Solitary Wave Solutions of a fifth order Equation

Chung-Hsien Tsai, Shy-Jen Guo, and Jeng-Chi Yan

Abstract—The objective of this research is to study the existence of solutions of a fifth order model equation for steady capillary-gravity waves over a bump with the Bond number near 1/3. We proved that there exist solitary wave solutions of equation (1).

Keywords—Classical fourth-order Runge-Kutta method, Green's function, Solitary wave solution, Steady capillary-gravity wave.

I. Introduction

PROGRESSIVE capillary-gravity waves on an irrotational incompressible inviscid fluid of constant density with surface tension in a two-dimensional channel of finite depth have been studied since nineteen century. Assume that a coordinate system moving with the wave at a speed is chosen so that in reference to it the wave motion is steady. Let H be the depth of water, g the acceleration of gravity, T the coefficient of surface tension, and ρ the constant density of the fluid. Then there are two nondimensional numbers which are important and defined as $F = c^2/(gH)$, the Froude number, and $\tau = T/(\rho g H^2)$, the Bond number.

When F is not close to 1, the linear theory of water waves is applicable. But when F approaches to 1, the solutions of linearized equations of water waves will grow to infinity (Peters and Stoker [17]). Therefore for F close to 1 nonlinear effect must be taken into account and thus $F = 1$ is a critical value. The first study of a solitary wave on water with surface tension is due to Korteweg and DeVries [11] after whom the K-dV equation with surface tension effect is named. A stationary K-dV equation with Bond number not near $1/3$ can also be formally derived by different approaches. However, if τ is close to 1, the formal derivation of the stationary K-dV equation fails. Thus $\tau = 1/3$ is also a critical value.

It becomes apparent that the problems for F near 1 and for τ near $1/3$ depend on each other and are difficult because they are not only strongly nonlinear, but also very delicate. Since the full nonlinear equations for the water waves are too complicated to study, it is of interest to study model equations. In Hunter and Vanden-Broeck's work [9], a fifth order ordinary differential equation considered as a perturbed stationary K-dV equation was obtained with the assumption that $F = 1 + F_2\epsilon^2$, $\tau = 1/3 + \tau_1\epsilon$ and ϵ is a small positive parameter. By integrating the fifth order ordinary differential equation once and set the con-stant of integration to be zero, then the model equation becomes

$$2F_2\eta - \frac{3}{2}\eta^2 + \tau_1\eta_{xx} - \frac{1}{45}\eta_{xxxx} = 0 .$$

The model equation has been studied extensively by many authors [1-7,9] and several types of solutions have been found, such as periodic solutions [1, 5, 6, 7], solitary wave solutions [2-7,9], generalized solitary wave solutions (solitary waves with osciallatory tails at infinity) in the parameter region $\tau_1 < 0$ and $F_2 > 0$ [1,9], etc.

We add a bump $y = b(x)$ at the bottom of the two-dimensional ideal fluid flow and then derive a forced model equation

$$2F_2\eta - \frac{3}{2}\eta^2 + \tau_1\eta_{xx} - \frac{1}{45}\eta_{xxxx} = b \qquad (1)$$

Equation (1) has been studied extensively by Tsai and Guo [21-27] and several types of solutions have been found.

In this paper, we shall prove that there exist solitary wave solutions of equation (1).

II. Derivation of the Model Equation

We consider the two-dimensional flow of an irrotational incompressible inviscid fluid of constant density ρ^* with surface tension T^* in a two-dimensional channel of finite depth. A rectangular coordinate system $(x^*,\ y^*)$ is chosen such that the flow is bounded above by the free surface $y^* = \eta^*(x^*, t^*)$ and below by the rigid horizontal bottom with a bump $y^* = -H + \mathbf{b}^*(x^*)$.

The governing equations are:

In $-\infty < x^* < \infty, -H + \mathbf{b}^*(x^*) < y^* < \eta^*$

$$\phi^*_{x^*x^*} + \phi^*_{y^*y^*} = 0 , \qquad (2)$$

at the free surface, $y^* = \eta^*$

$$\eta^*_{t^*} + \phi^*_{x^*}\eta^*_{x^*} - \phi^*_{y^*} = 0 , \qquad (3)$$

$$\phi^*_t + \frac{1}{2}({\phi^*_{x^*}}^2 + {\phi^*_{y^*}}^2) + g\eta^* - \frac{T^*\eta^*_{x^*x^*}}{\rho^*(1+{\eta^*_{x^*}}^2)^{\frac{3}{2}}} = \frac{B^{*2}}{2}. \qquad (4)$$

at the bottom, $y^* = -H + b^*(x^*)$

$$\phi^*_{y^*} - \phi^*_{x^*}\mathbf{b}^*_{x^*} = 0 \qquad (5)$$

CHUNG-HSIEN TSAI is with the Department of Information Technology, Overseas Chinese University, 407 Taichung, TAIWAN (R.O.C.) (e-mail: ctsai@ocu.edu.tw).

SHY-JEN GUO is with the Department of International Trade, National Taichung University of Science and Technology, 404 Taichung, TAIWAN (R.O.C.) (corresponding author. e-mail: ch861997@yahoo.com).

JENG-CHI YAN is with the Department of Information Technology, Overseas Chinese University, 407 Taichung, TAIWAN (R.O.C.) (e-mail: jengchi@ocu.edu.tw).

Where $\phi^*(x^*,y^*,t^*)$ is the potential function, B^* is an arbitrary constant, and H is the depth when the bump $\mathbf{b}^*$ is zero. In order to investigate long waves and derive asymptoyic solutions, it is conventient to introduce the following dimensionless variables:

$$\left.\begin{aligned} &x=\tfrac{x^*}{L},\, y=\tfrac{y^*}{H},\, t=(\tfrac{H}{L})^4\,\tfrac{(gH)^{\frac12}}{L}t^*,\\ &\eta(x,t)=\tfrac{\eta^*(x^*,t^*)}{A},\, B=\tfrac{B^*}{(gH)^{\frac12}}\\ &\phi(x,y,t)=\tfrac{H}{LA(gH)^{\frac12}}\phi^*(x^*,y^*,t^*),\\ &\tau=\tfrac{T^*}{\rho^* gH^2},\, \mathbf{b}(x)=\tfrac{(H/L)^{-2M}}{H}\mathbf{b}^*(x^*),\end{aligned}\right\} \tag{6}$$

where M is a positive integer to be chosen later.
In terms of the nondimensional variables (6), (2)-(5) become:

In $-\infty<x<\infty, -1+\beta^M\mathbf{b}(x)<y<\alpha\eta$

$$\beta\phi_{xx}+\phi_{yy}=0, \tag{7}$$

At the free surface, $y=\alpha\eta$

$$\beta^2\eta_t+\alpha\phi_x\eta_x-\beta^{-1}\phi_y=0, \tag{8}$$

$$\beta^2\phi_t+\frac{\alpha}{2}(\phi_x^2+\beta^{-1}\phi_y^2)+\eta-\frac{\beta\tau\eta_{xx}}{(1+\alpha^2\beta\eta_x^2)^{\frac32}}=\frac{B^2}{2\alpha}, \tag{9}$$

at the bottom, $y=-1+\beta^M\mathbf{b}(x)$

$$\phi_y-\beta^{M+1}\phi_x\mathbf{b}_x=0. \tag{10}$$

In (7)-(10), α,β , and τ are nondimensional parameters

$$\alpha=\frac{A}{H},\quad \beta=(\frac{H}{L})^2,\quad \tau=\frac{T^*}{\rho^* gH^2}. \tag{11}$$

We seek solutions for periodic water waves of wavelength λ^*, and introduce the dimensionless wavelength

$$\lambda=\frac{\lambda^*}{L}, \tag{12}$$

The Froude number F is defined as

$$F=\frac{c}{(gH)^{\frac12}}=\frac{\alpha}{\lambda}\int_0^\lambda\phi_x dx. \tag{13}$$

Since we are interested in small amplitude and shallow-water waves with τ near $\frac13$, in (7)-(10), we take

$$\alpha=\epsilon^2,\quad \beta=\epsilon. \tag{14}$$

and expand η,ϕ,τ , and B as

$$\left.\begin{aligned} &\eta=\eta_0+\epsilon\eta_1+\epsilon^2\eta_2+\cdots\\ &\phi=\tfrac{Bx}{\epsilon^2}+\phi_0+\epsilon\phi_1+\epsilon^2\phi_2+\cdots\\ &\tau=\tfrac13+\epsilon\tau_1+\epsilon^2\tau_2+\cdots\\ &B=B_0+\epsilon B_1+\epsilon^2 B_2+\cdots\\ &F=F_0+\epsilon F_1+\epsilon^2 F_2+\cdots\end{aligned}\right\} \tag{15}$$

Substituting (14) and (15) into (7)-(10), taking M = 4 in (10), and expanding at the boundary condition $y=0$ and $y=-1$, we obtain in $-\infty<x<\infty$, $-1<y<0$

$$\epsilon(\phi_{0_{xx}}+\epsilon\phi_{1_{xx}}+\epsilon^2\phi_{2_{xx}}+O(\epsilon^3))+(\phi_{0_{yy}}+\epsilon\phi_{1_{yy}}+\epsilon^2\phi_{2_{yy}}+O(\epsilon^3))=0, \tag{16}$$

at $y=0$,

$$\begin{aligned} &\epsilon^2(\eta_{0t}+\epsilon\eta_{1t}+O(\epsilon^2))\\ &+\epsilon^2\{(\frac{B_0+\epsilon B_1+\epsilon^2B_2+O(\epsilon^3)}{\epsilon^2}+\phi_{0_x}(x,0,t)+O(\epsilon))\}\\ &(\eta_{0x}+\epsilon\eta_{1x}+\epsilon^2\eta_{2x}+O(\epsilon^3))\\ &-\epsilon^{-1}\{(\phi_{0y}(x,0,t)+\epsilon^2(\eta_0+\epsilon\eta_1+O(\epsilon^2))\phi_{0yy}(x,0,t)\\ &+\epsilon(\phi_{1y}(x,0,t)+\epsilon^2(\eta_0+\epsilon\eta_1+O(\epsilon^2))\phi_{1yy}(x,0,t)\\ &+O(\epsilon^4))+\epsilon^2(\phi_{2y}(x,0,t)+O(\epsilon^2))+\epsilon^3(\phi_{3y}(x,0,t)\\ &+O(\epsilon^2))+O(\epsilon^4))\}=0,\end{aligned} \tag{17}$$

$$\begin{aligned} &\epsilon^2(\phi_{0t}(x,0,t)+\epsilon\phi_{1t}(x,0,t)+O(\epsilon^2))\\ &+\frac{\epsilon^2}{2}\{\{\frac{B_0+\epsilon B_1+\epsilon^2B_2+O(\epsilon^3)}{\epsilon^2}\\ &+(\phi_{0x}(x,0,t)+\epsilon^2\eta_0\phi_{0xy}(x,0,t))+\epsilon\phi_{1x}(x,0,t)\\ &+\epsilon^2\phi_{2x}(x,0,t)+O(\epsilon^3)\}^2+\epsilon^{-1}\{(\phi_{0y}+\epsilon^2\phi_{0yy})+\epsilon\phi_{1y}\\ &+\epsilon^2\phi_{2y}+O(\epsilon^3)\}^2\}+(\eta_0+\epsilon\eta_1+\epsilon^2\eta_2+O(\epsilon^3))\\ &-\epsilon(\frac13+\epsilon\tau_1+\epsilon^2\tau_2+O(\epsilon^3))(\eta_{0xx}+\epsilon\eta_{1xx}+\epsilon^2\eta_{2xx}\\ &+O(\epsilon^3))(1+O(\epsilon^{10}))\\ &=\frac{(B_0+\epsilon B_1+\epsilon^2B_2+\epsilon^3B_3+\epsilon^4B_4+O(\epsilon^5))^2}{2\epsilon^2},\end{aligned} \tag{18}$$

at $y=-1$

$$\begin{aligned} &(\phi_{0y}(x,-1,t)+\epsilon\phi_{1y}(x,-1,t)+\epsilon^2\phi_{2y}(x,-1,t)\\ &+\epsilon^3\phi_{3y}(x,-1,t)+O(\epsilon^4))-\epsilon^5(\frac{B_0+\epsilon B_1+\epsilon^2B_2+O(\epsilon^3)}{\epsilon^2})\\ &+\phi_{0x}(x,-1,t)+O(\epsilon))\mathbf{b}_x=0.\end{aligned} \tag{19}$$

From (16) to (19), we have

$O(\epsilon^{-1})$:

$$\phi_{0y}(x,0,t)=0. \tag{20}$$

$O(1)$:

$$\phi_{0yy}(x,y,t)=0, \tag{21}$$

$$B_0\eta_{0x}-\phi_{1y}(x,0,t)=0, \tag{22}$$

$$B_0\phi_{0x}(x,0,t)+\eta_0=0, \tag{23}$$

$$\phi_{0y}(x,-1,t)=0. \tag{24}$$

From (21) and by (22) or (24), it follows that

$$\phi_{0y}=0,\quad \phi_0(x,y,t)=\phi_0(x,t). \tag{25}$$

$O(\epsilon)$:

$$\phi_{0xx}(x,t)+\phi_{1yy}(x,y,t)=0, \tag{26}$$

$$B_0\eta_{1x}+B_1\eta_{0x}-\phi_{2y}(x,0,t)=0, \tag{27}$$

$$B_0\phi_{1x}(x,0,t)+B_1\phi_{0x}(x,0,t)+\tfrac{\phi_{0y}^2(x,0,t)}{2}+\eta_1-\tfrac13\eta_{0xx}=0, \tag{28}$$

$$\phi_{1y}(x,-1,t)=0. \tag{29}$$

From (26) and by (29), we found that

$$\phi_{1y}(x,y,t)=-\phi_{0xx}(x,t)(y+1), \tag{30}$$

and

$$\phi_{1x}(x,y,t)=-\phi_{0xxx}(x,t)(\frac{y^2}{2}+y)+R_{1x}(x,t), \tag{31}$$

From (22), (23), and by (30), we obtain

$$B_0=1, \tag{32}$$

$$\phi_{0x}=-\eta_0. \tag{33}$$

From (28) and by (25), (31), and (32), it follows that

$$\phi_{1xx}(x,0,t)=\frac{1}{3}\eta_{0xxx}-\eta_{1x}+B_1\eta_{0x} \tag{34}$$

$$=R_{1xx}(x,t). \tag{35}$$

$O(\epsilon^2)$:

$$\phi_{1xx}(x,y,t)+\phi_{2yy}(x,y,t)=0, \tag{36}$$

$$\eta_{0t}+B_0\eta_{2x}+B_1\eta_{1x}+(B_2+\phi_{0x})\eta_{0x}-\eta_0\phi_{1yy}-\phi_{3y}=0 \text{ at } y=0, \tag{37}$$

$$\phi_{0t}+\phi_{2x}+B_1\phi_{1x}-B_2\eta_0+\frac{\eta_0^2}{2}+\eta_2-\frac{1}{3}\eta_{1xx}-\tau_1\eta_{0xx}=0 \text{ at } y=0, \tag{38}$$

$$\phi_{2y}(x,-1,t)=0. \tag{39}$$

From (36), (39) and by (31), we found that

$$R_2(x,t)=-\frac{1}{3}\phi_{0xxxx}(x,t)-R_{1xx}(x,t), \tag{40}$$

$$\phi_{2y}(x,y,t)=\phi_{0xxxx}(x,t)(\frac{y^3}{6}+\frac{y^2}{2}+\frac{y}{3})+R_2(x,t)(y+1), \tag{41}$$

and

$$\phi_2(x,y,t)=\phi_{0xxxx}(x,t)(\frac{y^4}{24}+\frac{y^3}{6}+\frac{y^2}{6})+R_2(x,t)(\frac{y^2}{2}+y)+R_3(x,t) \tag{42}$$

From (27) and by (32),(41)

$$R_2(x,t)=\eta_{1x}+B_1\eta_{0x}. \tag{43}$$

From (37) and by (30),(32) ,(33),

$$\eta_{2x}=-\eta_{0t}-B_1\eta_{1x}-(B_2-2\eta_0)\eta_{0x}+\phi_{3y}(x,0,t) \tag{44}$$

Differentiating (38) about x and by (33) , (35) , (42)

$$\eta_{2x}=\eta_{0t}-R_{3xx}-B_1R_{1xx}+(B_2-\eta_0)\eta_{0x}+\frac{1}{3}\eta_{1xxx}+\tau_1\eta_{0xxx}. \tag{45}$$

By (34), (35), (40), and (43)

$$B_1=0. \tag{46}$$

By (44), (45), and (46)

$$\frac{1}{3}\eta_{1xxx}=-2\eta_{0t}-2B_2\eta_{0x}+3\eta_0\eta_{0x}-\tau_1\eta_{0xxx}+R_{3xx}+\phi_{3y}(x,0,t) \tag{47}$$

$O(\epsilon^3)$:

$$\phi_{2xx}(x,y,t)+\phi_{3yy}(x,y,t)=0, \tag{48}$$

$$\phi_{3y}(x,-1,t)=B_0b_x. \tag{49}$$

From (48), (49) and by (42), we obtain

$$\phi_{3y}(x,-1,t)=\frac{1}{45}\phi_{0xxxxxx}(x,t)-\frac{1}{3}R_{2xx}(x,t)+R_{3xx}(x,t)+\phi_{3y}(x,0,t) \tag{50}$$

By (32), (33), (43), (46), and (50), we have

$$2\eta_{0t}+2B_2\eta_{0x}-3\eta_0\eta_{0x}+\tau_1\eta_{0xxx}-\frac{1}{45}\eta_{0xxxxx}=\mathbf{b}_x. \tag{51}$$

The Froude number F is defined and expanded as

$$\begin{aligned}F&=F_0+\epsilon F_1+\epsilon^2F_2+O(\epsilon^3)\\&=\frac{\epsilon^2}{\lambda}\int_0^\lambda(\frac{B_0+\epsilon B_1+\epsilon^2B_2+O(\epsilon^3)}{\epsilon^2}+\phi_{0x}+O(\epsilon))dx\\&=B_0+\epsilon B_1+\epsilon^2B_2+\frac{\epsilon^2}{\lambda}\int_0^\lambda\phi_{0x}dx+O(\epsilon^3).\end{aligned} \tag{52}$$

By (33) and the mean value of periodic solution over a period is zero, we found that

$$\int_0^\lambda\phi_{0x}dx=-\int_0^\lambda\eta_0dx=0\,.$$

If η_0 is a solitary wave solution with the properity that

$$\int_0^\infty\phi_{0x}dx=-\int_0^\infty\eta_0dx<\infty\,, \tag{53}$$

then, with $\lambda=\infty$, the term

$$\frac{1}{\lambda}\int_0^\lambda\phi_{0x}dx$$

in (52) will be zero. We shall see that all the solitary wave solutions discovered in the following chapters will satisfy (53).

Therefore, we have

$$B_0=F_0,\ B_1=F_1,\ B_2=F_2.$$

and then (51) becomes

$$2\eta_{0t}+2F_2\eta_{0x}-3\eta_0\eta_{0x}+\tau_1\eta_{0xxx}-\frac{1}{45}\eta_{0xxxxx}=\mathbf{b}_x. \tag{54}$$

Next, we assume $\eta_{0t}=0$ in equation (54), integrate (54) once and set the constant of integration to be zero, then we have the following model equation

$$2F_2\eta_0-\frac{3}{2}\eta_{0x}^2+\tau_1\eta_{0xx}-\frac{1}{45}\eta_{0xxxx}=\mathbf{b}. \tag{55}$$

In the following sections, we shall use η for η_0 in equation (55), that is,

$$2F_2\eta-\frac{3}{2}\eta^2+\tau_1\eta_{xx}-\frac{1}{45}\eta_{xxxx}=\mathbf{b}. \tag{56}$$

and disscuss the solutions of the model equation (56).

III. Problem Formulation

We follow Zufiria [28] to construct a Hamiltonian associated to (1).

When $\mathbf{b}=0$, we rewrite (1) as

$$\eta_{xxxx}-45\tau_1\eta_{xx}-90F_2\eta+\frac{135}{2}\eta^2=0. \tag{57}$$

We multiply $-\eta_x$ to (57) and integrate the resulting equation, then equation (57) has first integral as

$$H = 45F_2\eta^2 + \frac{1}{2}\eta_{xx}^2 - \eta_{xxx}\eta_x + \frac{45}{2}\tau_1\eta_x^2 - \frac{45}{2}\eta^3, \qquad (58)$$

where H is a constant. Introducing the change of variables

$$\left.\begin{aligned} q_1 &= \eta, \quad p_1 = \eta_{xxx} - 45\tau_1\eta_{x,} \\ q_2 &= \eta_{xx}, \quad p_2 = \eta_{x,} \end{aligned}\right\}$$

then (58) becomes

$$H(q_1, q_2, p_1, p_2) = 45F_2q_1^2 + \frac{1}{2}q_2^2 - p_1p_2 - \frac{45}{2}\tau_1p_2^2 - \frac{45}{2}q_1^3, \qquad (59)$$

and we have

$$\frac{dz}{dx} = J\nabla_z H(z) = Az + g(z) \equiv f(z, \mu), \qquad (60)$$

where $\mu = (\tau_1, F_2) \in \mathbf{R}^2$,

$$z = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \in \mathbf{R}^4, \quad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad (61)$$

and

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -45\tau_1 \\ -90F_2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, g(z) = \begin{pmatrix} 0 \\ 0 \\ \frac{135}{2}q_1^2 \\ 0 \end{pmatrix}. \qquad (62)$$

Therefore (59) is a two degree of freedom Hamiltonian with two parameters τ_1 and F_2. Because different parameters (τ_1, F_2) in (59) give rise to different eigenvalues λ for the linearized system of (60) at the origin, we divide the parameter plane (τ_1, F_2) into following nine cases

Case 0 $(\tau_1 = 0, F_2 = 0)$: $\lambda = 0,0,0,0$.
Case 1 $(\tau_1 \in \mathbf{R}, F_2 > 0)$: $\lambda = \pm r, \pm wi;\ r, w > 0$.
Case 2 $(\tau_1 < 0, F_2 = 0)$: $\lambda = 0, 0, \pm wi;\ w > 0$.
Case 3 $(\tau_1 < 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 > 0)$:
$\lambda = \pm w_1 i, \pm w_2 i;\ w_1 > w_2 > 0$.
Case 4 $(\tau_1 < 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 = 0)$:
$\lambda = \pm wi, \pm wi;\ w > 0$
Case 5 $(\tau_1 \in \mathbf{R}, F_2 < 0,\ (45\tau_1)^2 + 360F_2 < 0)$:
$\lambda = \pm a \pm bi;\ a, b > 0$
Case 6 $(\tau_1 > 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 = 0)$:
$\lambda = \pm r, \pm r;\ r > 0$
Case 7 $(\tau_1 > 0, F_2 < 0,\ (45\tau_1)^2 + 360F_2 > 0)$:
$\lambda = \pm r_1, \pm r_2;\ r_1 > r_2 > 0$
Case 8 $(\tau_1 > 0, F_2 = 0)$: $\lambda = 0, 0, \pm r;\ r > 0$.

We rewrite (1) as follows,

$$\eta_{xxxx} - 45\tau_1\eta_{xx} - 90F_2\eta = -45(\mathbf{b}(x)) + \frac{3}{2}\eta^2) \equiv f, \qquad (63)$$

IV. Solitary wave solutions

We consider the problem

$$\eta_{xxxx} - 45\tau_1\eta_{xx} - 90F_2\eta = \tilde{\mathbf{b}}(x) - \frac{135}{2}\eta^2, -\infty < x < \infty \qquad (64)$$

$$\eta(\infty) = \eta(-\infty) = 0 \qquad (65)$$

where $\tilde{\mathbf{b}}(x) = -45\mathbf{b}(x)$ is even.

Since we are interested in even solutions, we shall only consider $x \in [0, \infty)$ hereafter.

Case 7: $\tau_1 > 0, F_2 = \rho\tau_1^2, \rho \in (-45/8, 0)$

First, we change (64) and (65) to an integral equation by constructing the Green's function $G(x,s)$ of

$$\eta_{xxxx} - 45\tau_1\eta_{xx} - 90F_2\eta = 0, \quad 0 < x < \infty \qquad (66)$$

$$\eta_x(0) = \eta_{xxx}(0) = \eta(\infty) = \eta_x(\infty) = 0. \qquad (67)$$

and obtain

$$G(x,s) = -\frac{1}{2r_1(r_1^2 - r_2^2)}(e^{-r_1|x-s|} + e^{-r_1|x+s|}) + \frac{1}{2r_2(r_1^2 - r_2^2)}(e^{-r_2|x-s|} + e^{-r_2|x+s|}) \qquad (68)$$

where

$$r_1 = \sqrt{(45\tau_1 + \sqrt{(45\tau_1)^2 + 360F_2})/2},$$
$$r_2 = \sqrt{(45\tau_1 - \sqrt{(45\tau_1)^2 + 360F_2})/2}, \quad r_1 > r_2 > 0. \qquad (69)$$

Hence (66) and (67) is equivalent to

$$\eta(x) = \int_0^\infty G(x,s)(\tilde{\mathbf{b}}(s) - \frac{135}{2}\eta^2(s))ds = I_1(x) + I_2(x) \qquad (70)$$

where

$$I_1(x) = \int_0^\infty G(x,s)\tilde{\mathbf{b}}(s)ds,$$
$$I_2(x) = \int_0^\infty G(x,s)(-\frac{135}{2}\eta^2(s))ds \qquad (71)$$

We denote by H_m the Banach space of even functions $f \in C^m(-\infty, \infty)$ with the norm

$$\| f \|_{H_m} = \sup_{0 \le k \le m} C_{\tilde{d}} f^{(k)}(x) \qquad (72)$$

where

$$C_{\tilde{d}} f^{(k)}(x) = \sup_{0 \le x < \infty} (e^{\tilde{d}x} | f^{(k)}(x)|),$$

and $\tilde{d}$ is a constant to be specified later.

Theorem 1 If $g(s) \in H_m$ and $Y(x) = \int_0^\infty G(x,s)g(s)\,\mathrm{ds}$, then $Y(x) \in H_{m+4}$ and

$$\| Y \|_{H_{m+4}} \le C_7 \| g(x) \|_{H_m}, \qquad C_7 = C/(r_2 - \tilde{d})$$

Hereafter we shall use C as a generic positive constant, which is independent of Y and g

Proof: From (64), it is straigthforward to check that $Y(x) \in H_{m+4}$ when $g(s) \in H_m$. Next we show that

$$\sup_{0 \le k \le m+4} (\sup_{0 \le x < \infty} e^{\tilde{d}x} |Y^{(k)}(x)|) \le C \|g(x)\|_{H_m}$$

Here we only consider the case $m = 0$. The proof for other cases are similar. From (68), we have

$$Y(x) + Y_1(x) + Y_2(x) + Y_3(x) + Y_4(x) + Y_5(x) + Y_6(x) + Y_7(x) + Y_8(x)$$

where

$$Y_1(x)=\int_0^x \frac{-1}{2r_1(r_1^2-r_2^2)}e^{-r_1(x-s)}g(s)ds,$$
$$Y_2(x)=\int_0^x \frac{-1}{2r_1(r_1^2-r_2^2)}e^{-r_1(x+s)}g(s)ds,$$
$$Y_3(x)=\int_0^x \frac{1}{2r_2(r_1^2-r_2^2)}e^{-r_2(x-s)}g(s)ds,$$
$$Y_4(x)=\int_0^x \frac{1}{2r_2(r_1^2-r_2^2)}e^{-r_2(x+s)}g(s)ds,$$
$$Y_5(x)=\int_x^\infty \frac{-1}{2r_1(r_1^2-r_2^2)}e^{-r_1(s-x)}g(s)ds,$$
$$Y_6(x)=\int_x^\infty \frac{-1}{2r_1(r_1^2-r_2^2)}e^{-r_1(x+s)}g(s)ds,$$
$$Y_7(x)=\int_x^\infty \frac{1}{2r_2(r_1^2-r_2^2)}e^{-r_2(s-x)}g(s)ds,$$
$$Y_8(x)=\int_x^\infty \frac{1}{2r_2(r_1^2-r_2^2)}e^{-r_2(x+s)}g(s)ds,$$

Estimating $|Y_1|$, we have

$$\begin{aligned}|Y_1| &\le \frac{1}{2r_1(r_1^2-r_2^2)}e^{-r_1x}\int_0^x e^{r_1s}|g(s)|ds\\ &\le Ce^{-r_1x}\|g\|_{H0}\int_0^x e^{(r_1s-\tilde{d}s)}ds\\ &\le \frac{C}{(r_1-\tilde{d})}\|g\|_{H_0}\ e^{-\tilde{d}x}\end{aligned}$$

Hence $C_{\tilde{d}}Y_1(x)\le C\|g\|_{H_0}/(r_1-\tilde{d})$ and it follows that $\|Y_1\|_{H_0}\le C_7\|g\|_{H_0}$, if we choose $d<r_2$. $Y_2,\ldots,Y_8$ can be estimated in the same manner and we obtain $|\tilde{d}|<r_2$.

Let

$$S_7=\left\{\eta\in H_m \|\eta\| H_m\le M,\ m\ge 4\right\}, \tag{73}$$

where M is positive and will be specified later. We also define an operator

$$Q_7(\eta)(x)=\int_0^\infty G(x,s)(\tilde{\mathbf{b}}(s)-\frac{135}{2}\eta^2(s))ds. \tag{74}$$

We want to show that the operator Q_7 maps S_7 into S_7 and it is a contraction. Then, (64) subject to (65) has a solitary wave solution.

Theorem 2 Assume $\tilde{b}\in H_{m-4},\ 0<\tilde{d}<r^2$ with M and $\|\tilde{b}\|_{H_{m-4}}$ satisfying (77), (78), and (79), then the operator Q_7 maps S_7 into S_7.

Proof: Assume $\eta\in S_7$ and by **Theorem 1**,

$$\|I_1(x)\|_{H_m}=\|\int_0^\infty G(x,s)\tilde{b}(s)ds\|_{H_m}\le C_7\|\tilde{b}\|_{H_{m-4}}. \tag{75}$$

Before estimating $\|I_2(x)\|_{H_m}$, we first show that $\|\eta^2\|_{H_m}\le 2^{m+1}M^2$. Since $\|\eta\|_{H_m}\le M$, we have

$$e^{\tilde{d}x}|\eta^{(k)}(x)|\le M,$$

for k = 0, 1,…,m and $0\le x<\infty$, If we choose $0<\tilde{d}<r_2$, then

$$e^{\tilde{d}x}|\eta^{(j)}(x)\|\eta^{(k)}(x)|\le e^{-\tilde{d}x}M^2\le M^2,$$

$0\le j+k\le m$, j, k = 0, 1,…, m,

and it follows that $\eta^2\in H_{m-4}$ and $\|\eta^2\|_{H_{m-4}}\le 2^{m-3}M^2$. Now,

$$\begin{aligned}\|I_2(x)\|_{H_m}&=\|-\frac{135}{2}\int_0^\infty G(x,s)\eta^2(s)ds\|_{H_m}\le\frac{135}{2}C_7\|\eta^2\|_{H_{m-4}}\\ &\le 135(2^{m-4})C_7M^2\end{aligned} \tag{76}$$

By (75) and (76), we have

$$\|Q_7\|_{H_m}\le\|I_1\|_{H_m}+\|I_2\|_{H_m}\le C_7\|\tilde{b}\|_{H_{m-4}}+135(2^{m-4})C_7M^2.$$

In order to have $\|Q_7\|_{H_m}<M$, we need to choose

$$M^-<M<M^+, \tag{77}$$

where

$$M^{\pm}=\tfrac{1\pm\sqrt{1-4M_1M_2}}{2M_1},\quad M_1=135(2^{m-4})C_7,\quad M_2=C_7\|\tilde{b}\|_{H_{m-4}}. \tag{78}$$

Also from $1-4M_1M_2\ge 0$, we obtain

$$\|\tilde{b}\|_{H_{m-4}}\le\frac{1}{135(2^{m-2})C_7^2}. \tag{79}$$

Furthermore, we have

Theorem 3 By following the same assumptions in Lemma 5, if (80) and (81) hold, then $Q_7:S_7\to S_7$ is a contraction.

Proof: Assume $\eta_1,\eta_2\in S_7$,

$$\begin{aligned}&\|Q_7(\eta_1)-Q_7(\eta_2)\|_{H_m}\\ &\le \|\int_0^\infty G(x,s)(\tilde{\mathbf{b}}(s)-\frac{135}{2}\eta_1^2(s))ds\\ &\quad-\int_0^\infty G(x,s)(\tilde{\mathbf{b}}(s)-\frac{135}{2}\eta_2^2(s))ds\|_{H_m}\\ &\le \frac{135}{2}\|\int_0^\infty G(x,s)(\eta_1^2(s)-\eta_2^2(s))ds\|_{H_m}\\ &\le \frac{135}{2}\|\eta_1^2-\eta_2^2\|_{H_{m-4}}\\ &\le \frac{135}{2}\|\eta_1^2-\eta_2^2\|_{H_m}\\ &\le 135(2^m)M\|\eta_1-\eta_2\|_{H_m}\end{aligned}$$

For choosing $135(2^m)\ M<1$ and having (77) satisfied, we need

$$M<min\{M^+,\frac{1}{135(2^m)}\}. \tag{80}$$

From (77) and (80), we also need

$$M^-<\frac{1}{135(2^m)},$$

that is,

$$1-\sqrt{1-4M_1M_2}<\frac{C_7}{8}=\frac{C}{8(r_2-\tilde{d})}. \tag{81}$$

The inequality (81) will hold if we choose

$$max\{0,r_2-\frac{C}{8(1-\sqrt{1-4M_1M_2})}\}<\tilde{d}<r_2. \tag{82}$$

Since $Q_7:S_7\to S_7$ is a contraction, and it follows that there exists a fixed point η_0 is S_7 such that

$$\eta_0=Q_7(\eta_0).$$

For Case 5 and Case 6, the proofs of the existence of a solitary wave solution of equation (1) are similar to Case 7.

Thus we only give a brief discussion for Case 5 and Case 6 and the associated Green's function.

Case 5:

Here $F_2 < -\frac{45}{8}\tau_1^2$, the Green's function is given by

$$G(x,s) = \frac{1}{4ab\sqrt{a^2+b^2}}(cos(b\,|\,x-s\,|-\theta)e^{-a|x-s|} + cos(b\,|\,x+s\,|-\theta)e^{-a|x+s|}), \qquad (83)$$

where $0 < x < \infty$,

$$a = \sqrt{(45\tau_1 + \sqrt{-360F_2})/2} > 0,$$

$$b = \sqrt{(-45\tau_1 + \sqrt{-360F_2})/2} > 0,$$

$$sin(\theta) = \frac{a}{\sqrt{a^2+b^2}},$$

$$cos(\theta) = \frac{b}{\sqrt{a^2+b^2}}.$$

Case 6:

Here in this case, $F_2 = -\frac{45}{8}\tau_1^2, \tau_1 > 0$, the Green's function becomes

$$G(x,s) = -\frac{1}{4r^3}((1+r\,|\,x-s\,|)e^{-r|x-s|} + (1+r\,|\,x+s\,|)e^{-r|x+s|}), \quad (84)$$

where $r = \sqrt{\frac{45\tau_1}{2}}$, and $Y(x)$ in **Theorem 1** becomes

$$Y(x) = Y_1(x) + Y_2(x) + Y_3(x) + Y_4(x) + Y_5(x) + Y_6(x) + Y_7(x) + Y_8(x)$$

where

$$Y_1(x) = \tfrac{1}{4r^3}\int_0^x e^{-r(x-s)}g(s)ds,$$

$$Y_2(x) = \tfrac{1}{4r^3}\int_0^x e^{-r(x+s)}g(s)ds,$$

$$Y_3(x) = \tfrac{1}{4r^3}\int_0^x r(x-s)e^{-r(x-s)}g(s)ds,$$

$$Y_4(x) = \tfrac{1}{4r^3}\int_0^x r(x+s)e^{-r(x+s)}g(s)ds,$$

$$Y_5(x) = \tfrac{1}{4r^3}\int_x^\infty e^{-r(s-x)}g(s)ds,$$

$$Y_6(x) = \tfrac{1}{4r^3}\int_x^\infty e^{-r(x+s)}g(s)ds,$$

$$Y_7(x) = \tfrac{1}{4r^3}\int_x^\infty r(s-x)e^{-r(s-x)}g(s)ds,$$

$$Y_8(x) = \tfrac{1}{4r^3}\int_x^\infty r(x+s)e^{-r(x+s)}g(s)ds.$$

Estimating $|Y_3|$, we have

$$|Y_3| \le \frac{1}{4r^2}e^{-rx}\int_0^x (x-s)e^{rs}\,|\,g(s)\,|ds$$

$$\le C_1e^{-rx}\,\|\,g\,\|_{H_0}\int_0^x (x-s)e^{(r-\tilde{d})s}ds$$

$$= C_1\,\|\,g\,\|_{H_0}\,(\frac{1}{(d-r)^2}(e^{-\tilde{d}x} - e^{-rx}(1+(r-d)x))$$

$$\le C\,\|\,g\,\|_{H_0}\,e^{-\tilde{d}x}.$$

Hence $C_{\tilde{d}}Y_3(x) \le C\,\|\,g\,\|_{H_0}$ and it follows that $\|\,Y_3\,\|_{H_0} \le C\,\|\,g\,\|_{H_0}$, if we choose $\tilde{d} < r$.

$$|Y_7| \le \frac{1}{4r^2}e^{rx}\int_x^\infty (s-x)e^{-rs}\,|\,g(s)\,|\,ds$$

$$\le C_1e^{rx}\,\|\,g\,\|_{H_0}\int_x^\infty (s-x)e^{-(r+\tilde{d})s}ds$$

$$= \frac{C_1}{(d+r)^2}\,\|\,g\,\|_{H_0}\,e^{-\tilde{d}x}$$

$$\le C\,\|\,g\,\|_{H_0}\,e^{-\tilde{d}x}$$

Hence $C_{\tilde{d}}Y_7(x) \le C\,\|\,g\,\|_{H_0}$, and it follows that $\|\,Y_7\,\|_{H_0} \le C\,\|\,g\,\|_{H_0}$ if we choose $\tilde{d} > -r$. Other Y_i, i = 1, 2, 4 to 6, and 8 can be estimated in the same manner and we obtain $|\,\tilde{d}\,| < r$.

V. NUMERICAL RESULTS

In this section, we shall give solitary wave solutions of equation (1) numerically by using classical fourth-order Runge-Kutta method.

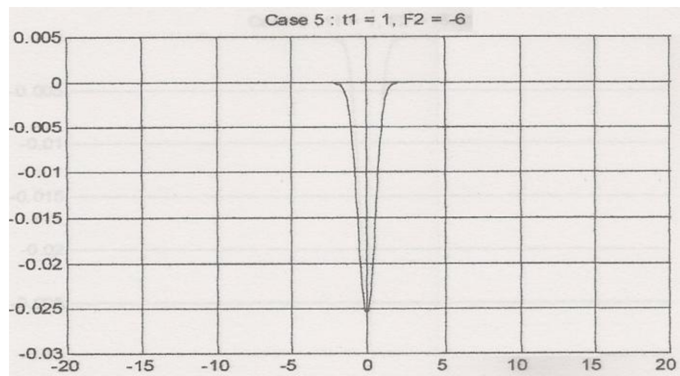

Figure 1: A solitary solution of equation (1) for Case 5 with $\tau_1 = 1, F_2 = -6$, and compact bump $b(x) = \exp(1/(x^2-1))$ on interval $(-1,1)$.

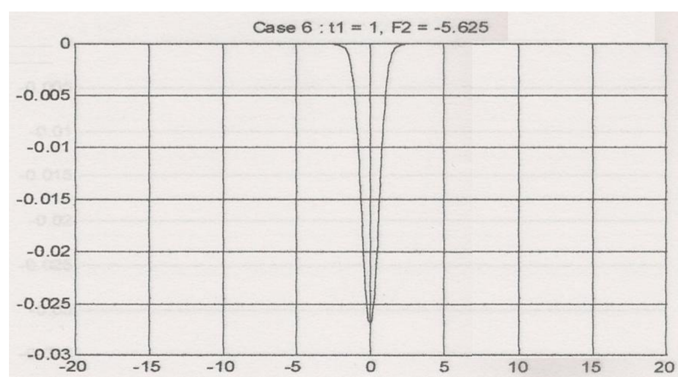

Figure 2: A solitary solution of equation (1) for Case 6 with $\tau_1 = 1, F_2 = -45/8$, and compact bump $b(x) = \exp(1/(x^2-1))$ on interval $(-1,1)$.

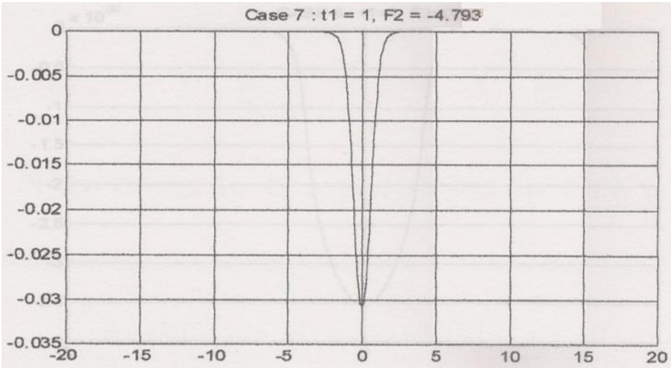

Figure 3: A solitary solution of equation (1) for Case 7 with $\tau_1 = 1, F_2 = -810/169 \approx -4.793$, and compact bump $b(x) = \exp(1/(x^2 - 1))$ on interval $(-1,1)$.

VI. Conclusion

We showed the existence of solitary wave solutions of a fifth order model equation for steady capillary-gravity waves over a bump with the Bond number near 1/3.

References

[1] Amick, C.J. & Mcleod, J.B.," A singular perturbation problem in waterwaves," Stability Appl. Anal. Continuous Media 1, 127-148 (1991)

[2] Amick, C.J. & Toland, J.F.," Solitary waves with surface tension I: Trajectories homoclinic to periodic orbits in four dimensions," Arch. Rat. Mech. Anal. 118, 37-69 (1992)

[3] Amick, C.J. & Toland, J.F.,"Homoclinic orbits in the dynamic phasespace analogy of an elastic strut," Euro. Jnl of Applied Mathematics 3, 97-114 (1992)

[4] B. Buffoni, A.R. Champneys, A.R., Toland, J.F.," Bifurcation and colescence of a plethora of homoclinic orbits for a Hamiltonian system," J. Dyn. Differential Equations 8, 221-281 (1996)

[5] Champneys, A.R. "Homoclinic orbits in reversible systems and their applications in mechanics, fluids and optics," Physica D 112, 158-186 (1998)

[6] Coppel, W.A. "Stability and Asymptotic Behavior of Differential equations," D.C. HEATH AND COMPANY, (1965)

[7] Coppola, V.T. & Rand, R.H. "Computer Algebra Implemention of Lie Transforms for Hamilton- ian Systems: Application to the Nonlinear Stability of L_4," ZAMM. Z. angew. Math. Mech 69, 275-284 (1989)

[8] Hirsch, M. W. & Smale, S."Differential Equations, Dynamical Systems and Linear Algebra," Academic Press, New York, (1974)

[9] Hunter J.K., and Vanden-Broeck, J.M ., "Solitary and periodic gravity-capillary waves of finite amplitude," J. Fluid Mech. 134, 205-219 (1983)

[10] Iooss, G. & Kirchgässner, K. "Water waves for small surface tension: an approach via normal form," Proceedings of the Royal of Edinburgh 122A, 267-299 (1992)

[11] Korteweg, P.J. and de Vries, G. "On the change of the form of long waves advancing in a rectangular canal, and on a new type of long stationary waves," Phil. Mag. 39, 422-443 (1895)

[12] Markeev, A.P. "Stability of a canonical system with two degrees of freedom in the presence of resonance," Journals of Applied Mathematics and Mechanics 32, 766-722 (1968)

[13] Merkin, D.R. "Intrduction to the Theory of Stability," Springer-Verlag, TAM 24, (1997)

[14] Meyer, K.R. & Schmidt, D.S."The stability of the Lagrange triangular point and a theorem of Arnold," J. Diff. Eqs. 62, 222-236 (1986a)

[15] Meyer, K.R. & Hall, G.R. "Intrduction to Hamiltonian Dynamical Systems and the N-Body Problem," Springer-Verlag, 90, (1990)

[16] Moser, J."New aspects in the theory of stability of Hamiltonian systems," Commun. Pure Appl. Math. Vol.11, No.1, (1958)

[17] Peters, A.D.; Stoker, J.J."Solitary waves in liquids having non-constant density," Comm. Pure Appl. Math. 13, 115-164 (1960)

[18] Sokol'skii, A.G. "On the stability of an autonomous Hamiltonian system with two degrees of freedom in the case of equal frequencies," J. Appl. Math. Mech., 38, 741-749 (1975)

[19] Sokol'skii, A.G. "On stability of an autonomous Hamiltonian system with two degrees of freedom under first-order resonance," J. Appl. Math. Mech., 41, no 1,20-28 (1977)

[20] Sokol'skii, A.G. "On stability of self-contained Hamiltonian system with two degrees of freedom in the case of zero frequencies," J. Appl. Math Mech., 45, no 3,321-327 (1982)

[21] Tsai, Chung-Hsien & Guo, Shy-Jen " Unsymmetric Solitary Wave Solutions of a Fifth Order Model Equation for Steady Capillary-Gravity Waves over a Bump with the Bond Number Near 1/3.," Proceedings of the 6th WSEAS International Conference on FLUID MECHANICS (FLUIDS'09), pp.94-99, Ningbo, China January 10-12 (2009).

[22] Tsai, Chung-Hsien & Guo, Shy-Jen (2009) " The existence of generalized solitary wave solutions of a fifth order model equation for steady capillary-gravity waves over a bump with the Bond number near 1/3.," Proceedings of the 6th WSEAS International Conference on FLUID MECHANICS (FLUIDS'09), pp.86-89 Ningbo, China January 10-12 (2009).

[23] Tsai, Chung-Hsien & Guo, Shy-Jen " Solitary Wave Solutions of a Fifth Order Model Equation for Steady Capillary - Gravity Waves over a Bump with the Bond Number Near 1/3.," Proceedings of the 7th WSEAS International Conference on FLUID MECHANICS (FLUIDS'10), pp. 236-240, University of Cambridge, UK, February 23-25 (2010).

[24] Tsai, Chung-Hsien & Guo, Shy-Jen " The existence of generalized solitary wave solutions of a fifth order model equation for steady capillary-gravity waves over a bump

with the Bond number near 1/3.," International Conference on APPLIED and COMPUTATIONAL MATHEMATICS (ICACM'11), pp.49-53, Lanzarote, Canary Islands, Spain, May 27-29 (2011).

[25] Tsai, Chung-Hsien & Guo, Shy-Jen (2011) " Asymptotic Solutions and Unsymmetric Solutions of a Fifth Order Ordinary Differential Equation," WSEAS TRANSACTIONS on FLUID MECHANICS 6, 1-11 (2011)

[26] Tsai, Chung-Hsien & Guo, Shy-Jen (2012) "Bounded solutions of a fifth order model equation for steady capillary-gravity waves over a bump with the Bond number near 1/3," Proceedings of the AMERICAN CONFERENCE on APPLIED MATHEMATICS (AMERICAN-MATH '12) ,pp. 145-148, Harvard, Cambridge, January 25-27 (2012)

[27] Chung-Hsien Tsai, Shy-Jen Guo, and Jeng-Chi Yan (2013) "Stability of Fixed Points of a Fifth Order Equation," 18th International Conference on APPLIED MATHEMATICS (AMATH '13), pp. 189-193, Budapest, Hungary, December 10-12 (2013)

[28] Zufiria, J.A. "Weakly nonlinear non-symmetric gravity waves on water of finite depth," J. fluid Mech. 180, 371-385 (1987)