4s^2+q_5s+q_6 \\ q_7s^2+q_8s+q_9 & q_{10}s^2+q_{11}s+q_{12} \end{bmatrix} \tag{31}$$

The solution of the Diophantine equation results in a set of algebraic equations with unknown controller parameters. Using matrix notation, the algebraic equations are expressed in the following form.

$$\begin{bmatrix} 1 & 0 & b_1 & 0 & 0 & b_3 & 0 & 0 \\ a_1 & a_3 & b_2 & b_1 & 0 & b_4 & b_3 & 0 \\ a_2 & a_4 & 0 & b_2 & b_1 & 0 & b_4 & b_3 \\ 0 & 0 & 0 & 0 & b_2 & 0 & 0 & b_4 \\ 0 & 1 & b_5 & 0 & 0 & b_7 & 0 & 0 \\ a_5 & a_7 & b_6 & b_5 & 0 & b_8 & b_7 & 0 \\ a_6 & a_8 & 0 & b_6 & b_5 & 0 & b_8 & b_7 \\ 0 & 0 & 0 & 0 & b_6 & 0 & 0 & b_8 \end{bmatrix}\begin{bmatrix} p_1 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{bmatrix}=\begin{bmatrix} m_1-a_1 \\ m_2-a_2 \\ m_3 \\ m_4 \\ -a_5 \\ -a_6 \\ 0 \\ 0 \end{bmatrix} \tag{32}$$

$$\begin{bmatrix} 1 & 0 & b_1 & 0 & 0 & b_3 & 0 & 0 \\ a_1 & a_3 & b_2 & b_1 & 0 & b_4 & b_3 & 0 \\ a_2 & a_4 & 0 & b_2 & b_1 & 0 & b_4 & b_3 \\ 0 & 0 & 0 & 0 & b_2 & 0 & 0 & b_4 \\ 0 & 1 & b_5 & 0 & 0 & b_7 & 0 & 0 \\ a_5 & a_7 & b_6 & b_5 & 0 & b_8 & b_7 & 0 \\ a_6 & a_8 & 0 & b_6 & b_5 & 0 & b_8 & b_7 \\ 0 & 0 & 0 & 0 & b_6 & 0 & 0 & b_8 \end{bmatrix}\begin{bmatrix} p_2 \\ p_4 \\ q_4 \\ q_5 \\ q_6 \\ q_{10} \\ q_{11} \\ q_{12} \end{bmatrix}=\begin{bmatrix} -a_3 \\ -a_4 \\ 0 \\ 0 \\ m_1-a_7 \\ m_2-a_8 \\ m_3 \\ m_4 \end{bmatrix} \tag{33}$$

The control law is defined by the differential equations

$$\begin{aligned} &u_1''' +(p_1+p_4)u_1'' +(p_1p_4\text{-}p_2p_3)u_1' = \\ &=q_1e_1''' +(q_1p_4+q_2\text{-}q_4p_3)e_1'' +(q_2p_4+q_3\text{-}q_5p_3)e_1' + \\ &+(q_3p_4\text{-}q_6p_3)e_1 ++q_4e_2''' +(q_4p_1+q_5\text{-}q_1p_2)e_2'' \\ &+(q_5p_1+q_6\text{-}q_2p_2)e_2' +(q_6p_1\text{-}q_3p_2)e_2 \end{aligned} \tag{34}$$

$$\begin{aligned} &u_2''' +(p_1+p_4)u_2'' +(p_1p_4\text{-}p_2p_3)u_2' = \\ &=q_7e_1''' +(q_7p_4+q_8\text{-}q_{10}p_3)e_1'' +(q_8p_4+q_9\text{-}q_{11}p_3)e_1' + \\ &+(q_8p_4+q_9\text{-}q_{11}p_3)e_1 ++q_{10}e_2''' +(q_{10}p_1+q_{11}\text{-}q_7p_2)e_2'' \\ &+(q_{11}p_1+q_{12}\text{-}q_8p_2)e_2' +(q_{11}p_1+q_{12}\text{-}q_8p_2)e_2 \end{aligned} \tag{35}$$

For purposes of simulation, the controller was realized in the Matlab/Simulink environment as an S-function. It was then necessary to obtain its state equations. Further there it is introduced a conversion of the first differential equation (34) to the state equations. The second differential equation (35) was converted similarly. Equation (34) can be itemized as follows

$$\begin{aligned} &u'''_{1A} + (p_1 + p_4)u''_{1A} + (p_1 p_4 - p_2 p_3)u'_{1A} = \\ &= q_1 e'''_1 + (q_1 p_4 + q_2 - q_4 p_3)e''_1 + (q_2 p_4 + q_3 - q_5 p_3)e'_1 + \\ &+ (q_3 p_4 - q_6 p_3)e_1 \end{aligned} \quad (36)$$

$$\begin{aligned} &u'''_{1B} + (p_1 + p_4)u''_{1B} + (p_1 p_4 - p_2 p_3)u'_{1B} = \\ &= q_4 e'''_2 + (q_4 p_1 + q_5 - q_1 p_2)e''_2 + (q_5 p_1 + q_6 - q_2 p_2)e'_2 + \\ &+ (q_6 p_1 - q_3 p_2)e_2 \end{aligned} \quad (37)$$

Equation (36) can be transcribed to the transfer function. It is also possible to establish an auxiliary variable Z

$$\begin{aligned} G(s) &= \frac{q_1 s^3 + (q_1 p_4 + q_2 - q_4 p_3)s^2 + (q_2 p_4 + q_3 - q_5 p_3)s + (q_3 p_4 - q_6 p_3)}{s^3 + (p_1 + p_4)s^2 + (p_1 p_4 - p_2 p_3)s} = \\ &= \frac{U_{1A}}{E_1} = \frac{U_{1A}}{Z}\frac{Z}{E_1} \end{aligned} \quad (38)$$

By means of the variable Z it is possible to define following equations

$$\begin{aligned} &q_1 z''' + (q_1 p_4 + q_2 - q_4 p_3)z'' + (q_2 p_4 + q_3 - q_5 p_3)z' + \\ &+ (q_3 p_4 - q_6 p_3)z = u_{1A} \end{aligned} \quad (39)$$

$$z''' + (p_1 + p_4)z'' + (p_1 p_4 - p_2 p_3)z' = e_1 \quad (40)$$

Equation (40) can be converted to a set of differential equations of the first order (state equations). Choice of the state variables is as follows

$$\begin{aligned} x_1 &= z \\ x_2 &= z' \\ x_3 &= z'' \end{aligned} \quad (41)$$

And the state equations are

$$\begin{aligned} x'_1 &= x_2 \\ x'_2 &= x_3 \\ x'_3 &= e_1 - (p_1 + p_4)x_3 - (p_1 p_4 - p_2 p_3)x_2 \end{aligned} \quad (42)$$

On the basis of the state variables, which are substituted to equation (39), it is possible to derive the first part of the manipulated variable u_{1A}

$$\begin{aligned} u_{1A} &= q_1(e_1 - (p_1 + p_4)x_3 - (p_1 p_4 - p_2 p_3)x_2) + (q_1 p_4 + q_2 - q_4 p_3)x_3 + \\ &+ (q_2 p_4 + q_3 - q_5 p_3)x_2 + (q_3 p_4 - q_6 p_3)x_1 \end{aligned} \quad (43)$$

Similarly it is possible to transcribe equation (37)

$$\begin{aligned} &q_4 z''' + (q_4 p_1 + q_5 - q_1 p_2)z'' + (q_5 p_1 + q_6 - q_2 p_2)z' + \\ &+ (q_6 p_1 - q_3 p_2)z = u_{1B} \end{aligned} \quad (44)$$

$$z''' + (p_1 + p_4)z'' + (p_1 p_4 - p_2 p_3)z' = e_2 \quad (45)$$

State variables were chosen as in the previous case (41). The state equations are as follows

$$\begin{aligned} x'_1 &= x_2 \\ x'_2 &= x_3 \\ x'_3 &= e_2 - (p_1 + p_4)x_3 - (p_1 p_4 - p_2 p_3)x_2 \end{aligned} \quad (46)$$

The second part of the manipulated variable u_{1B} can be computed similarly like the part u_{1A} by substitution of the state variables to equation (44)

$$\begin{aligned} u_{1B} &= q_4(e_2 - (p_1 + p_4)x_3 - (p_1 p_4 - p_2 p_3)x_2) + (q_4 p_1 + q_5 - q_1 p_2)x_3 + \\ &+ (q_5 p_1 + q_6 - q_2 p_2)x_2 + (q_6 p_1 - q_3 p_2)x_1 \end{aligned} \quad (47)$$

The manipulated variable u_1 is then defined by the following sum

$$u_1 = u_{1A} + u_{1B} \quad (48)$$

An expression for computation of the manipulated variable u_2 is obtained similarly on the basis of differential equation (35).

IV. System Identification

The control algorithm was applied as a self-tuning controller. Self-tuning control is based on the online identification of a model of a controlled process. Each self – tuning controller consists of an on – line identification part and a control part.

Various discrete linear models are used to describe dynamic behaviour of controlled systems; see for example the overview in [14]. The most widely applied linear dynamic model is the ARX model. Usually the ARX model is tested first and more complex model structures are only examined if it does not perform satisfactorily. However, the ARX model matches the structure of many real processes. The parameters can be easily estimated by a linear least-squares technique.

A. Identification of Discrete Model

The ARX model describing the TITO process is defined as

$$\begin{aligned} y_1(k) &= \boldsymbol{\Theta}_1(k)\phi(k-1) + e_{s1}(k) \\ y_2(k) &= \boldsymbol{\Theta}_2(k)\phi(k-1) + e_{s2}(k) \end{aligned} \quad (49)$$

where $e_{s1}(k)$, $e_{s2}(k)$ are non-measurable disturbances.

Parameter vectors are specified as follows:

$$\begin{aligned} \boldsymbol{\Theta}_1^T(k) &= [a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4] \\ \boldsymbol{\Theta}_2^T(k) &= [a_5, a_6, a_7, a_8, b_5, b_6, b_7, b_8] \end{aligned} \quad (50)$$

The data vector is

$$\begin{aligned} \phi^T(k-1) = [&y_1(k-1), y_1(k-2), y_2(k-1), \\ &y_2(k-2), u_1(k-1), u_1(k-2), u_2(k-1), u_2(k-2)] \end{aligned} \quad (51)$$

The aim of the identification is a recursive estimation of unknown model parameters $\boldsymbol{\Theta}$ on the basis of the inputs and the outputs considering the time moment k t_k, $\{y(i), u(i), i = k, k-1, k-2, ..., k_0\}$ (where k_0 is an initial time of the identification). We are looking for a vector $\hat{\boldsymbol{\Theta}}$ minimizing the criterion

$$J_k(\boldsymbol{\Theta}) = \sum_{i=k_0}^{k} e_s^2(i) \quad (52)$$

where

$$e_s(i) = y(i) - \boldsymbol{\Theta}^T \phi(i) = \begin{bmatrix} 1 & -\boldsymbol{\Theta}^T \end{bmatrix} \begin{bmatrix} y(i) \\ \phi(i) \end{bmatrix} \tag{53}$$

When using the least squares method, the influence of all measured input and output samples to the parameter estimates is the same. This is inconvenient for the identification of nonlinear systems, where changes in the identified parameters are expected. Tracking of changes of the parameters can be achieved using exponential forgetting. This technique ensues from the assumption that new data describe the dynamics of an object better than older data, which are multiplied by smaller weighting coefficients. However, if the identified plant is insufficiently activated, the input and output signals are steady (this situation is typical for closed control systems), and the exponential forgetting factor can cause numerical instability of the identification algorithm. A possible solution of this problem is the application of adaptive directional forgetting [15]. This technique changes the forgetting factor according to the level of information in the data. In view of the parameter changes in the nonlinear coupled-drives apparatus and the expected insufficient activation of the controlled system, the recursive least squares method with adaptive directional forgetting was applied. Then we minimize a modified criterion

$$J_k(\boldsymbol{\Theta}) = \sum_{i=k_0}^{k} \varphi^{2(k-i)} e_s^2(i) \tag{54}$$

where $0 \langle \varphi^2 \leq 1$ is the exponential forgetting factor.

The vector of parameters is updated according to the following recursive expression

$$\hat{\boldsymbol{\Theta}}(k) = \hat{\boldsymbol{\Theta}}(k-1) + \frac{\boldsymbol{C}(k-1)\phi(k-1)}{1+\xi(k-1)} \hat{e}(k-1) \tag{55}$$

Where

$$\xi(k-1) = \phi^T(k-1)\boldsymbol{C}(k-1)\phi(k-1) \tag{56}$$

is an auxiliary scalar and

$$\hat{e}(k-1) = y(k) - \hat{\boldsymbol{\Theta}}^T(k-1)\phi(k-1) \tag{57}$$

is a prediction error. If $\xi(k-1) > 0$, then the square covariance matrix C is updated according to following expression

$$\boldsymbol{C}(k) = \boldsymbol{C}(k-1) - \frac{\boldsymbol{C}(k-1)\phi(k-1)\phi^T(k-1)\boldsymbol{C}(k-1)}{\varepsilon^{-1}(k) + \xi(k-1)} \tag{58}$$

Where

$$\varepsilon(k) = \varphi(k) - \frac{1-\varphi(k)}{\xi(k-1)} \tag{59}$$

If $\xi(k-1) = 0$ then

$$\boldsymbol{C}(k) = \boldsymbol{C}(k-1) \tag{60}$$

The directional forgetting factor is computed in each sampling period according to the expression

$$\varphi(k) = \left\{1 + (1+\rho)[\ln(1+\xi(k-1))] + \left[\frac{(\upsilon(k-1)+1)\eta(k-1)}{1+\xi(k-1)+\eta(k-1)} - 1\right]\frac{\xi(k-1)}{1+\xi(k-1)}\right\}^{-1} \tag{61}$$

Where

$$\eta(k) = \frac{\hat{e}^2(k)}{\lambda(k)} \tag{62}$$

$$\upsilon(k) = \varphi(k)[(\upsilon(k-1)+1] \tag{63}$$

$$\lambda(k) = \varphi(k)\left[\lambda(k-1) + \frac{\hat{e}^2(k-1)}{1+\xi(k-1)}\right] \tag{64}$$

are auxiliary variables.

B. *Identification of Continuous-Time Model*

It is not possible to measure directly input and output derivatives of a system in case of continuous – time control loop. One of the possible approaches to this problem is establishing of filters and filtered variables to substitute the primary variables. This approach is described in detail in [16], [17], [18]. The filtered variables are then used in the recursive identification procedure.

Let us consider a linear continuous – time ARX model in a form of differential equation

$$A(\sigma)y(t) = B(\sigma)u(t) + n(t) \tag{65}$$

where $n(t)$ is a random continuous – time variable and σ is the derivative operator. After the Laplace transform we obtain

$$A(s)Y(s) = B(s)U(s) + N(s) + O_1(s) \tag{66}$$

where the polynomial O_1 represents the Laplace transform of initial conditions. The output of the system is than given as

$$Y(s) = \frac{B(s)}{A(s)}U(s) + \frac{N(s)}{A(s)} + \frac{O_1(s)}{A(s)} \tag{67}$$

In order to obtain approximations of derivatives of the continuous – time variables it is necessary to establish filters using differential equations

$$C(\sigma)u_f(t) = u(t); \quad C(\sigma)y_f(t) = y(t) \tag{68}$$

where $C(\sigma)$ is a stable polynomial and u_f is a filtered input and y_f is a filtered output. After the Laplace transform we obtain

$$C(s)U_f(s) = U(s) + O_2(s); \quad C(s)Y_f(s) = Y(s) + O_3(s) \tag{69}$$

where O_2(s) is a polynomial of initial conditions for the filtered input and O_3(s) is a polynomial of initial conditions for the filtered output. The degree of the polynomial c must be greater or equal to the degree of the polynomial A (*deg* $C(s)$> *deg* $A(s)$). It is profitable to choose *deg* $C(s)$ = *deg* $A(s)$ (the lower is the degree of the polynomial C, the faster is the dynamics of the filter). Time constants of the filters must be lower than time constants of the plant. A right choice of the filter's constants makes convergence of the parameters faster.

After substitution of the filtered variables to the equation (66) we obtain

$$A\left[CY_f(s)-O_3\right]=B\left[CU_f-O_2\right]+N(s)+O_1 \tag{70}$$

After modification and substitution

$$AY_f(s)=BU_f(s)+\frac{O_1-BO_2+AO_3+N(s)}{C} \tag{71}$$

and substitution

$$O=\frac{O_1-BO_2+AO_3}{C} \tag{72}$$

we obtain

$$Y_f(s)=\frac{B}{A}U_f(s)+\frac{O}{A}+\frac{1}{A}N(s) \quad \Rightarrow \quad G_f(s)=\frac{B}{A}=G(s) \tag{73}$$

Expression (73) proves that the transfer behaviour between the filtered and between the non – filtered variables is equivalent. Different are only initial conditions for the filtered and original variables. This fact enables to employ the filtered variables for the model parameter estimation.

After transformation to the time domain we obtain the following equation

$$A(\sigma)y_f(t)=B(\sigma)\,u_f(t)+n(t) \tag{74}$$

The filtered variables are taken in discrete time intervals *tk* = *kTs*, *k* = 0,1,2, …, where *Ts* is the sampling period. The equation (74) can be modified to the form suitable for the model parameters estimation

$$y^{(n)}{}_f(t_k)=-\sum_{i=0}^{n-1}a_i y_f^{(i)}(t_k)+\sum_{j=0}^{m}b_j u_f^{(j)}(t_k)+n(t_k) \tag{75}$$

The parameters of the model are estimated by the recursive method described in the previous section according to expressions (55) - (64). For the considered continuous – time model given by expressions (9) - (12) the equation (75) takes following form

$$\begin{aligned}y''_{1f}(t_k)=&-a_1y'_{1f}(t_k)-a_2y_{1f}(t_k)-a_3y'_{2f}(t_k)-a_4y_{2f}(t_k)+\\&+b_1u'_{1f}(t_k)+b_2u_{1f}(t_k)+b_3u'_{2f}(t_k)+b_4u_{2f}(t_k)+\varepsilon_1(t_k)\end{aligned} \tag{76}$$

$$\begin{aligned}y''_{2f}(t_k)=&-a_5y'_{1f}(t_k)-a_6y_{1f}(t_k)-a_7y'_{2f}(t_k)-a_8y_{2f}(t_k)+\\&+b_5u'_{1f}(t_k)+b_6u_{1f}(t_k)+b_7u'_{2f}(t_k)+b_8u_{2f}(t_k)+\varepsilon_2(t_k)\end{aligned} \tag{77}$$

The regression vector and the vector of parameters are

$$\begin{aligned}\phi_{1,2}{}^T(t_k)=&[-y'_{1f}(t_k),-y_{1f}(t_k),-y'_{2f}(t_k),-y_{2f}(t_k),\\&-u'_{1f}(t_k),-u_{1f}(t_k),-u'_{2f}(t_k),-u_{2f}(t_k),1]\end{aligned} \tag{78}$$

$$\boldsymbol{\Theta}_1{}^T(t_k)=\left[a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4,d_1\right] \tag{79}$$

$$\boldsymbol{\Theta}_2{}^T(t_k)=\left[a_5,a_6,a_7,a_8,b_5,b_6,b_7,b_8,d_2\right] \tag{80}$$

Considering the order of the system, the filters for both variables were chosen to have second order.

$$\begin{aligned}&y''_{1f}(t)+c_1y'_{1f}(t)+c_0y_{1f}(t)=y_1(t)\\&y''_{2f}(t)+c_1y'_{2f}(t)+c_0y_{2f}(t)=y_2(t)\\&u''_{1f}(t)+c_1u'_{1f}(t)+c_0u_{1f}(t)=u_1(t)\\&u''_{2f}(t)+c_1u'_{2f}(t)+c_0u_{2f}(t)=u_2(t)\end{aligned} \tag{81}$$

A right choice of the coefficients of the filter's polynomials and choice of the sampling period are the ruling factors for the speed of the parameter's convergence. Time constants of the filters must be lower than time constants of the plant.

V. Simulation Verification

The proposed controllers were verified by simulation. Verification by simulation was carried out on a range of plants with various dynamics.

A. Simulation of Discrete Control

As a simulation example for the discrete controller it is shown control of a system which represents a linear model of a coupled drives process obtained by the recursive identification for a particular steady state [13].

$$\boldsymbol{A}(z^{-1})=\begin{bmatrix}1-0.5827z^{-1}+0.1745z^{-2} & -0.0220z^{-1}+0.1797z^{-2}\\ 0.0167z^{-1}-0.0886z^{-2} & 1-0.4564z^{-1}-0.0830z^{-2}\end{bmatrix} \tag{82}$$

$$\boldsymbol{B}(z^{-1})=\begin{bmatrix}-0.0035z^{-1}+0.0955z^{-2} & 0.1484z^{-1}+0.2197z^{-2}\\ 0.2783z^{-1}+0.3107z^{-2} & -0.0371z^{-1}-0.3489z^{-2}\end{bmatrix} \tag{83}$$

The step response of the system is in Fig. 2.

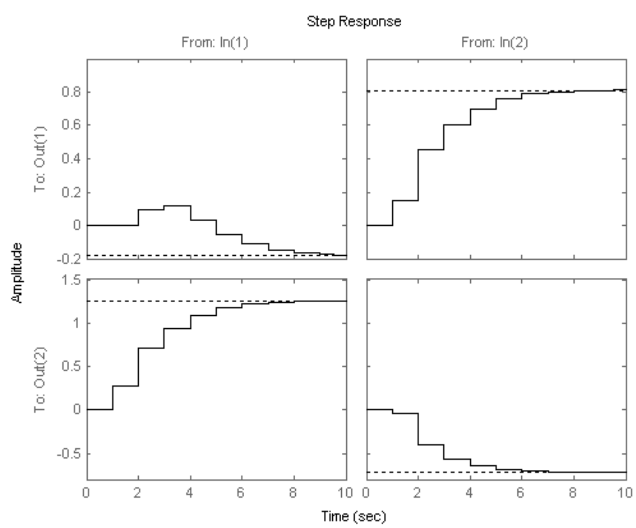

Fig. 2 Step response of the discrete system

The tuning parameter is the matrix ***M***. A suitable pole-placement (matrix ***M***) was chosen experimentally. At first, a multiple pole was chosen on the real axis. A suitable position of the multiple pole was chosen by experiments and comparison of control results. Then it was searched a suitable combination of various poles in the neighbourhood of the multiple pole.

$$\boldsymbol{M}\left(z^{-1}\right)=\begin{bmatrix} 1-0.7z^{-1}+0.01z^{-2}- \\ -0.1z^{-3}-0.05z^{-4}+ & 0 \\ +0.0001z^{-5} & \\ & 1-0.7z^{-1}+0.01z^{-2}- \\ 0 & -0.1z^{-3}-0.05z^{-4}+ \\ & +0.0001z^{-5} \end{bmatrix} \tag{84}$$

The time responses of the control are shown in Fig. 3-4

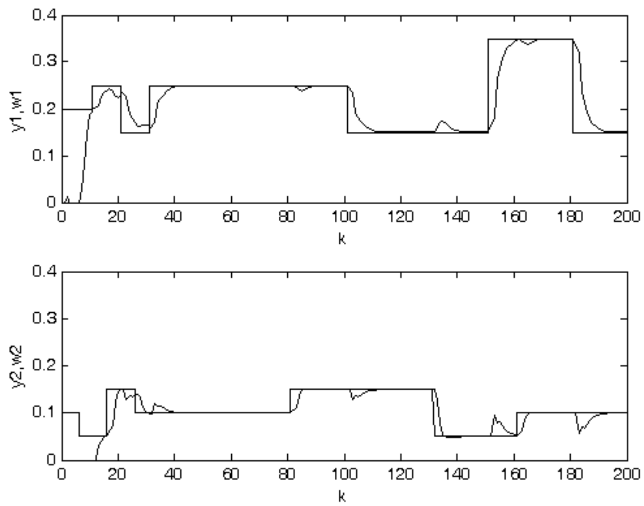

Fig. 3 Adaptive control with discrete controller

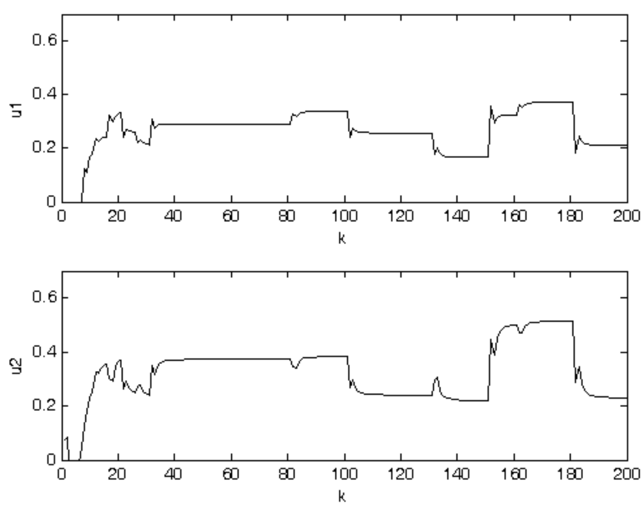

Fig. 4 Adaptive control with discrete controller-manipulated variables

B. Simulation of Continuous-Time Control

A continuous-time model in the form of the matrix fraction obtained by a possible conversion of the discrete model does not need to have the structure on which it is based the computation of the control law. The model obtained by this way would by then unusable.

It is shown control of the following continuous-time system

$$\boldsymbol{A}(s)=\begin{bmatrix} s^2+2s+0{,}7 & 0{,}2s+0{,}4 \\ -0{,}5s-0{,}1 & s^2+2s+0{,}7 \end{bmatrix} \tag{85}$$

$$\boldsymbol{B}(s)=\begin{bmatrix} 0{,}5s+0{,}2 & 0{,}1s+0{,}3 \\ 0{,}5s+0{,}1 & 0{,}3s+0{,}4 \end{bmatrix} \tag{86}$$

Fig. 5 shows the plant's step response

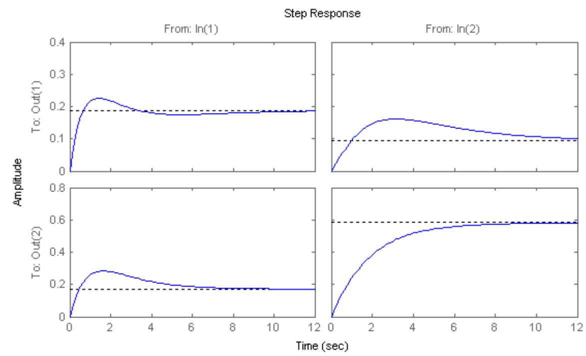

Fig. 5 Step response of the continuous-time system

The matrix $\boldsymbol{M}$ was obtained as follows

$$\boldsymbol{M}(s)=\begin{bmatrix} s^5+5s^4+10s^3+ & 0 \\ +10s^2+5s+1 & \\ 0 & s^5+5s^4+10s^3+ \\ & +10s^2+5s+1 \end{bmatrix} \tag{87}$$

The time responses of the control are shown in Fig. 6-7.

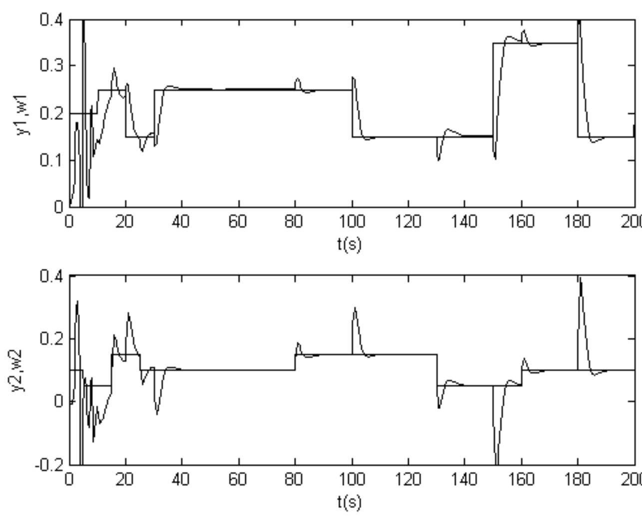

Fig. 6 Adaptive control with continuous-time controller

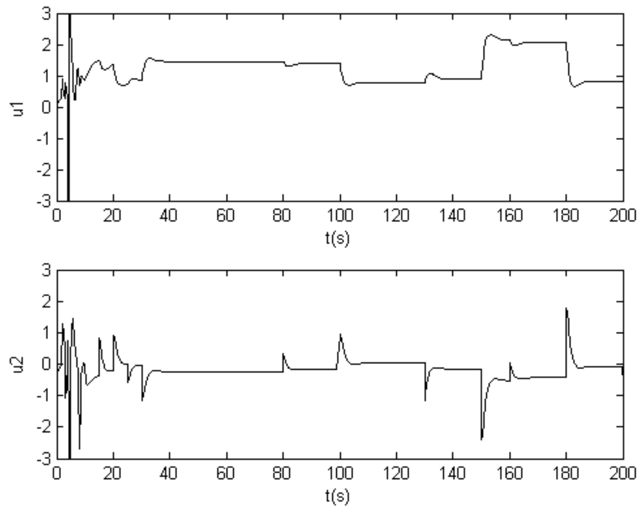

Fig. 7 Adaptive control with continuous-time controller-manipulated variables

From the courses of the variables in Fig.4-7 it is obvious that the basic requirements on control were satisfied. The system was stabilized and the asymptotic tracking of the reference signals was achieved.

VI. Conclusion

The 1DOF TITO controller was designed and implemented both in discrete and continuous-time versions. General principles were elaborated on a specific system with two inputs and two outputs that is often applicable in industrial practice. Control law based on specific model was derived in the form of self-contained expressions that is especially useful for practical applications of control on common industrial devices. An advantage of the proposed strategy lies in its simplicity and applicability.

It is necessary to recognize that self-tuning controllers do not work satisfactorily in the initial adaptation phase if the initial parameter estimates are chosen without a priori information. However, the most important property for practical use of self-tuning controllers is their performance after the adaptation phase.

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