On a research of symmetric equations of Volterra type

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***Abstract*—** As is known, solving of some problems in the ecosystem, communications, computational biology, mechanics and etc. are reduced to the computation of symmetric integrals. In some cases, traditional methods are applied to the calculation of these integrals, not keeping the suitable results. Therefore appears a need to constructing special methods for the solving of symmetric integrals. Here for these aims are proposed to using of multistep methods with constant coefficients. It is obviously that the advantage of the proposed methods are shown and constructed here the specific available methods advantages for which are illustrated by the some model equations, as well as the results obtained here also have compares by the known.

***Keywords*—** Symmetric integrals, the initial value problem for ODE, quadrature methods, order and degree, multistep methods with the intermediate points.

I. INTRODUCTION

APPLICATION of the integral equations with the variable boundaries to solving practical problems starts from the known works of Abel (see [1, p.12]). Fundamental research of the integral equations with the variable boundaries and its application to solving of some science and practical problems are proposed by Volterra. The scientists from the different country have called those integrals the Volterra integral equations (see for example [1, p.13]).

Consider the following nonlinear integral equation with the variable boundaries:

$$y(x) = f(x) + \int_{-x}^{x} K(x,s,y(s))ds, \qquad x \in [x_0, X]. \quad (1)$$

Note that the equation (1) is a nonlinear symmetric Volterra integral equation. Suppose that the equation (1) has a unique solution defined on the interval $[x_0, X]$. For construction of numerical method with the order of accuracy of p for solving of the equation (1), imposes some limitation on its kernel of function $K(x,s,y)$.

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Let jointly continuous function $K(x,s,y)$ is defined in the closed set $G = \{x_0 \le x \le X; |s| \le x; |y| \le a\}$ and it has the continuous partial derivatives until up to order of p, inclusively. And sufficiently smooth function $f(x)$ is defined on the interval $[x_0, X]$. We denote by y_i approximately, but by the through $y(x_i)$ - the exact value of the solution of the equation (1) at the mesh point $x_i = x_0 + ih (i = 0,1,2,..., N)$. Here $h > 0$ is a step size for the partition of the segment $[x_0, X]$ to N - equal parts. Note that the linear integral equations of type (1) in the most general form are investigated by Volterra (see 19-25).

As is well known, to find the exact solution of the equation (1), even in the linear case, is not always possible. Therefore, scientists for research mathematical models of the practices problems suggested using of approximate methods. And for this aim the Volterra used quadrature methods. There are many works dedicated to the study of the solutions of equations of type (1), by using the different modifications of the quadrature methods (see for example [3] – [15]). Note that to solving of the equations of type (1) one can be applied spline function, collocations method, operator methods, and etc. (see for example [16] – [21]).

Consider the application of the quadrature method to the calculation of symmetric integrals, which in one variant, can be written in the following form:

$$\int_{-x_n}^{x_n} g(s)ds = \sum_{i=-n}^{n} A_i g(x_i) + R_n , \qquad (2)$$

here the real numbers $A_i (i = -n, -n+1,..., n)$ are coefficients, and R_n - remainder term of the quadrature formula.

Note that when $A_n = A_{-n} = 0$ quadrature formula (2) is open, and the corresponding methods are explicitly. But $|A_n| + |A_{-n}| \ne 0$ quadrature formula (2) is a closed-type, and the corresponding to them are implicit methods. As is known the implicit methods are usually more accurate than explicit. Even the explicit and the implicit methods have the same accuracy, then applied implicit methods to solving some problems give the best results. The known representative of such methods are the trapezoidal and Simpson method. The following explicit method is also popular:

$$\int_{-x_n}^{x_n} g(s)ds = \sum_{i=-n+1/2}^{n-1/2} g(x_i) + R_n . \qquad (3)$$

Method (3) usually called as the method of rectangles, the order of accuracy of which coincides with the order of accuracy of the trapezoidal method and is explicit. Method rectangles (3) differ from the methods (2) that, in the method (3) use the calculation of value of the kernel at intermediate points $x_i + h/2 \,(i = -n, -n+1, ..., n-1)$.

As is known, that in the hybrid methods used the computing of the kernel of the integral at the intermediate and the nodal points (see [10,11]). If we assume that

$$K(x,s,y) = \varphi(s,y),$$

then considering that the solutions of equation (1)is known and one can be write the following:

$$\begin{aligned} &y'(x) = f'(x) + \varphi(x, y(x)) + \varphi(-x, y(-x)), \\ &y(x_0) = f(x_0). \end{aligned} \qquad (4)$$

This is initial value problem for the ordinary differential equations. By using this given equivalence between the differential and integral equations, scientists have applied the methods from solving of ODE to solve of integral equations. Here at first consider to replace of the application of multistep methods to solving of the equation (1) on the interval $[x_0, X]$ by the finding the solution of equation (1) on the interval $[x_m, x_{n+m}]$ $(m \ge 1)$.Then construct multistep methods for solving Volterra integral equations.

II. A way for constructing multistep methods for solving of the equation (1).

It is easy to see that the equation (1) can be written as follows:

$$\begin{aligned} y(x) = f(x) + \int_0^x K(x,s,y(s))ds + \\ + \int_0^x K(x,-s,y(-s))ds \end{aligned} \qquad (5)$$

Consequently, the solution of equation (1) is equivalent to the solution of the following Volterra integral equation:

$$y(x) = f(x) + \int_0^x M(x,s,y(s))ds , \qquad (6)$$

here

$$M(x,s,y(s)) = K(x,s,y(s)) + K(x,-s,y(-s))$$

Let us solving equation (6) by used of the multistep method from the work [9]. Then we have:

$$\begin{aligned} \sum_{i=0}^k \alpha_i y_{n+i} = \sum_{i=0}^k \alpha_i f_{n+i} + h \sum_{j=0}^k \sum_{i=0}^k \beta_i^{(j)} (K(x_{n+j}, x_{n+i}, y_{n+i}) \\ + K(x_{n+j}, x_{-n-i}, y_{-n-i})). \end{aligned} \qquad (7)$$

Hence it was obvious that the application of the above named method to calculation of integrals in equation (5) are used the same method. Note that the method (7) reminds the symmetric methods. But that is not symmetrical. Hence thus from the method (7) one can be receive a symmetric methods.

Now consider to the construction methods of the type (7).

As follows from equation (3), the main disadvantage of the quadrature method is the increasing volume of calculations be the increasing values of the variable n. To release this drawback, here offers to determine the relationship between the quantities $y(x+h)$ and $y(x)$. For this purpose, consider the following relation:

$$\begin{aligned} y(x+h) - y(x) = f(x+h) - f(x) + h\int_{-x}^{x} K'_x(\xi, s, y(s))ds + \\ \int_{-x-h}^{-x} K(x+h, s, y(s))ds + \\ + \int_{x}^{x+h} K(x+h, s, y(s))ds, \quad x < \xi < x+h. \end{aligned} \qquad (8)$$

Note that when applying the quadrature method to the definition of the values of the two last integrals in (8) the volume of calculations does not increases by the increasing of the values of variable x.Therefore consider to the calculation of the following integral

$$h\int_{-x}^{x} K'_x(\xi, s, y(s))ds .$$

Similarly equation as (8) was investigated in [22] for the finite elements.

Suppose that by the someway found the solution of equations (1), taking into which in the equation (1) we obtain the identity. Then from this identity we can write the following:

$$\begin{aligned} h\int_{-x}^{x} K'_x(\xi,s,y(s))ds = h(y'(\xi) - f'(\xi)) - h(K(\xi,\xi,y(\xi)) + \\ + K(\xi,-\xi,y(-\xi))) - \\ - h\int_{-\xi}^{-x} K'_x(\xi,s,y(s))ds - h\int_{x}^{\xi} K'_x(\xi,s,y(s))ds. \end{aligned}$$

By using obtained in (8) we have:

$$\begin{aligned} y(x+h) - y(x) = f(x+h) - f(x) + h(y'(\xi) - f'(\xi)) - \\ - h(K(\xi,\xi,y(\xi) + K(\xi,-\xi,y(-\xi)) + \\ + \int_{x}^{\xi} K((x,s,y(s)) + K(x,-s,y(-s)))ds + \\ + \int_{\xi}^{x+h} (K(x+h,s,y(s)) + K(x+h,-s,y(-s)))ds. \end{aligned}$$

Some experts may consider that

$$y(x+h) - y(x) = hy'(\xi), \; x < \xi < x+h. \qquad (10)$$

However, we have assumed above that

$$K(x+h,s,y(s))-K(x,s,y(s))=hK'(\xi,s,y(s)),\quad x<\xi<x+h. \tag{11}$$

Consider the case $K(x,s,y(s))=y(x)b(s,y(s))$ - then receive that the solving of integral equation (1) is equivalent to solving the following system

$$y(x)=f(x)+y(x)v(x), \tag{12}$$

$$v'(x)=b(x,y(x))+b(-x,y(-x)), v(0)=0. \tag{13}$$

From equations (12) we have: $y(x)=f(x)/(1-v(x))$. Consequently, $v(x)\neq 1$. Note that if $v(x)=const$, then by using the uniqueness of the solution of problem (13), we find that $v(x)\equiv 0$. One can prove that the condition (10) and the following

$$f(x+h)-f(x)=hf'(\xi);\ y(x+h)v(x+h)-y(x)v(x)=h(y'(\xi)v(\xi)+y(\xi)v'(\xi)) \tag{14}$$

equivalent. It follows that the condition (10) is not always true. Note that by the selection of the function $b(\xi,y(\xi))$, can ensure one of the above mentioned equality. Hence, therefore we can assume that $y(x+h)-y(x)=hy'(\xi)$. Then in equation (9) by replacing the value of the first derivatives of the function by the one sums of its values at the mesh points and replacing integrals with some integral sum (see for example [7], [9]) we have:

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=h\sum_{i=0}^{k}\alpha_i f_{n+i}+h\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)}(K(x_{n+j},x_{n+i},y_{n+i})+\gamma_i^{(j)}K(x_{n+j},x_{n-i},y_{n-i})) \tag{15}$$

By choosing of the coefficients in equation (15) can be obtained the different methods with the in different properties. To determine the coefficients in the method (15), consider the special case and set:

$$K(x,s,y)=\varphi(s,y).$$

In this case from the equation (1) can be write:

$$y'(x)-f'(x)=\varphi(x,y(x))+\varphi(-x,y(-x)),\quad y(x_0)=f(x_0). \tag{16}$$

By using sufficiently smooth functions $y(x)$ and $f(x)$ to solving of the problem(16) we can apply the following finite-difference method

$$\sum_{i=0}^{k}\alpha_i z_{n+i}=h\sum_{i=0}^{k}\beta_i z'_{n+i},\quad (z(x)=y(x)-f(x)). \tag{17}$$

It is known that in order to the method (17) had a degree p, its coefficients must satisfy the following conditions:

$$\sum_{i=0}^{k}\alpha_i=0;\ \sum_{i=0}^{k}\beta_i=\sum_{i=0}^{k}i\alpha_i;$$

$$\sum_{i=0}^{k}i^{l-1}\beta_i=\frac{1}{l}\sum_{i=0}^{k}i^l\alpha_i\quad (l=2,3,...,p). \tag{18}$$

Note that here the notion, of the degree and of the stability for the method (15) are defined by analogy the relevant concepts for the method (17). The system (18) is investigated by different authors (see for example [23]-[27]). According to this definition, consider the connection between the coefficients β_i and $\beta_i^{(j)}$ $(j,i=0,1,...,k)$.

It is not difficult prove that if are known the values of the coefficients β_i $(i=0,1,...,k)$, then the values of the variable $\beta_i^{(j)}$ $(j,i=0,1,...,k)$ can be defined by the following system:

$$\sum_{j=0}^{k}\beta_i^{(j)}=\beta_i\quad (i=0,1,...,k). \tag{19}$$

Note that the solutions of the system (19) are not unique. Consider the construction methods of the type (15) for $k=1$ and $k=2$.

As is consequent from the method (15) we must proposed the formula for calculation of the values y_{n+i} and y_{-n-i}. Then the method of trapezoidal type for calculation of the variables y_{n+i} and y_{-n-i} can be written as:

$$\begin{aligned}y_{i+1}=y_i+f_{i+1}-f_i&+h(K(x_{i+1},x_i,y_i)+K(x_i,x_i,y_i)+\\&+K(x_{i+1},x_{i+1},y_{i+1})+K(x_i,x_{i+1},y_{i+1}))/4+\\&+h(K(x_{i+1},x_{-i},y_{-i})+K(x_i,x_{-i},y_{-i})+\\&+K(x_{i+1},x_{-i-1},y_{-i-1})+K(x_i,x_{-i-1},y_{-i-1}))/4;\end{aligned} \tag{20}$$

$$\begin{aligned}y_{-i-1}=y_{-i}+f_{-i-1}-f_{-i}&-h(K(x_{-i-1},x_i,y_i)+K(x_{-i},x_i,y_i)+\\&+K(x_{-i-1},x_{i+1},y_{i+1})+K(x_{-i},x_{i+1},y_{i+1}))/4+\\&+h(K(x_{-i-1},x_{-i},y_{-i})+K(x_{-i},x_{-i},y_{-i})+\\&+K(x_{-i-1},x_{-i-1},y_{-i-1})+K(x_{-i},x_{-i-1},y_{-i-1}))/4;\end{aligned} \tag{21}$$

Now consider to construction method for $k=2$ and for this aim uses the midpoint rule. Then can be write the following:

$$\begin{aligned}y_{i+2}=y_i+f_{i+2}-f_i&+h(K(x_{i+1},x_{i+1},y_{i+1})+K(x_{i+2},x_{i+1},y_{i+1}))+\\&+h(K(x_{i+1},x_{-i-1},y_{-i-1})+K(x_{i+2},x_{-i-1},y_{-i-1})),\end{aligned} \tag{22}$$

$$\begin{aligned}y_{-i-2}=y_{-i}+f_{-i-2}-f_{-i}&-h(K(x_{-i-1},x_{i+1},y_{i+1})+K(x_{-i-2},x_{i+1},y_{i+1}))-\\&-h(K(x_{-i-1},x_{-i-1},y_{-i-1})+K(x_{-i-2},x_{-i-1},y_{-i-1})).\end{aligned} \tag{23}$$

For application these implicit methods to solving some practical problem one may be used the Euler explicit method. The compares above constructed methods with the known in the following we applied the hybrid methods to solving nonsymmetrical Volterra integral equation.

III. Construction the Hybrid Method for Solving Nonlinear Volterra Integral Equation.

Note that to construction methods with the higher order of accuracy the last time dedicated some works of different authors. For these aim here to solving the following integral equation:

$$y(x)=f(x)+\int_{x_0}^{x}K(x,s,y(s))ds, \qquad (24)$$

proposed use the hybrid method. The hybrid methods can be constructing in some variants. Here we investigate to application of the following method to solving of the integral equation (24):

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=h\sum_{i=0}^{k}\beta_i y'_{n+i}+h\sum_{i=0}^{k}\gamma_i y'_{n+i+\nu_i},$$
$$(|\nu_i|<1; i=0,1,...,k). \qquad (25)$$

This method has applied to solving of the next problem (see for example [19]):

$$y'=f(x,y), y(x_0)=y_0, x_0\le x\le X.$$

The method (25) by depending of its application usually has called as the multistep methods with the constant coefficients or the finite difference methods. Therefore some authors are called the correlation of (25) as the finite difference equation (see for example [20],[21]). Note that the method (25) in the work [19] has applied to solving of the initial-value problem for the ordinary differential equation of the first order and proof, that if the method (25) is convergence, then its coefficients satisfies the following conditions:

A: The coefficients $\alpha_i,\beta_i,\gamma_i$ and $l_i,(i=0,1,2,...,k)$ are the real numbers; moreover, $\alpha_k\neq 0$.

B: The characteristic polynomials

$$\rho(\lambda)\equiv\sum_{i=0}^{k}\alpha_i\lambda^i;\ \sigma(\lambda)\equiv\sum_{i=0}^{k}\beta_i\lambda^i;\ \gamma(\lambda)\equiv\sum_{i=0}^{k}\gamma_i\lambda^{i+l_i}$$

have no common multiple different from constant.

C: $\sigma(1)+\gamma(1)\neq 0$ and $p\ge 1$.

Here we, assume that the conditions are holds. Note that using of the method (25) to solving equation (1) we must determine the values y_{n+i} and $y_{n+i+\nu_i}$ of the function y(x). It is known that for the application of multistep method to solving of some problems, the values y_i (i=0,1,…,k-1) must be known. Therefore if are known the values y'_{n+i} and $y'_{n+i+\nu_i}$ (i=0,1,…,k) of the function y(x),then we can be applied the method (25) to solving of the equation (1). So for the contraction method of type (25) to solving of the equation (24), considers to the following expansion of Taylor:

$$y(x+ih)=y(x)+ihy'(x)+\frac{(ih)^2}{2!}y''(x)+...+$$
$$+\frac{(ih)^p}{p!}y^{(p)}(x)+O(h)^{p+1}. \qquad (26)$$

$$y'(x+ih)=y'(x)+ihy''(x)+\frac{(ih)^2}{2!}y'''(x)+...+$$
$$+\frac{(ih)}{(p-1)!}^{p-1}y^{(p)}(x)+O(h)^p \qquad (27)$$

$$y'(x+(i+\nu_i)h)=y'(x)+(i+\nu_i)hy''(x)+...+$$
$$+\frac{((i+\nu_i)h)^{p-1}}{(p-1)!}y^{(p)}(x)+O(h)^p \qquad (28)$$

here $x=x_0+nh$ is the fixed point.

As mentioned above, the aim of this work is the construction of the methods with the high order of accuracy. It is known that in the theory of multistep methods the notion “order of accuracy” is replace to the notion of the degree of the multistep methods. The degree of the method (25) can be defined as the following:

Definition 1. If for the sufficiently smooth function y(x) and for the integer value quantity p the following is holds:

$$\sum_{i=0}^{k}\left(\alpha_i y(x+ih)-h\beta_i y'(x+ih)-h\gamma_i y'(x+(i+\nu_i)h)\right)=$$
$$=O(h)^{p+1}, h\to 0 \qquad (29)$$

then p is called the degree of the method (25).

For the definition of the coefficients of the method (25) one can be used the next lemma.

Lemma. Let y(x) be a sufficiently smooth function, and assume that condition A,B and C are holds. For the method (25) to have the degree p, satisfies the following conditions of its coefficients are necessary and sufficient:

$$\sum_{i=0}^{k}\alpha_i=0;\quad \sum i\alpha_i=\sum_{i=0}^{k}(\beta_i+\gamma_i), \qquad (30)$$

$$\sum_{i=0}^{k}(i^{l-1}\beta_i+(i+\nu_i)^{l-1}\gamma_i)=l\sum_{i=0}^{k}i^l\alpha_i,\quad (l=2,3,...,p).$$

It is easy to determine that for the chosen values $\nu_i=0\ (i=0,1,...,k)$, the system (30) is linear and coincides with the known systems (18). Subject to the conditions from $|\nu_0|+|\nu_1|+...+|\nu_k|\neq 0$, the system (30) is nonlinear. This system contains $p+1$ equations in $4k+4$ unknowns and is homogeneous; it must possess the zero solution, and for system (30) has a non-zero solution, suppose that the condition $4k+4>p+1$ is holds. Hence, we obtain that there are methods of type (25) with the degree $p\le 4k+2$.

Consider the construction methods of type (25) for $\beta_i=0(i=0,1,...,k)$ and suppose that $k=1$.

Note that the method with the degree $p=4$ can be written as follows:

$$y_{n+1}=y_n+h(y'_{n+l_0}+y'_{n+1+l_1})/2. \qquad (31)$$

Here $l_1 = -l_0;\ l_0 = (3-\sqrt{3})/6,\ 1+l_1 = (3+\sqrt{3})/6$. Remark that for applying the hybrid method (31) to solving some problems it is needed known of the values $y_{n+1/2+\sqrt{3}/6}$, $y_{n+\frac{1}{2}}$ and $y_{n+1/2-\sqrt{3}/6}$. Note, that these variables are independent from y_{n+1}, because that method (31) is explicit. But there is, exist implicit hybrid methods of type (25). For example, consider the following method:

$$y_{n+1} = y_n + h(3y'_{n+1/3} + y'_{n+1})/4.$$

This method is implicit and has the degree $p=3$.

Now, let us consider to construction of the method of type (25) for the case $k=1$. In this case, assuming that $\alpha_1 = -\alpha_0 = 1$, the corresponding method to the method (15) can be written as follows:

$$\begin{aligned} y_{n+1} = y_n + f_{n+1} - f_n + h(K(x_{n+1}, x_{n+l_0}, y_{n+l_0}) + \\ K(x_{n+l_0}, x_{n+l_0}, y_{n+l_0}) + \\ + K(x_{n+1}, x_{n+1-l_0}, y_{n+1+l_0}) + K(x_{n+1-l_0}, x_{n+1-l_0}, y_{n+1-l_0}))/4. \end{aligned} \tag{32}$$

For the construction of the more accurate methods to consider of the construction method of type (25) for the $k=1$. Then receive the next method

$$\begin{aligned} y_{n+1} = y_n + h(y'_{n+1} + y'_n)/12 + \\ + 5h(y'_{n+1/2-\sqrt{5/10}} + y'_{n+1/2+\sqrt{5/10}})/12,\ (p=6). \end{aligned} \tag{33}$$

To apply hybrid methods to solving of some problems, we should know the values of $y_{n+1/2-\sqrt{5/10}}$ and $y_{n+1/2+\sqrt{5/10}}$, and the accuracy of these values should have order at least $O(h^6)$. Note that hybrid method (33) is implicit and that when applying it to solving of equation (24), is used a predictor-corrector scheme containing only one explicit method. For solving equation (24) the corresponding to the method (33) may be written as the follows:

$$\begin{aligned} y_{n+1} = y_n + f_{n+1} - f_n + h(2K(x_{n+1}, x_{n+1}, y_{n+1}) + \\ + K(x_{n+1}, x_n, y_n) + K(x_n, x_n, y_n))/24 + \\ + 5h(K(x_{n+1}, x_{n+\beta}, y_{n+\beta}) + K(x_{n+\beta}, x_{n+\beta}, y_{n+\beta}) + \\ + K(x_{n+1}, x_{n+\hat{\beta}}, y_{n+\hat{\beta}}) + K(x_{n+\hat{\beta}}, x_{n+\hat{\beta}}, y_{n+\hat{\beta}}))/24 \end{aligned} \tag{34}$$

Here $\beta = 1/2 - \sqrt{5}/10$ and $\hat{\beta} = 1/2 + \sqrt{5}/10$.

For the illustration of the results received here, we have considered to application of the method (31) to solving the following equations:

1. $y(x) = 1 + x^2/2 + \int_0^x y(s)ds$ the exact solution is $y(x) = 2e^x - x - 1$.

Step size	Variable x	Example 1	Example 2 ($\lambda = -1$)
$h=0,1$	0.10	0.3E-6	0.2E-5
	0.40	0.3E-5	0.6E-4
	0.70	0.1E-4	0.2E-3
	1.00	0.1E-4	0.6E-3

2. $y(x) = e^{-x} + \int_0^x e^{-(x-s)} y^2(s)ds$ the exact solution is $y(x) = 1$.

3. $y(x) = \int_0^x (1+y^2(s))ds/(1+s^2)$ the exact solution is $y(x) = x$.

The obtained results, place in the following table.

Step size	Variable x	Example 1	Example 2	Example 3
$h=0,01$	0.10	0.26E-10	0.15E-06	0.1E-16
	0.20	0.54E-10	0.54E-06	0.1E-16
	0.30	0.83E-10	0.10E-05	0.1E-16
	0.40	0.11E-10	0.17E-05	0.1E-16
	0.50	0.14E-09	0.24E-05	0.1E-16
	0.60	0.18E-09	0.32E-05	0.1E-16
	0.70	0.21E-09	0.40E-05	0.1E-16
	0.80	0.25E-09	0.48E-05	0.1E-16
	0.90	0.30E-09	0.57E-05	0.1E-16
	1.00	0.34E-09	0.66E-05	0.2E-16

Remark. Consider the computation of the symmetric integral under the assumption that the kernel can be represented as:

$$K(x,s,y) = b(x)a(s,y).$$

Then we have:

$$\varphi(x) = \int_{-x}^{x} K(x,s,y(s))ds = b(x)\int_{-x}^{x} a(s,y(s))ds.$$

We denote

$$v(x) = \int_{-x}^{x} a(s,y(s))ds.$$

Consider the function $v(-x)$. Then we have:

$$v(-x) = \int_{x}^{-x} a(s,y(s))ds = -\int_{-x}^{x} a(s,y(s))ds = -v(x).$$

Therefore $v(x)$ is odd function, т.е. $v(-x) = -v(x)$. But, $v'(x)$ is even function. Indeed,

$$v'(x) = a(x,y(x)) + a(-x,y(-x)).$$

If in this equality is replacing x through $-x$, then we have:

$$v'(-x) = a(-x,y(-x)) + a(x,y(x)) = v'(x).$$

This implies that $v'(x)$ is even function.

Thus we receive that solving of the equation (1) can be change by solving of the next system:

$$y(x) = f(x) + b(x)v(x)$$

$$v'(x) = a(x, y(x)) + a(-x, y(-x)),\ v(x_0) = 0.$$

Calculation of the $y(x)$ depend from the values of the function $v(x)$. Consequently, we receive that the investigation of the equation (1) on the interval $[-X, X]$ can be reduced to investigation that on the interval $[x_0, X]$. But research of the solution of the equation (1) the function $y(x)$ on the interval $[x_0, X]$ indeed, for the calculation of the value $y(-x)$ can be used the equality

$$y(-x) = f(-x) - b(-x)\ v(x).$$

This implies that for finding the values of the functions $y(x)$ defined on the interval $[-X, X]$ can be its values from the segment $[x_0, X]$, if the kernel of the integral is degenerate, it is to say, that $K(x,s,y) = b(x)a(s,y)$.

From here it is easy to determine that if the kernel of the integral can be represented as follows:

$$K(x,s,y) = \sum_{i=1}^{n} b_i(x)a_i(s,y),$$

then holds up the results obtained.

Analogy investigation has holds in [27]. Note that in the work [29] considered to application symmetric discrete difference equation to solving Lotka-Volterra equation.

For illustration results from the paragraph 1 to consider application of the methods to solving the following simple examples:

1. $y(x) = \frac{1}{2}\int_{-x}^{x} \cos s ds.$ (In this case, the exact solution is $y(x) = \sin x$).

2. $y(x) = e^{-\lambda x} + \lambda \int_{-x}^{x} y(s)ds.$ The exact solution is $y(x) = \exp(\lambda x)$.

In the first to solving these equations we used the trapezoidal methods by using ideas from the Remark. Results for which can be found from the following table:

Step size	Variable x	Example 1	Example 2 ($\lambda = -1$)
$h = 0{,}1$	0.10	0.8E-4	0.1E-3
	0.40	0.3E-3	0.6E-3
	0.70	0.5E-3	0.1E-2
	1.00	0.7E-3	0.1E-2

Now consider applications methods (20-23) and the method 34. Then we receive the next results.

IV. Conclusion

As is known when calculating of the volume of the rotation body, some areas and the power of signal, and also when solving of some scientific and practical problems, are faced with the computation of symmetric integrals. Symmetric integrals with variable boundaries investigated relatively little. In this regard, we have tried to, in a sense fill this gap. To this end, we offered here some modification of known methods for application them to solving of symmetric integrals with variable boundaries. And also describes a method for constructing stable methods with the high order of accuracy. To illustrate the obtained here results; we constructed concrete methods that have been applied to solving some simple equations of type (1). In some cases scientists, have constructed more effective methods by using the some properties of kernel of the integral. Therefore, here for solving symmetric integrals with degenerate kernel, are constructed effective multistep methods with the constant coefficients, and are given way for compare of the constructed methods with the known. As seen from the above received results and the presented numerical calculations, proposed here methods are more promising for solving symmetric integrals with variable boundaries

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