To the theory of nonlinear dynamic equations for the long elastic rod in viscous media

S.O. GLADKOV

Moscow Aviation Institute (National Research University) (MAI)

4 Volokolamskoe shosse, A-80, GSP-3, Moscow 125993

Russian Federation

sglad@newmail.ru

Abstract **–** The classical action for the long elastic rod was constructed and the resistance force accounted due to the formula for f_{fr}. The both of the nonlinear dynamic equation and the transversal condition for free end of the elastic rod has been found. In accordance by Navier - Stokes equation the force of resistance f_{fr} of cylindrical body as a function of its radius R was calculated. In this case of the distance range δr measured from the surface are $n^{-\frac{1}{3}} << \delta r << \frac{\nu}{u}$, where $n-$ molecular concentration, $\nu-$ kinematic viscosity and $u-$ is the velocity of the flow.

Keywords **–** Viscosity, laminar regime, heterogeneity displacement, nonlinear equations, transversal condition.

I. INTRODUCTION

While getting acquainted with the theoretical basis of viscous liquids hydrodynamics [1-3] a very interesting detail has been found. For example, monograph [1] states that the resisting force at the laminar flow of the body by a steady liquid flow moving with the **u** rate will be determined by the formula $f_i = \eta a_{ik} u_k$, where $\eta-$ the dynamic viscosity of the medium and the $a_{ik}-$ tensor is simply a constant value related to the shape of the body. On the other hand, as shown in monograph [2], the resisting force of cylindrical body should be described by the Lamb–Oseen formula with the nonlinear form of the dependence, namely $f_i = \eta \delta_{ik} \frac{u_k}{\ln u}$, where $\delta_{ik}-$ Kroneker's symbol. However, both monographs deal with laminar flow, i.e. they speak about relatively low flow rates and relatively low Reynolds numbers. This contradiction has forced us to return to this classic issue again and consider the problem of calculating the resisting force of the cylindrical body f_i, but in terms of an approach slightly different than that of the Stokes model, albeit in a potential flow approximation. Reynolds number $\mathrm{Re} = \frac{Lu}{\nu}$, where $L-$ characteristic linear scale and $\nu-$ kinematic viscosity, then, choosing, for example, water parameters, for which the kinematic viscosity $\nu = 10^{-2}\frac{cm^2}{s}$, and the body rate is low $u = 1cm\cdot s^{-1}$ for a linear scale $L \approx 1cm$ we find that $\mathrm{Re} = 100$. Such an assessment does not quite fit the picture of the low Reynolds numbers for which the inequality $\mathrm{Re} << 1$ should be realized. It means that either a linear dimension of the body should be in the range of nanoparticles dimensions and the speed is not too low, for example $u \le 1cm\cdot s^{-1}$ or, on the other hand, the body dimensions may be quite adequate and be in the range of , for example, $1cm$, but with the very rate low at the same time. In our opinion, both cases are of no interest from the physical point of view. So it would be very interesting to find a solution of the Navier - Stokes equations disregarding the substantial derivative and in the case when the δr ,characteristic distances measured from the surface of the body are within the range $n^{-\frac{1}{3}} << \delta r << \frac{\nu}{u}$, where $n-$ the density of liquid molecules. As can be seen, this inequality is easy to realize for relatively low rates and then we really may disregard the left side of the Navier - Stokes equation. In this case the equation will have the following form $-\nabla P + \nu\Delta\mathbf{V} = 0$. We should note that it is valid for bodies of arbitrary shape. As part of our objective we will provide a solution for the case of laminar flow under the condition of close proximity to the boundary of the body $n^{-\frac{1}{3}} << \delta r << \frac{\nu}{u}$.

If a relatively small piece is selected on a thin rod, it can be regarded as a cylindrical body in case of a very good approximation. It is for this piece of cylinder we will first calculate the resisting force as a function of its radius R. Please note, that none of the monographs known to us [1 – 3] does not give the solution of this problem, except for the Lamb–Oseen formula. In the case of cylindrical bodies, the dependence of the resisting force on the flow rate must also be linear, as in the well-known Stokes problem solved for a spherical body. Although the final expression for the resisting force, which we will strictly get, appears to be proportional to ηu product, where $\eta-$ the dynamic viscosity (which, by the way, was noted in [1]), we have been able to calculate the dependence of the resisting force of the cylinder also on its radius R.

The first part of our research will be devoted to the analysis of this particular case. The second part will be devoted to the application of the established formula for calculating the resisting force for a slender column at its arbitrary (not small!) and heterogeneity displacements.

II. BASIC EQUATIONS AND THEIR ANALYSIS FOR A CYLINDRICAL BODY

In steady case the Navier - Stokes equation may be written as $\rho(\mathbf{V}\nabla)\mathbf{V} = -\nabla P + \eta\Delta\mathbf{V}$, where P – pressure, ρ – density and $\mathbf{V}$ – steady flow rate. As we described above, in the area close to the surface of the body the equation may be written as follows:

$$-\nabla P + \eta\Delta\mathbf{V} = 0. \qquad (1)$$

We will look for its solution in a not quite standard form, i.e.

$$\mathbf{V} = \mathbf{u} + rotrotrotrotf\mathbf{u}, \qquad (2)$$

where the function f – is to be found.

Inserting (2) subject to $\mathbf{u} = const$ into equation (1) we find $\Delta^3 rotf\mathbf{u} = \left[\mathbf{u}\times\Delta^3\nabla f\right] = 0$. Thus, function f should satisfy the following equation:

$$\Delta^3 f = 0, \qquad (3)$$

where the constant of integration is set to zero. If now we insert (2) into equation (1), we get $-\nabla P + \eta\Delta^2\left(\mathbf{u}\Delta f - \Delta^2\nabla(\mathbf{u}\nabla f)\right) = 0$. Therefore, pursuant to equation (3) this implies $\nabla P = -\eta\Delta^2\nabla(\mathbf{u}\nabla\mathbf{f})$. So, after integration, we have

$$P = P_0 - \eta\Delta^2(\mathbf{u}\nabla f), \qquad (4)$$

where P_0 – external pressure. Due to the symmetry of the problem, it is convenient to move to the cylindrical coordinate system where the z axis is directed along the axis of the cylinder and the polar angle углом φ is measured from the x axis, as shown in Figure 1.

We have chosen x axis along the flow direction, i.e. according to $\mathbf{u}$ rate. Due to the symmetry of the problem, the function f will depend only on the radial coordinate r. Therefore, if $f = f(r)$ and Laplace operator in the cylindrical coordinate system for the radial component is $\Delta f = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial f}{\partial r}$, equation (3) after six-time integration gives rise to the following solution:

$$f = C_1 r^4(\ln r - 3/4) + C_2 r^5 + C_3 r^2\ln r + $$
$$+ C_4 r^2 + C_5\ln r + C_6,$$

Assuming here that $C_1 = C_2 = C_6 = 0$, and re-setting constant values we receive

$$f = Ar^2\ln r + B\ln r + Cr^2. \qquad (5)$$

Of (2) we find

$$\mathbf{V} = \mathbf{u} + \mathbf{u}\Delta^2 f - \Delta\left[\frac{f'}{r}\mathbf{u} + \left(f'' - \frac{f'}{r}\right)\mathbf{n}(\mathbf{un})\right]. \qquad (6),$$

where the unitary vector $\mathbf{n} = \frac{\mathbf{r}}{r}$ is normal to the outer surface of a cylindrical body. We use the following formulas to find the radial and tangential rate components:

$$V_r = u_r - (\Delta\mathbf{G})_r,$$
$$V_\varphi = u_\varphi - (\Delta\mathbf{G})_\varphi, \qquad (7),$$

where vector $\mathbf{G} = \left(f'' - \frac{f'}{r}\right)\mathbf{n}(\mathbf{un}) + \mathbf{u}\frac{f'}{r}$. According to the general formulas of differential geometry we have

$$(\Delta\mathbf{G})_r = \Delta G_r - \frac{G_r}{r^2} - \frac{2}{r^2}\frac{\partial G_\varphi}{\partial\varphi},$$
$$(\Delta\mathbf{G})_\varphi = \Delta G_\varphi - \frac{G_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial G_r}{\partial\varphi}.$$

After plugging in here the functions $G_r = u\cos\varphi f''$, $G_\varphi = u\sin\varphi\frac{f'}{r}$ we have

$$V_r = (\mathbf{un})\left[1 - \left(f^{(4)} + \frac{f'''}{r} - \frac{f''}{r^2} - 2\frac{f'}{r^3}\right)\right],$$
$$V_\varphi = (\mathbf{u}\tau)\left[1 - \left(\frac{f'}{r}\right)'' - \frac{1}{r}\left(\frac{f'}{r}\right)' + 2\frac{f''}{r^2} + \frac{f'}{r}\right].$$

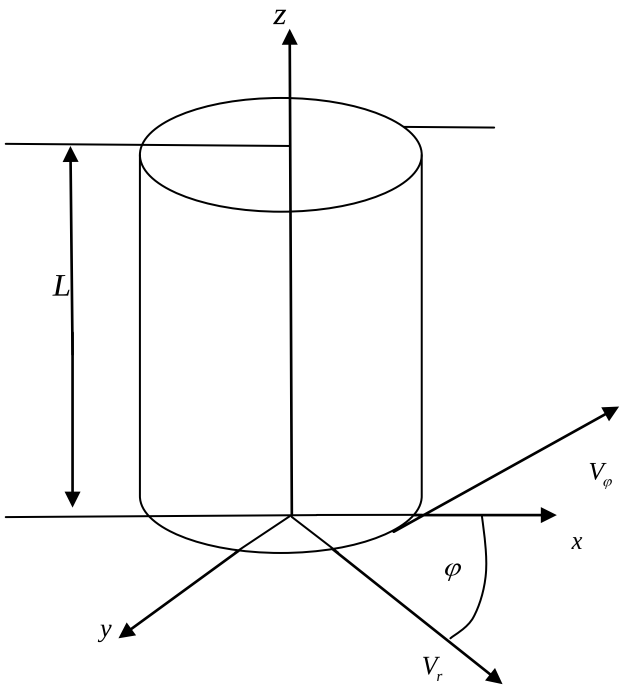

Figure 1. Schematic interpretation of coordinate system and direction of velocity flow.

Finally, plugging in solution (5), we find

$$V_r = u\cos\varphi\left(1 + 5\frac{B}{r^4} + 6\frac{A\ln r}{r^2} + 5\frac{A}{r^2} + 6\frac{C}{r^2}\right),$$
$$V_\varphi = u\sin\varphi\left(1 + 6\frac{A\ln r}{r^2} + 7\frac{A}{r^2} - 5\frac{B}{r^4} + 6\frac{C}{r^2}\right) \qquad (8)$$

According to conditions where $V_r\big|_{r=R} = 0,\ V_\varphi\big|_{r=R} = 0$ we easily find that

$$C = 0,\ A = -\frac{R^2}{6(1+\ln R)},\ B = -\frac{R^4}{30(1+\ln R)}. \qquad (9)$$

Thus, solution (8) will look like

$$V_r = u\cos\varphi\left(1 - \frac{R^4}{6r^4(1+\ln R)} - \frac{R^2\ln r}{r^2(1+\ln R)} - \right.$$
$$\left. - \frac{5R^2}{6r^2(1+\ln R)}\right), \qquad (10)$$
$$V_\varphi = u\sin\varphi\left(1 + \frac{R^4}{6r^4(1+\ln R)} - \frac{R^2\ln r}{r^2(1+\ln R)} - \right.$$
$$\left. - \frac{7R^2}{6r^2(1+\ln R)}\right).$$

As can be seen from (10), the equation $div\mathbf{V} = 0$ is satisfied automatically. Now, to calculate the resisting force determined by the close proximity to the surface of the body and directed opposite to the x axis, we need to use the general expression [1] $F_i = \int_S \sigma_{ik}dS_k$, wheree S – body surface and the stress tensor is given by the equation $\sigma_{ik} = -P\delta_{ik} + \eta\left(V_{i,k} + V_{k,i}\right)$, where $V_{i,k}$ – covariant derivative. Hence,

$$F_i = \eta\int_S \left(V_{i,k} + V_{k,i}\right)dS_k - \int_S PdS_i \cdot \tag{11}$$

Thus it is left only to calculate the dependency of the pressure. To find it we apply equation (4), according to which $P = P_0 - \eta\Delta^2(\mathbf{u}\nabla f)$, and plug solution (5) in it with the ratios determined by (9). Then

$$f = -\frac{R^2}{6\left(1+\ln R\right)}\left(r^2 - \frac{R^2}{5}\right)\ln r$$

and due to rather tedious computations we find that

$$P = P_0 + \frac{4\eta R^2(\mathbf{un})}{3r^3\left(1+\ln R\right)}\left(1 + 2\ln r - \frac{4}{5}\frac{R^2}{r^2}\right). \tag{12}$$

On the boundary, i.e. at $r = R$ we thus have

$$P = P_0 + \frac{4\eta(\mathbf{un})}{3R\left(1+\ln R\right)}\left(0.2 + 2\ln R\right)\cdot \tag{13}$$

The vales of the viscous stress tensor components on the surface of the body will be as follows:

$$\begin{aligned} \sigma_{rr}\big|_{r=R} &= \frac{2\eta u\cos\varphi}{3R\left(1+\ln R\right)}\left(2+3\ln R\right), \\ \sigma_{r\varphi}\big|_{r=R} &= \frac{\eta u\sin\varphi}{3R\left(1+\ln R\right)}\left(1+3\ln R\right). \end{aligned} \tag{14}$$

And, therefore, according to (11) the force projection to x axis will be

$$F_x = \int_S \left(\sigma_{rr}\cos\varphi + \sigma_{r\varphi}\sin\varphi\right)rd\varphi dz - \int_S P_0\cos\varphi rd\varphi dz$$

The last integral vanishes and the first two lead us to the formula for the resisting force per unit length of the column

$$f_x = \frac{F_x}{L} = \frac{\pi\eta u\left(4.2+\ln R\right)}{3\left(1+\ln R\right)}. \tag{15}$$

As is clear from the formula found, we see the rapid $f_x(R)$ function increase (four times approximately) at $R \sim 1$.

III. RESISTING FORCE FOR A LONG THIN ROD

In accordance with the general principles of viscous liquids hydrodynamics (see [1]), now we can calculate the resisting force affecting a cylindrical body in case of its arbitrary vibrational displacements using the general form of the viscous stress tensor [1 – 3]: $\sigma_{ik} = -P\delta_{ik} + \eta\left(V_{i,k} + V_{k,i}\right)$, where δ_{ik} is Kronecker's symbol. Whereas, according to our knowledge, the resisting force is defined as $F_i^{fr} = \int_S \sigma_{ik}dS_k$, the, according to (11), we get $F_i^{fr} = \eta\int_S \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)dS_k$. In case of a long slender column, a surface element may be represented as $dS = 2\pi r(l)dl$, where $r(l)$ is a cross-sectional radius varying along the length, and a length element may be represented as $dl = dy\sqrt{1+\xi'^2}$, where the "prime mark" means the differentiation by x. The ξ value defines the column displacement towards the vertical axis, specified as x axis. We assume that the column is fixed in one upper point and its second end moves in a completely random way with each length element characterized by the deviation $\xi = \xi(x,t)$. The problem to be solved is merely two - dimensional. Thus the resisting force acting strictly according to normal drawn to the surface can be calculated by the following formula:

$$F_n^{fr} = \frac{\pi\eta\left(4.2+\ln R\right)}{3\left(1+\ln R\right)}\pi\eta\int_l r(l)\frac{\partial u_n}{\partial l_n}dl, \tag{16}$$

where $u_n -$ rate along the normal. AS a result, the resisting force per unit length of the column is

$$f_n^{fr} = \frac{\pi\eta\left(4.2+\ln R\right)}{3\left(1+\ln R\right)}r\left(l\right)\frac{\partial u_n}{\partial l_n}. \tag{17}$$

The formula (17) also answers the question regarding the normal resisting force per unit length of a flexible and arbitrarily curving slender column in case of its arbitrary changing cross-sectional radius $r(l)$. It is understood that the resistance according to (16) should be defined as an integral along the column length, i.e. $F_n^{fr} = \int_l f_n^{fr}dl = K\int_l r(l)\frac{\partial}{\partial l_n}\left(\dot\xi\sqrt{1+\xi'^2}\right)dl$, where coefficient $K = \frac{\pi\eta\left(4.2+\ln R\right)}{3\left(1+\ln R\right)}$.

If the radius is constant, i.e. $r(l) = R = const$, it follows from here that

$$F_n^{fr} = \frac{\pi\eta\left(4.2+\ln R\right)}{3\left(1+\ln R\right)}R\int_l dl\frac{\partial}{\partial l_n}\left(\dot\xi\sqrt{1+\xi'^2}\right)\cdot \tag{18}$$

We see that expression (18) is nonlocal and describes the full resisting force affecting the column. The expressions received also allow us to record general dynamic motion equations for the column introducing the component, related to the resisting force (18), to the classical operation. According to the general definition of a classical operation S (see, e.g., [4]) of (18) we come to

$$\begin{aligned} \Delta S = \Delta\dot Q &= \int_{t_0}^{t_1}dt\int_l\int F_n^{fr}dl_n dl = \\ &= \frac{\pi\eta R\left(4.2+\ln R\right)}{3\left(1+\ln R\right)}\int_{t_0}^{t_1}dt\int_l dl\int dl_n\frac{\partial}{\partial l_n}\left(\dot\xi\sqrt{1+\xi'^2}\right) \end{aligned} \tag{19}$$

where $\Delta\dot Q -$ dissipation function. The inner integral is a total differential and thus, remembering that the length element is $dl = dx\sqrt{1+\xi'^2}$, we immediately find $\Delta\dot Q = \Delta S = \pi\eta R\frac{4.2+\ln R}{3(1+\ln R)}\int_{t_0}^{t_1}dt\int_{x_0}^{x_1}\dot\xi\left(1+\xi'^2\right)dx$ out of (19).

Following the general properties of S action [4], we may omit the first summand proportional to the derivative with time and to write such expression down for a dissipation function $\Delta\dot Q = \Delta S$

$$\Delta S = \pi\eta R\frac{4.2+\ln R}{3(1+\ln R)}\int_{t_0}^{t_1}dt\int_{x_0}^{x_1}\dot\xi\xi'^2dx\cdot \tag{20}$$

Using expression (20) it is easy to find a variation of the displacement operation $\delta\xi(t,x)$ and thus receive the relevant equation of motion and transversal conditions for the non-fixed end considering resisting forces of the medium. Moreover, we can represent the general classical operation in case of arbitrary displacements of the rod using (20) as

$$\begin{aligned} S = \frac{\rho}{2}\int_{t_0}^{t_1}dt\int_{x_0}^{x_1}dx\Big[\sqrt{1+\xi'^2}\Big[\left(\dot\xi^2 - u_0^2 - 2gx\right) + \\ + \frac{2\pi\eta R}{3\rho}\cdot\frac{4.2+\ln R}{1+\ln R}\dot\xi\xi'^2 + \frac{B}{r_0^{\,2}}\Big] \end{aligned}, \tag{21}$$

where $g-$ gravity acceleration and a relevant summand in (21) determines the contribution of gravity to the operation and $\rho-$ linear density of column material. Its dimensions are $\left[\frac{g}{cm}\right]$. The constant B represents the column stiffness and r_0- radius of curvature at an arbitrary column displacement point. In the flat Euclidean space it is expressed by $r_0=\frac{\left(1+\xi'^2\right)^{\frac{3}{2}}}{\xi''}$. In three dimensions case its expression is more tedious but is also easy to write down.

Classical problems connected with investigation of small vibrations of a rod (ore string) or thin membrane are described practically in all known text-books on partial differential equations (see [9-12]). In derivation of a corresponding linear differential equation most authors use principles of Newton's classical mechanics, which results in obtaining of hyperbolic equation sought for. It should be noted, that the least action method (often called Lagrange's method), when the derivation is based on obtaining the functional proportional to kinetic T and potential U energies difference, has much more possibilities (see [4]). Note, however, that the question of what is responsible for such vibration motion of a thin string or membrane from physical point of view has not been analyzed in any known literature source. That is why we would like to dwell upon this issue in more details and to illuminate physics of such dynamical phenomenon. So, the general principle, according to which any vibration equation may be derived, is Maupertuis's principle. Indeed, in vibration of a string (or membrane, or something like that) its total energy E is conserved, where $E=T+U$, but at $E=const$ classic operation $S=\int_{t_0}^{t_1}(T-U)dt$ may be presented as $S=\int_{t_0}^{t_1}(T-U)dt=\int_{t_0}^{t_1}2Tdt-E(t_1-t_0)$. Remember, that such approach is referred to as "short-cut action method". Variation of the equation $S-S_0=\int_{t_0}^{t_1}2Tdt=\int \mathbf{p}d\mathbf{r}$, where $S_0=-E(t_1-t_0)$, $\mathbf{p}=m\mathbf{u}$ is momentum, m is mass, $\mathbf{u}-$ is a velocity, gives the equation of motion in the form $\mathbf{a}=\mathbf{n}\frac{u^2}{r_0}+\mathbf{t}\frac{du}{dt}$, where $\mathbf{n}-$ is unit vector of the normal to the trajectory, $\mathbf{t}-$ is tangent to it, u is absolute value of mass m movement speed, r_0- is radius of trajectory curvature in the given point. This equation is the differential equation by which we may describe any dynamics of the system in non-dissipative case, if in it there is the smallest hint on bend. Now let us check up how the equation (1) "works" as applied to known particular cases. In two-dimensional case (thin string), if movement speed of the corresponding vibration is constant, then $\frac{du}{dt}=0$ and $u=u_0=const$, equation (1) takes the form $\mathbf{a}=\mathbf{n}\frac{u_0^2}{r_0}$. If the vibrations are small, the unit vector of the normal $\vec{n}$ "outlooks" along the axis y and hence the acceleration $\mathbf{a}=(0,a_y)=(0,\ddot{y})$. In string vibration theory displacement along the axis y is usually denoted as $\xi(x,t)$. Using this notation we shall write, therefore, that $a_y=\frac{\partial^2\xi}{\partial t^2}$. Since the curvature radius in plane case for a random trajectory is $r_0=\frac{\left(1+y'^2\right)^{3/2}}{y''}$, for small vibrations it will be approximately $\frac{1}{r_0}\approx\frac{\partial^2\xi}{\partial x^2}$. Consequently, the equation of small string vibrations may be presented as $\frac{\partial^2\xi}{\partial t^2}=u_0^2\frac{\partial^2\xi}{\partial x^2}$. Due to such method we can write any small vibrations of the different bodes. As an example else one we shall briefly dwell upon small vibrations of a membrane. For it the situation is quite analogous, and for its small displacements relative to some mean membrane plane the average (non-Gaussian) curvature will be $K=\frac{1}{R}=\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\approx\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}$ by definition. As it has always been considered that movement speed of surface membrane points is constant and equal to u_0, and acceleration is $a_y=\frac{\partial^2\xi}{\partial t^2}$, then in the framework of equation (1) the equation of motion acquires standard form which can be found practically in any text-book in mathematical physics $\frac{\partial^2\xi}{\partial t^2}=u_0^2\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}\right)$. Which was to be shown. The purpose of the present paper is to derive dynamic equations of string motion, the rod being nearly fixed at one end. The term " nearly fixed" means that this end may be abruptly moved and then returned to the initial point, whereby the rod will get its initial shape which should be considered as the initial condition (see below). Further motion of both the rod as a whole and its free end will be described in time and space. In case of a stretchy rod (e.g. rubber rod) acceleration components must be predetermined as $\mathbf{a}=\left(\frac{\partial^2\xi_x}{\partial t^2},\frac{\partial^2\xi_y}{\partial t^2}\right)$, where ξ_x,ξ_y are displacements in the directions of x and y axes respectively. If a rod is not stretchy, the acceleration component $\frac{\partial^2\xi_x}{\partial t^2}$ vanishes because $\xi_x=0$. Note, that the case of a stretchy elastic rod is rather complicated, for its solution requires taking into account internal deformations provided for by the equations of elasticity theory in terms of which the value ξ_y should be identified with the displacement vector of internal points of the rod $\mathbf{u}$. Such a problem is a subject of a separate report and need not be considered in the present paper. Below we shall study only the case of a non-stretchy rod, when point displacement takes place only along the axis x, and which will be denoted as $\xi_x=\xi_x(t,y)$. In order to account the resistance force of viscous medium we can use the integral (18).

IV. NONLINEAR EQUATIONS OF DYNAMICS OF THIN ELASTIC ROD

If we account (19) and (20) we find the classical action in the view as

$$S=\frac{\rho}{2}\int_{t^*}^{t_1}dt\int_{y_0}^{y_1}dy\sqrt{1+\xi_y'^2}\left[\left(\dot{\xi}_x^2-u_0^2-2gy\right)+\right.$$
$$\left.+\frac{2\pi\eta R}{3\rho}\cdot\frac{4.2+\ln R}{1+\ln R}\dot{\xi}_x\xi_x'^2\right] \qquad (22)$$

Variation of action (22) under the condition of the point $M(\ y_1,\xi_x(t,y_1)\ )$ mobility results in two equations. Equation of motion

$$\frac{\partial}{\partial t}\left(\dot{\xi}_x\sqrt{1+\xi_x'^2}\right)+\frac{\partial}{\partial y}\left[\frac{\xi_x'\left(\dot{\xi}_x^2-u_0^2-2gy\right)}{2\sqrt{1+\xi_x'^2}}\right]+.$$
$$+\frac{2\pi\eta R}{3\rho}\cdot\frac{4.2+\ln R}{1+\ln R}\left(2\xi_x'\dot{\xi}_x'+\dot{\xi}_x\xi_x''\right)=0 \qquad (23)$$

And equation of a transversal

$$\left.\begin{array}{l}\left(\dot{\xi}_x^2-u_0^2-2gy\right)\left(1+\varphi'\xi_x'\right)+\\ +\dfrac{2\pi\eta R}{3\rho}\cdot\dfrac{4.2+\ln R}{1+\ln R}\xi_x'\dot{\xi}_x\left(2\varphi'-\xi_x'\right)\sqrt{1+\xi_x'^2}\end{array}\right|_{y_1}=0, \qquad (24)$$

where $\varphi(y)$ is pre-assigned law of unfixed of a thin rod and trajectory change. In details form equation (23) is reduced to the following:

$$\ddot{\xi}_x+\frac{2\dot{\xi}_x\xi_x'\dot{\xi}_x'}{1+\xi_x'^2}+\beta\frac{\left(2\xi_x'\dot{\xi}_x'+\xi_x''\dot{\xi}_x\right)}{\sqrt{1+\xi_x'^2}}=$$
$$=\frac{\xi_x''\left(u_0^2+2gy-\dot{\xi}_x^2\right)}{2\left(1+\xi_x'^2\right)^2}, \qquad (25)$$

where friction coefficient $\beta=\frac{2\eta}{3\rho_b R}\cdot\frac{4.2+\ln R}{1+\ln R}$, and volume density of the rod $\rho_b=\frac{m}{V_b}=\frac{m}{\pi R^2 l}$ (not to confuse with linear density ρ !). If the mixed derivative $\frac{\partial^2\xi_x}{\partial t\partial y}=0$ (we shall stress that it refers only to those rod segments for which the tangent angle to the curvature line in the given point is not in close vicinity of $\frac{\pi}{2}$), and hence the first equation in the system (11a) becomes simplified and we have

$$\ddot{\xi}_x+\beta\frac{\xi_{\div}''\dot{\xi}_{\div}}{\sqrt{1+\xi_x'^2}}=\frac{\xi_x''\left(u_0^2+2gy-\dot{\xi}_x^2\right)}{2\left(1+\xi_x'^2\right)^2}. \qquad (26)$$

The transversal condition is

$$\left.\begin{array}{l}\left(\dot{\xi}_x^2-u_0^2-2gy\right)\left(1+\varphi'\xi_x'\right)+\\ +\dfrac{2\pi\eta R}{3\rho}\cdot\dfrac{4.2+\ln R}{1+\ln R}\xi_x'\dot{\xi}_x\left(2\varphi'-\xi_x'\right)\sqrt{1+\xi_x'^2}\end{array}\right|_{y_1}=0. \qquad (27)$$

V. CONCLUSION

Summing up, we turn our attention to the following moments.

1. One more new solution of the Navier - Stokes equation for the laminar flow of the near-surface layer of a cylindrical body for the area $n^{-\frac{1}{3}}<<\delta r<<\frac{\nu}{u}$ has been found;
2. It has been demonstrated that in such a case the resisting force per unit length of a cylindrical body is proportional to the multiplier $\frac{4.2+\ln R}{3(1+\ln R)}$. As to the long elastic rod as we see from expression (18) the force of resistance is proportional $\frac{4.2+\ln R}{3(1+\ln R)}\cdot R$. And when $R\to 0$ we have $F^{fr}\to 0$. It's a correct result;
3. In the general case of arbitrary curves of a thin rod fixed at one end, an expression is given for a classical operation, taking into account both the gravity and the resisting force exerted by the viscous continuum.

We should emphasize that the extended version of our research will also include the computation of the resisting force as well as the dissipative function if the thin rod motion is not inertial and its rate in each local arear is time-dependent.

4. The both of the nonlinear move equation and transversal condition was find from the classical action (22).

REFERENCES

[1]. L.D. Landau, E.M. Lifshic. Theoretical physics. V. 6. Hydrodynamics. M.: Science. 2010.
[2]. N.E. Kochin, I.A. Kibel, N.V. Rose. Theoretical hydromechanics. M.: Fizmatlit. 1963. V. 1.
[3]. L.I. Sedov. The mechanics of continuously media. V. 1, 2. M.: Science. 1973.
[4]. L.D. Landau, E.M. Lifshic. Theoretical physics. V. 1. Mechanics. M.: Science. 2008.
[5]. S.O. Gladkov. On the connection of Joule - Thompson's coefficient with dissipation properties of filtrating media. The European Physical Journal E. 2003. V. 10. P. 171 - 174.
[6]. Vikas Poply, Phool Singh, K.K. Chaudhary. Analysis of Laminar Boundary Layer Flow along a Stretching Cylinder in the Presence of Thermal Radiation. WSEAS Transactions on fluid mechanics. V. 8. 2013. PP. 159 – 164.

[7]. K. Adhan, A.H. Al-Salihi, A.H. Hasmani. M.G. Timol. Similarity Methods in the Analysis for Laminar Forced Convection on a Horizontal Plate. WSEAS Transactions on Heat And Mass Transfer. Issue 1. V. 8. January 2013. PP. 1-6.

[8]. Zodwa G. Makukula, Precious Sibanda, Sandile S. Motsa. On New Solutions for Heat Transfer in a Visco-Elastic Fluid between Parallel Plates. International journal of mathematical models and methods in applied sciences. Issue 4. V. 4. 2010. PP. 221-230.

[9]. A.N. Tihonov, A.A. Samarskii. Equations of mathematical physics. Gostechizdat. 1953.
[10]. S.L. Sobolev Equations of mathematical physics. Gostechizdat. 1954.
[11]. M.M. Smirnov Second-order partial differential equations Moscow. Science (Nauka). 1964.
[12]. N.S. Koshlyakov, E.B. Gliner , M. M. Smirnov Partial differential equations of mathematical physics. M.: High school 1970.