Numerical Analysis of Gravity and Parabolic Catenaries

J. Vasek, O. Sucharda

***Abstract*—** This paper analyses gravity and parabolic catenaries. It discusses development of an algorithm for tasks and optimising of the calculation. Different iteration methods have been used in order to optimise the calculation. The iteration methods are: direct iteration, regula falsi, bisection and Newton methods. Development of the algorithm is used in different tasks. For each method, attention is paid to suitability of application, needed time and number of calculation steps needed in order to achieve the correct result. Matlab was used for development of algorithms for tasks.

***Keywords*—** gravity catenary, parabolic catenary, iteration methods, development of an algorithm

I. Introduction

THIS paper analyses the gravity and parabolic catenaries. Catenaries are commonly used analysis methods for cable structures discussed in [1, 2]. Several approaches based on numerical methods with special catenary element are available [3, 4] when analysing the catenary or cable structures. Nonlinear analysis of cable structures is shown in [5, 6, 7]. The solution can be made on the basis of discrete analysis [8]. The Finite Element Method is used when dealing with cable structures for bridges [9].

This paper uses four interaction methods for analysis of gravity and a parabolic catenary. Computational complexity is compared for those methods. The methods are direct iteration, regula falsi, bisection and tangential methods.

The iteration method has been used for calculation of some model tasks. Then, the optimum method which is suitable for a general solution has been chosen. The Matlab software [10] was used for computations. Algorithms for civil engineering tasks were developed in Matlab, for instance, in [11] and [12]. The development of an algorithm was based on methods described in [13].

When analysing the steel structures, it is recommended in some cases to use the probabilistic approach [14].

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II. Theory of the Parabolic Catenary

The studied problem was a parabolic catenary in the Fig. 1 with an additional condition – the total length.

$$L = \frac{H}{2q}\left[\lambda_a\sqrt{1+\lambda_a^2} + \ln\left(\lambda_a + \sqrt{1+\lambda_a^2}\right) + \lambda_b\sqrt{1+\lambda_b^2} + \ln\left(\lambda_b + \sqrt{1+\lambda_b^2}\right)\right] \quad (1)$$

$$\lambda_a = \frac{qx_d}{H},\ \lambda_b = \frac{q(l - x_d)}{H} \quad (2)$$

and

$$x_d = \frac{l}{2} + \frac{Hh}{ql} \quad (3)$$

The cable which is suspended in two joints and loaded with a continuous load applied onto the horizontal projection is referred to as a parabolic catenary [15] and [16]. Because of a very low bending stiffness, the load-carrying cable is considered in calculations to be an element which does not bear bending moments.

The only internal force which arises in the structure is tensile force. All geometric and static quantities are expressed by means of horizontal reaction. Of importance for description of the structure is determination of the horizontal force. The horizontal force cannot be described using conditions of balance only. It is necessary to choose an additional condition. In this case, the additional condition is the specified length of the cable. When calculating the horizontal force, the equation for the cable length (1) is taken as a basis. For more details about calculation see (2) and (3). The horizontal force H was determined using iteration methods.

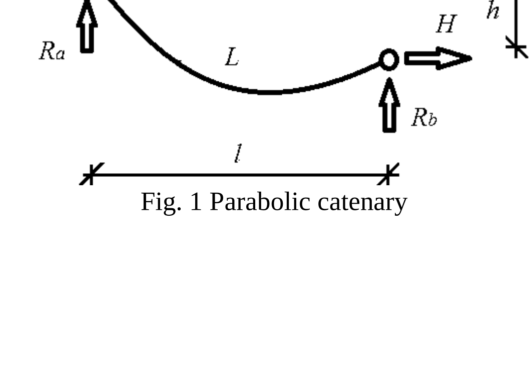

Fig. 1 Parabolic catenary

III. THEORY OF THE GRAVITY CATENARY

The gravity catenary can be regarded as a perfectly flexible cable which is not able to transfer other internal forces than tensile normal forces. In this case, a cable without prolongation has been considered. The difference between the parabolic catenary and gravity catenary is the way of loading. The fibre in Fig. 2 is loaded with a continuous load applied onto the fibre axis.

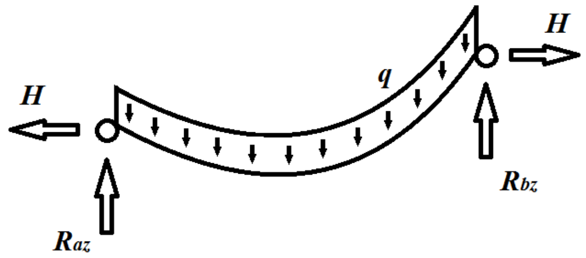

Fig. 2 Loading of the loaded catenary

This type of load can be represented by the dead load. Effects of the dead load depend on cable deflection. In general, the higher the deflection is, the bigger the effects of the dead weight are. For this reason, it is not always possible to simplify the situation and use a parabolic catenary. In this case, it is also assumed that the both ends of the fibre are fixed to non-displaceable supports.

$$z = a.\cosh\frac{x}{a} \tag{4}$$

$$z" = \frac{1}{a}\sqrt{1+z'} \tag{5}$$

The dead weight of each cable make the cable to shape as a catenary. From the mathematical point of view, the catenary shape can be described using (4). Because it is mathematically easy to derive a differential equation (5) of a genuine gravity catenary, such a coordinate system is chosen where the gravity catenary crosses the vertical axis in the lowest point.

The coordinate of the point of intersection of the catenary and the axis is (a) - this parameter is used then to derive all other geometric quantities which characterise the catenary. Fig. 3 shows this location in the coordinate system.

As this is not statically determined task, it is essential to include an additional condition into the calculation. In the case, the additional condition is the know length of the cable. Mathematical description of the cable length is obtained by (6). If coordinates and and cable length are known, the equations (4) and (6) can be used to derived (7).

Because the parameter (a) cannot be expressed explicitly, iteration methods should be used.

$$l = \int_{x_B}^{x_A}\sqrt{1+\left(\frac{dz}{dx}\right)^2}\,dx = \int_{x_B}^{x_A}\sqrt{1+\sinh\left(\frac{x}{a}\right)}dx = \left[a.\sinh\left(\frac{x}{a}\right)\right]_{x_A}^{x_B} \tag{6}$$

$$\sqrt{l^2-h^2} = 2.a.\sinh\frac{L}{2.a} \tag{7}$$

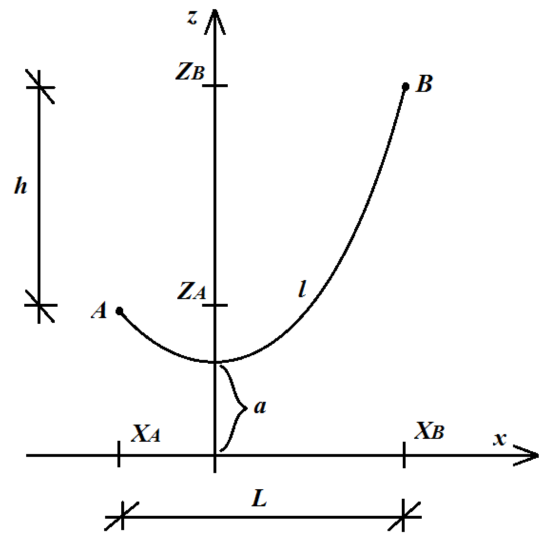

Fig. 3 Location of the catenary in the coordinate system

IV. CALCULATING THE PARABOLIC CATENARY

A. A model case of the parabolic catenary

The distance between the suspended points for this structure is l = 30 m and the difference in height is h = 1 m. The continuous load applied onto the cable projection is q = 0.8 kN/m. The additional condition – the length of the cable – is L = 33 m.

B. Iteration methods

All iteration methods are based on the equations (1) and (2). Another condition for those methods is selection of specific criteria, for instance, the value of the first approximation or the termination condition. In order to compare the solutions, same values for identical criteria were maintained. The use of the numerical methods was based on [17] and [18].

C. Direct iteration

For a graphical representation of this method see Fig. 4. The initial equation (1) was modified and one side shows the horizontal force only – that side of the equation represents the linear function (dotted), while the other side of the equation comprises other input parameters. This is the intersection of the solid line and dotted curve.

Then, the zero approximation and iteration cycles result in the final value. This is the intersection of the solid line and dotted curve.

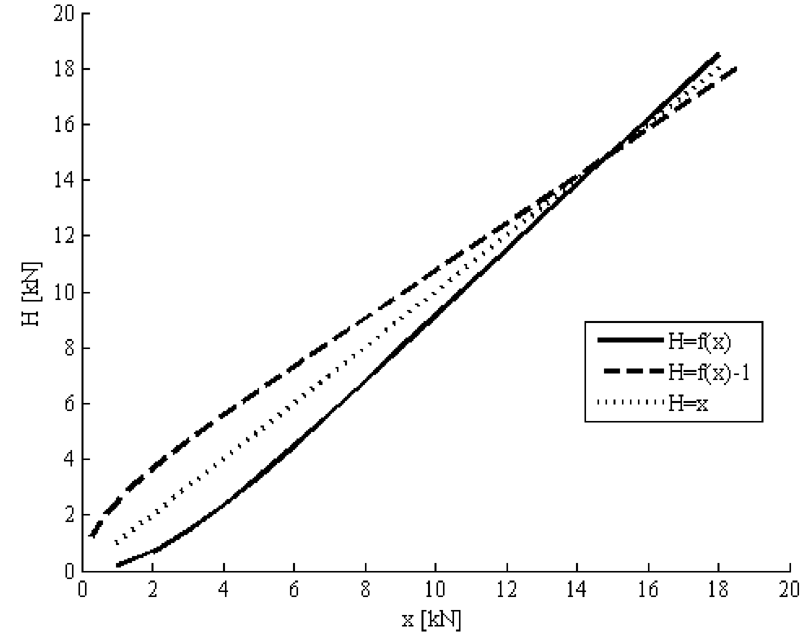

Fig. 4 Parabolic catenary – results Direct iteration

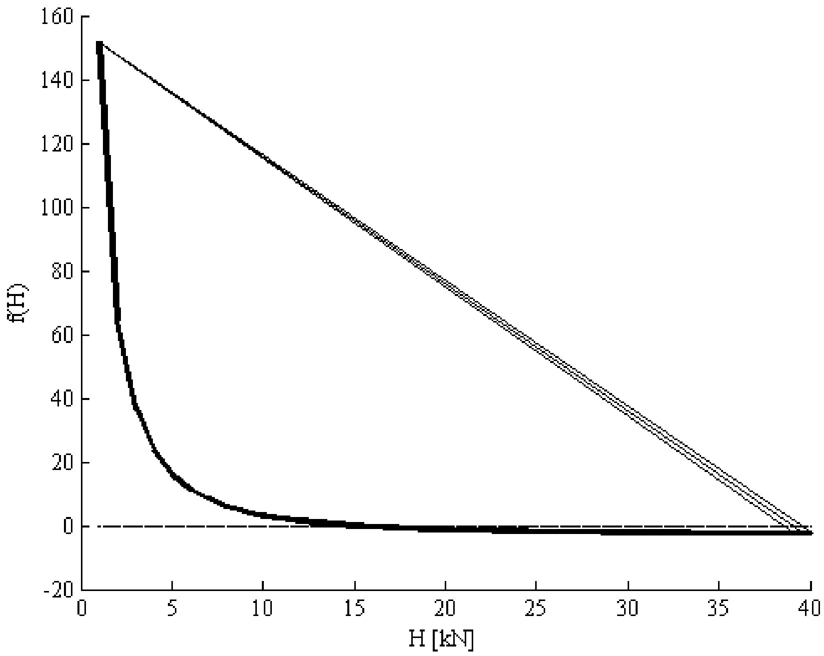

Fig. 5 Parabolic catenary – results for regula falsi

The zero approximation needed for iteration is 1 kN. Two termination conditions were specified. The first termination condition is the exact number of iteration steps being 100. The second termination condition is the deviation between two subsequent calculated values being 0.001 kN. With this method, divergence was an issue. Because the curve which represented the modified equation (1) was convex, it was moving towards infinity in each subsequent iteration step. Therefore, the algorithm was modified in order to use an inverse function (dashed curve). It was not necessary to determine the entire inversion function. It is, however, more efficient for the calculation to add a double of the difference between the original function and I and III quadrant axis to the original function. This resulted in convergence. For the required deviation the horizontal force was 14.8853 kN. For the required number of steps, 100, the horizontal force was 14.8911 kN.

D. Regula falsi

This method is sometimes referred to as the false position method or the chord method. For general background see Fig. 5. Once the equation (1) is adjusted to be homogeneous, this method gives intersection of a curve with a horizontal axis. The initial condition is the interval in which the required value is located. In every subsequent iteration step, a chord line is created between the outer points. Then, the value of the outer point in the interval is replaced with the value obtained by intersection of the chord and horizontal axis. The first three iteration steps are represented by the chords of the curve. Because of the shape of the curve under investigation, this method iterates very slowly. In order to accelerate convergence of this method, it would help making the input interval narrower so that the outer point could be as close as possible to the required value. The input interval comprises the required result and is limited by the lower boundary 1 kN and by the upper limit 40 kN. The condition which will stop the iteration is the deviation between the two subsequent calculated values. In order to keep the input conditions, this deviation is again 0.001 kN. The horizontal force calculated using this method is 14.8937 kN.

E. Bisection method

The bisection method or the interval dividing method is similar to the regula falsi because of its input criteria. The interval under investigation is limited again by 1 kN and 40 kN.

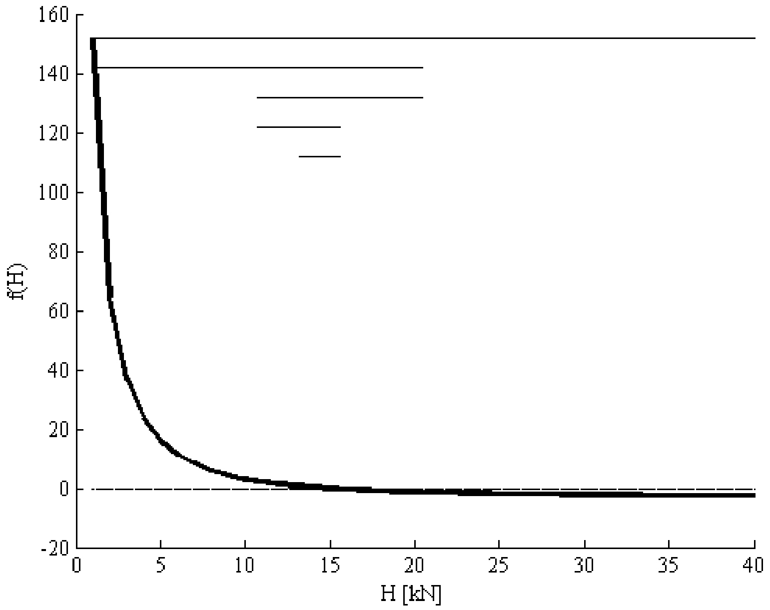

Fig. 6 Parabolic catenary – results Bisection

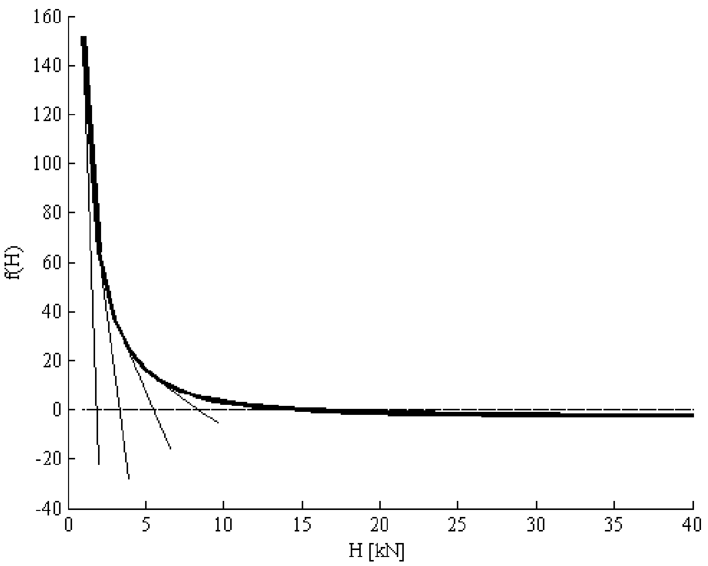

Fig. 7 Parabolic catenary – results Newton method

The termination condition is again the difference between two subsequent iteration values: 0.001 kN. The equation for the cable length (1) was modified and is homogeneous now. In the next iteration step, the previous interval with the required value is reduced. The value of the outer point of the interval changes after the functional value of the function under investigation is compared in the half which precedes the interval. If the difference against the next value is within the specified termination deviation, the calculation will be interrupted. The bisection method is described in Fig. 6. The first five iteration steps are described there as horizontal lines. It is evident that the interval with the required value (14.8910 kN) becomes smaller more quickly. Unlike the regula falsi method, the bisection method does not depend much on the shape of the curve under investigation and iterates considerably faster.

F. Newton method

This iteration method is shown in Fig. 7. The initial and termination conditions are identical with those used in the direct iteration method. In the zero approximation point, the tangent to the curve under investigation is found. Then, the intersection with the horizontal axis is found. The next tangent is constructed in the functional value of that point. The tangent represents the next iteration step. In the chart, the first four iteration steps are visible. The approach the final value, 14.8905 kN, relatively quickly. In order to develop an algorithm for this method it is necessary that derivations should be calculated in each iteration point [5]. The model was calculated using the three-point forward formula (8) with the 0.01 differentiation. If other methods were used, the time needed for the modelling by means of the Newton method did not extend. The number of iteration steps were not be influenced too.

$$f'(x) = \frac{-3.f\ (x) + 4.f\ (x + \mathrm{dif}) - f\ (x + 2.\mathrm{dif})}{2.\mathrm{dif}} \tag{8}$$

G. Comparison of the iteration methods

When using the methods described above, the resulting H was 14.89 kN for the given input values. Using this value, other geometric and force parameters can be determined. Fig. 8 shows deflection of the cable for the specified values of the structure. Fig. 9 shows how the calculated horizontal forces depend on the number of steps of each method. The zero step represents the initial approximation values. In terms of necessary steps, the regula falsi method is the most demanding – it requires 276 steps. The reason for such a high value is the shape of the function under investigation and the initial values. The result is also proved by the chart which shows the iteration steps used in the regula falsi method, see Fig. 5. The least number of iterations (7 steps) was needed by the Newton method. Tab. 1 shows the number of iteration methods in each method as well as the time needed for the calculation. Except for the Newton method, the time correlates with the number of steps. The reason for more time needed in the Newton method is a rather long operation in one step, the reason being calculation of derivations in each point. The shortest time needed for calculation of the horizontal reaction was for the bisection method. Unlike the Newton method, the bisection method does not have enough input conditions. It is necessary to specify the interval where the required value is located. For this reason, it is recommended to use the tangent method. In this method, it is sufficient to determine the zero approximation and the difference. The direct iteration needed also a shorter time than the Newton method. But the direct iteration faces a similar problem as the bisection method.

H. Comparison of the iteration methods

The Matlab software [10] was used to model by means of iteration methods the horizontal reaction of a planar parabolic catenary.

Method	*H* [kN]	Calc. steps	Time (s)
Iteration (step)	14.8911	100	0.1393
Iteration (deviation)	14.8853	44	0.0602
Regula falsi	14.8937	276	0.5492
Bisection	14.8910	16	0.0349
Newton method	14.8905	7	0.0777

Table 1. Comparison of the methods – results

It is generally assumed that the cable is a perfectly bendable and non-flexible fibre. The structure is supported in two suspended points by means of solid joints the difference in the height of which is 1 m. The plan distance between the supports is 30 m. The cable structure is loaded with a continuous load where 0.8 kN/m is applied onto ground projection. Algorithms were developed for the following methods: the direct iteration, bisection, regular falsi and Newton method. Attention was paid to the time needed for calculations and for the number of iteration steps. The least time needed for obtaining the result was measured for the bisection method. This method, however, requires that the interval be known where the value is located. Therefore, the Newton method is a better choice for general development of an algorithm. The number of iteration steps is the lowest for the tangent method, even if the time needed for the calculation is rather long. The advantage of the tangent method over the bisection method is that it needed only to enter the first approximation method and difference. The reason for more time needed by the Newton method is the more extensive iteration step which calculates derivations in each point.

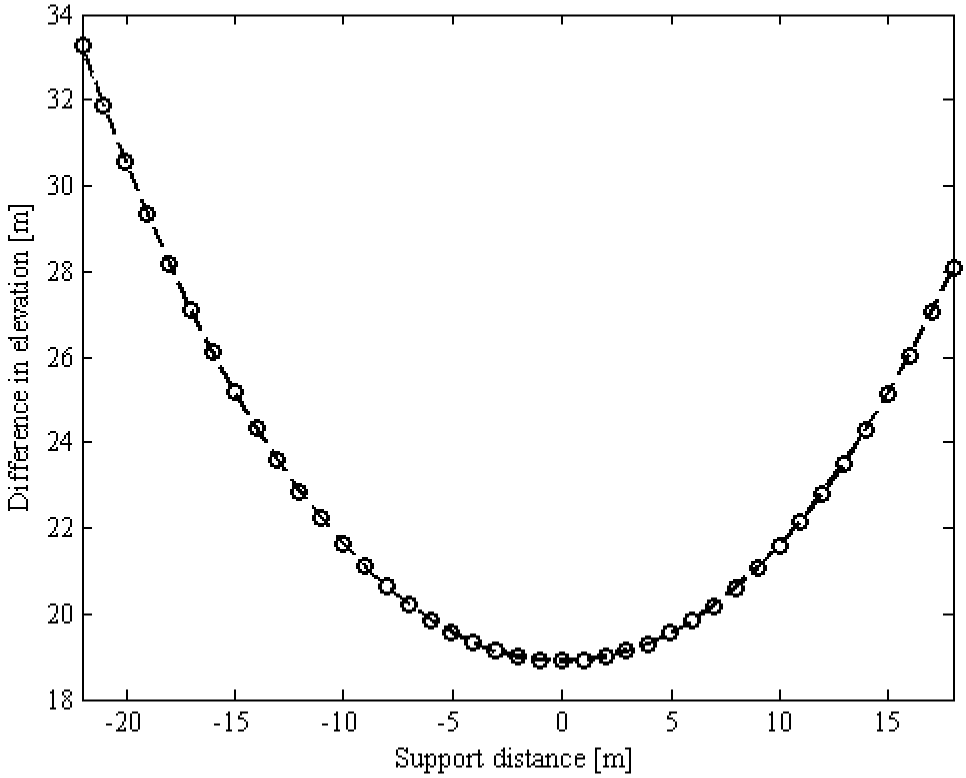

Fig. 8 Parabolic catenary – cable deflection

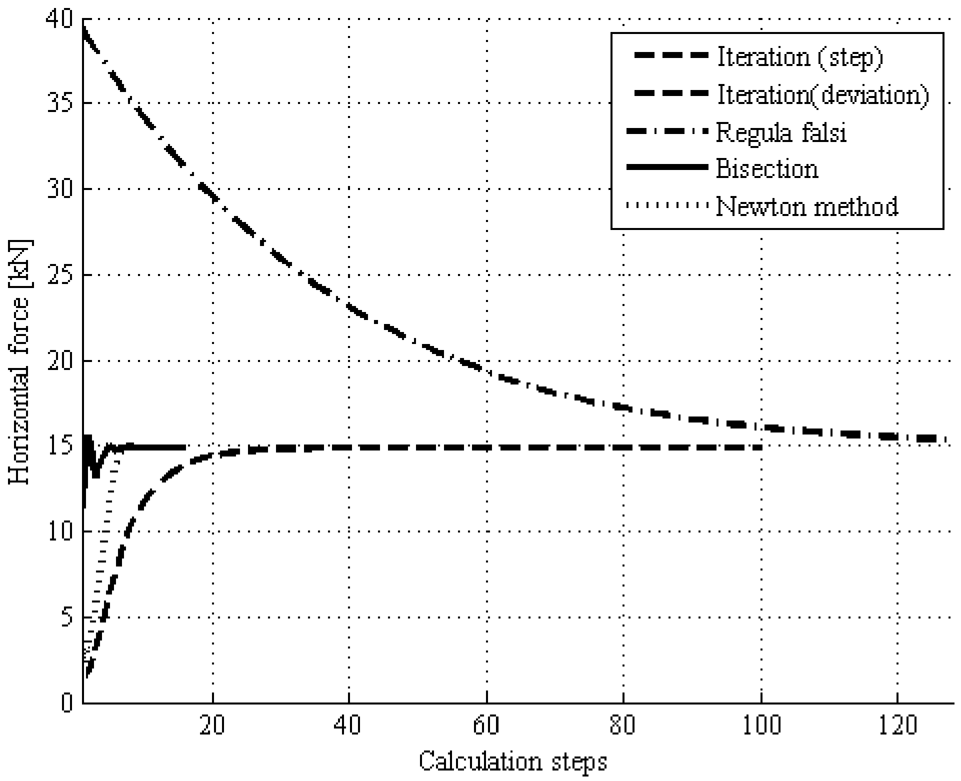

Fig. 9 Parabolic catenary – comparison of the methods

V. A MODEL CASE OF THE GRAVITY CATENARY

A. A model case of the gravity catenary

A model case has been used to support the application of the iteration method for a catenary. Only three input parameters were used in the calculation. Difference in heights between the supports was 2 m. The ground distance between the supports was 40 m and the length of the cable was 45 m. It is necessary to know the load in order to calculate the dimensioning force.

This quantity, however, does not need to be known for determination of (a) – that is why, this quantity was not taken into account.

B. Conditions for iteration methods

The initial and end conditions are essential for the iteration methods. Because the goal is to compare the iteration methods, identical conditions have been used for all methods. The initial condition of 10 is the zero approximation value. This condition is necessary for following methods: direct iteration and Newton method. The regula falsi and bisection methods require an interval which comprises the solution. Limit values in the interval were 10 and 100. The Newton method required for derivation a three-point forward formula with a 0.1 difference. The end condition was identical in all cases. The difference between the two subsequent steps should be below 0.001.

C. Comparison of the iteration methods

It is essential to modify (7) in order to calculate (a) and apply the iteration methods. The dependence of (a) was determined for the direct iteration using (9). Fig. 10 shows the graphic chart. The solution is a point which is located in the intersection of the curve and the axis in the first and third quadrants.

In case of other methods, it was necessary to adjust (7) into a homogeneous form – (10). Fig. 11 shows this equation – the point of intersection with the horizontal axis is the solution.

$$a = \frac{\sqrt{l^2 - h^2}}{2.\sinh\frac{L}{2.a}} \tag{9}$$

$$2.a.\sinh\frac{L}{2.a} - \sqrt{l^2 - h^2} = 0 \tag{10}$$

Table 2 shows the values of (a) and complexity of calculation in terms of needed calculation steps. Fig. 12, which corresponds with results in Table 2, shows development of the method, depending on the number of steps.

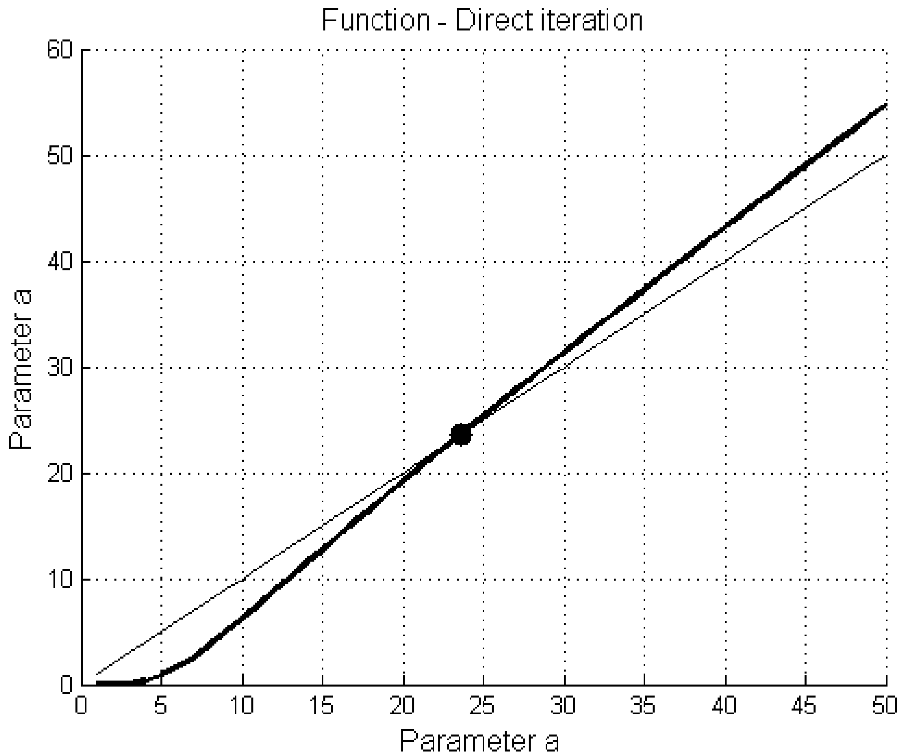

Fig. 10 Function – direct iteration

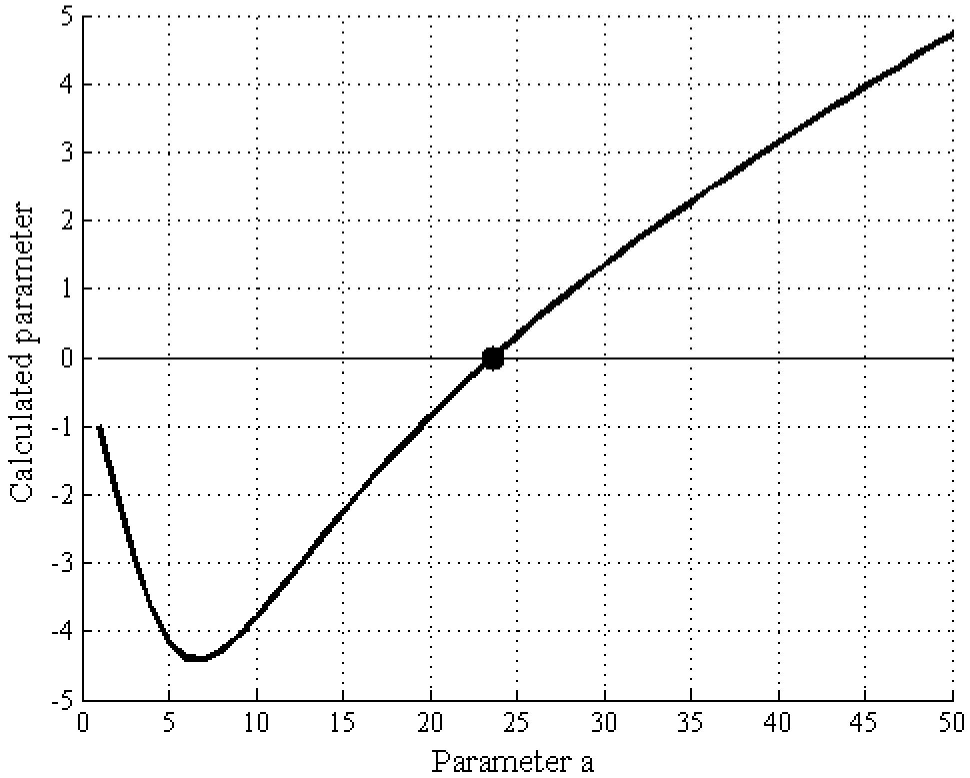

Fig. 11 Function – homogeneous

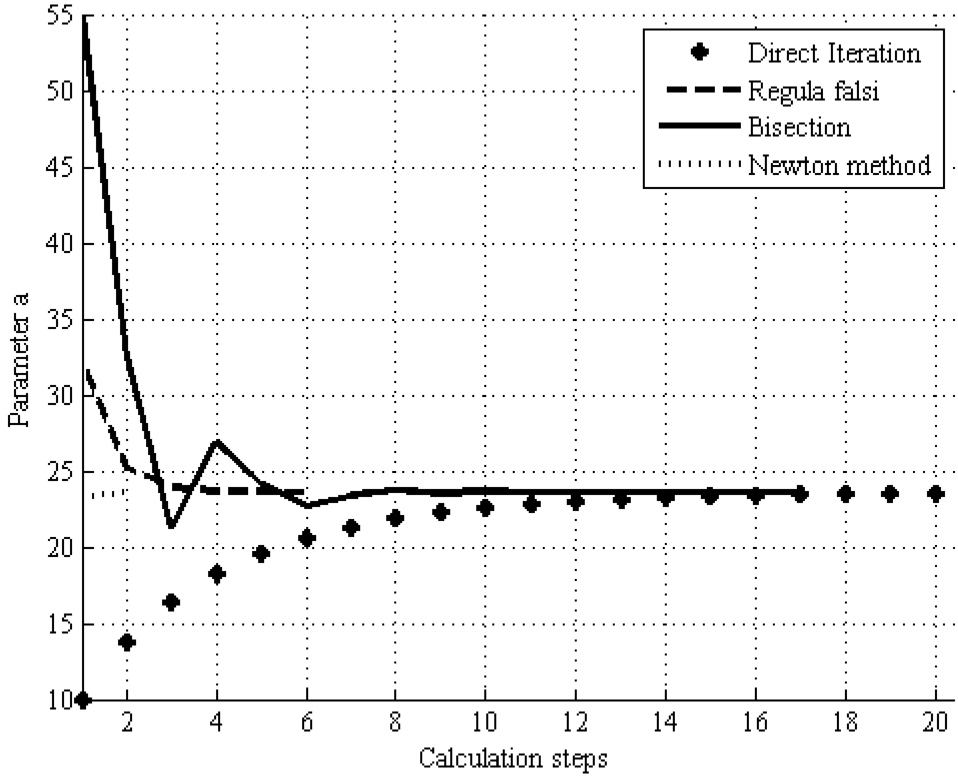

Fig. 12 Course of the iterations

Method	Parameter (a)	Calculation steps	Time (s)
Direct Iteration	23.6125	23	0.0752
Regula falsi	23.6184	6	0.0408
Bisection	23.6169	17	0.0509
Newton method	23.6153	2	0.0696

Table 2. Comparison of the methods - results

It follows from the results above that the rounded off value of (a) is 23.61. Regarding the calculation complexity, the most iteration steps were needed in order to reach the result for the direct iteration method.

The least calculation steps were needed for the Newton method. However, a big issue is there the input condition for the zero approximation. The curve in Fig. 11 shows that the initial value should be bigger than the minimum value of (10). For this reason, that method cannot be applied in general cases. Therefore, the most suitable method seems to be the regula falsi iteration method. A disadvantage is knowledge of an interval with the known value. The calculations included simulations with the increasing limit value of the input interval up to 10^6. Even with such an extremely high value, the time needed for calculation or the number of calculation steps have not increased considerably.

VI. Conclusion

Algorithms based on the following methods were developed in Matlab [10]: direct iteration, regula falsi, and bisection and Newton methods. The methods were used for calculation of the parabolic and gravity catenaries. The best method suitable for the gravity catenary in general applications has appeared to be the regular falsi method.

As far as the parabolic catenary is concerned, the choice of an optimum method depends on criteria. The least number of iterations was needed for the Newton method. The shortest time needed for calculation of the horizontal reaction was reached for the bisection method.

The authors will focus on further research on the use of special cable elements [19] and application of probabilistic methods [20].

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