Structural analysis of Historical Masonry Arches: state-of-the-art and recent developments

Lucio Nobile, Veronica Bartolomeo

***Abstract*—**The aim of this paper is to give an overview of the main analytical and numerical methods for the assessment of masonry arches , highlighting strengths and weaknesses. The methods are mainly three: i) the Thrust Line Analysis Method; ii) the Mechanism Method; iii) the Finite Element Method. The Thrust Line Analysis Method and the Mechanism Method are analytical methods and are based on two of the fundamental theorems of the Plastic Analysis, while the Finite Element Method is a numerical method that uses different strategies of discretization to analyze these structures.

***Keywords*—**Masonry arch, Structural Models, Discrete limit analysis, Collapse Mechanism.

I. Introduction

As reported in a previous paper [1], Benvenuto [2] gave us the historical perspective of the first static theories regarding the masonry arch..

Between the seventeenth and eighteenth century, the geometric and the empiric rules reported in the ancient treatises were replaced by a real static theory on the stability of the arches. De La Hire [3] was the first developing an innovative approach, which remained the same through the eighteenth century. The arch was considered as a series of rigid blocks of well-defined geometry and specific weight. However his model neglected the friction, which was taken into account by Coulomb Model.

Only around the fifties of this century, the problem was taken up and dealt with a more congenial method. Attempts in the twenties to adapt the elastic theory to the masonry arch were not very successful. The weak points of these attempts were to assume the masonry material as elastic and to consider valid the results even if the thrust line was external to the core in some points. The turning point of the fifties was the introduction of the limit design and of its increasing applications in structural analysis. The theorems of limit analysis are admirably suited to the determination of the collapse load of masonry arches.

So nowadays the engineering methods of assessment for arch bridges mainly rely on the pioneering works by Pippard and Ashby [4,5]. These researchers determined the load required, at a given location, to cause the formation of two additional hinges, and hence a mechanism, in a two hinged arch. The method guaranteed that an equilibrium configuration exists for the considered structural model but gave only rough estimates of the limit load. Following this approach and Drucker's studies, Kooharian [6] published the first modern paper on this topic, followed one year later by Onat and Prager paper [7].

Another milestone was Heyman publication [8], in which he explained for the first time the applicability of ultimate load theory for any masonry loadbearing structure.

Heyman did not introduce anything new, but formalized in a clear way some hypotheses on the material that formed the basis for the calculation of the arches in the XVIII and XIX century. These assumptions enabled Heyman to frame the masonry action in the plastic theory and to formulate the famous safe theorem that will be explained later on.He introduced three hypotheses for the determination of the admissibility domain of the masonry material.

The three hypotheses are: (i) *the masonry has no tensile strength*; (ii) *the masonry has infinite compression strength*; (iii) *sliding failure doesn't occur*. The first assumption that does not always adhere to the reality, but it is a safety benefit. It is strictly true only if the masonry is made by dry- stone blocks or with weak mortar: however, in most cases, the adherence between mortar and masonry blocks is negligible because the mortar may decay in time. Therefore, whatever is the ultimate tensile strength of the individual blocks, the masonry may be considered a non resistant tensile material (NRT material). The hypothesis of infinite compression strength is a valid approximation only if the ratio between the average compression stress and the masonry compression strength is a negligible value compared to the unit.

A reduction of the resistant section occurs in a NRT material with a consequent redistribution of the compression stresses leading to an increase of the peak values. In normal conditions of service, stresses are so low that any phenomenon of crushing failure does not occur.

The assumption of absence of sliding failure implies that

Prof. L. Nobile is with the Dept. of Civil, Chemical, Environmental, and Materials Engineering (DICAM) of the University of Bologna-Campus of Cesena, via Cavalcavia 61, 47521 Cesena, ITALY (phone: +390547338311; fax: +390547338307; e-mail: lucio.nobile@ unibo.it).

V. Bartolomeo, PhD, is with the Dept. of Civil, Chemical, Environmental, and Materials Engineering (DICAM) of the University of Bologna-Campus of Cesena, via Cavalcavia 61, 47521 Cesena, ITALY.

the shear component of the stress exerted between two adjacent voussoirs can never exceed the friction resistance between them. In fact, low compression stresses allow developing high friction forces that prevent voussoirs from losing cohesion and sliding. The validity of this hypothesis can be verified considering the slope of the thrust line with respect to the joint lines: if the thrust line is perpendicular to the joints, there is no mutual sliding between the voussoirs. If it forms an angle minor than 90°, the voussoirs tend to slide downwards or upwards.

Concerning Heyman's hypotheses, the collapse mechanism of the arch is then identified by the progressive formation of hinges that coincide with the points where the thrust line is tangent to the intrados or extrados of the arch. The mechanism for formation of hinges is not the only possible for the arch, but the experimental studies of Hendry [9] show that it can be considered as the most likely collapse mechanism for arches well buttressed. The analogy between the rotation failure mechanics of the arch and that of the steel frames allows Heyman to apply the fundamental theorems of the plastic analysis, including the safe theorem:

"If any equilibrium state can be found that is one for which a set of internal forces is in equilibrium with the external loads, and, further, for which every internal portion of the structure satisfies a strength criterion, then the structure is safe".

The safe theorem allows remedying the vagueness connected to the true thrust line location between infinite numbers of possibilities: an arch is safe simply if a thrust line can be drawn inside his thickness.

The thrust line has not to go out of the masonry thickness: to this end, it is interesting to study its two extreme positions that represent two states still in equilibrium. In fact, when the thrust line touches the lower or the upper boundary of the arch, the masonry finds itself at the limit of the admissible states region and the eccentricity is such that promotes the formation of hinges. In particular, in the two extreme conditions, the thrust line gives the location of three hinges that open: in this way, the value of the horizontal abutment thrust can be calculated, as shown in Fig.1.

In the two extreme positions of the thrust line, the horizontal abutment thrust is : a) minimum; b) maximum. The minimum horizontal thrust is obtained when the arch acts on the environment: for example, by removing the centring that supports the masonry, an arch will thrust on the abutments and these one will open slightly. In minimum thrust state, or passive state, the thrust line will have the greatest rise and the smallest clear span; it will touch the extrados at the key and intrados at the back. The maximum horizontal thrust will be obtained when the environment acts on the arch: for example, when two abutments move closer to each other, the arch span diminishes. In state of maximum thrust state, or active state, the thrust line will have the smallest rise and the greatest clear span; it will touch the extrados at the crown and the intrados down. Three hinges will open if one is at the key; on the contrary, four hinges form.

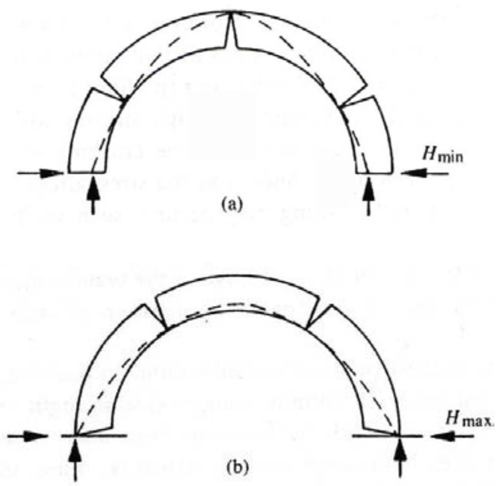

Fig.1 (a) Minimum abutment thrust (b) Maximum abutment thrust

It is important to know the two extreme positions of the thrust line, because the real thrust of the arch can't be calculated, but the upper and the lower limits can be fixed.

The collapse of a masonry arch does not involve an absence of strength, but rather a loss of stability. In fact the collapse takes place when a thrust line can't be finding within the arch boundaries. The crisis is connected with the formation of a fourth hinge that transforms the stable arch in an unstable mechanism of collapse. The four hinges open in alternating way in the intrados and in the extrados, following a pattern that is function of the arch shape and the working loads. In case of symmetrical load, a fifth hinge can open, but generally slight geometrical failings make the structure to behave asymmetrically.

A masonry arch has to support two main types of loads: i) the self-weight; ii) the additional loads. The additional point loads have a thrusting nature and can cause collapses because their action move the thrust line out of the arch, generating the fourth hinge. On the contrary, the self-weight is the resistant load of any masonry structure and offers resistance to any mechanism of collapse.

The catenary is the arch true shape. Arches with other shape stand up because catenaries are included in their thickness. The thrust line shape is the mathematical catenary if the self-weight is equally distributed around the arch. There is a minimum thickness of semicircular arch that just contains a catenary. The limit arch has exactly this minimum thickness and is in unstable equilibrium. The ratio between the real arch thickness and the limit arch one defines the safety factor that is of geometric nature. Heyman suggests 2 as safe practical value: that is, if you're able to draw a thrust line in the middle half of the arch, the arch is safe. So the thrust line can be perceived as an index of the stability condition of the arch.

The research of Heyman highlights that an elastic analysis is problematic for masonry structures because there isn't a unique calculable equilibrium state. On the contrary, the limit analysis allows considering the structure only in relation to its ultimate state, using few material parameters and neglecting the initial stress state. Some of the principal methods for the assessment of masonry arch bridges are based on the fundamental theorems of Limit Analysis.

A summary of the basic rules that apply in the theory of plasticity can be found in the work of Horne [10]. In the context of masonry arches, there are fundamentally three main considerations to apply the theorems of plastic limit analysis: i) *the internal actions must be in equilibrium with the external loads*; ii) *there must be a sufficient number of hinges to transform the structure into a mechanism*; iii) *the maximum stresses must be less than or equal to the material strength*.

The three fundamental theorems of plastic analysis can be stated in a simplified form as:

• Static or lower bound theorem. If the equilibrium and yield conditions are everywhere satisfied, then the load factor λ_s is less than or equal to the failure load factor λ_p;

• Kinematic or upper bound theorem. If the equilibrium and the mechanism conditions are everywhere satisfied, then the load factor λ_k is greater than or equal to the failure load factor λ_p;

• Uniqueness theorem. If the internal stress state is such that the three conditions of equilibrium, mechanism, and yield are satisfied then that load factor is the collapse load factor λ_p.

The aim of this paper is to give an overview of the main analytical methods based on Heyman's theories for the assessment of masonry arch bridges, highlighting strengths and weaknesses.

II. Methods for the Assessment of the Masonry Arch Bridges

Structural analysis is a general term describing the operations to represent the real behavior of a construction. The analysis can be founded on mathematical models created on theoretical bases or on physical models tested in laboratory. In both cases, the models try to individuate the load carrying capacity of the structure, identifying the stress state, the strain and the internal forces distribution of the entire structure or of its parts. Besides, the models proposed for arch structures try to indicate the failure mode and the location of plastic hinges.

As previously seen, among the three fundamental structural criteria (strength, stiffness and stability), it is the stability that governs the life of the masonry arches because the average medium stresses are low and the strains are negligible. So the most important methods for the evaluation of masonry arch bridges are based on Heyman's theories and on the fundamental theorems of the Plastic Analysis. They are: i) the thrust line analysis method; ii) the mechanism method.

The Thrust Line Analysis Method is based on the lower bound theorem or "safe" theorem and defines the limits for the thrust line location. It uses a static approach and defines the limit load that ensures the equilibrium of the arch bridge analyzed. On the contrary, the Mechanism Method is based on the upper bound theorem and studies the number of plastic hinges needed to transform the arch in a mechanism. In this case, the stability of the arch is analyzed with regards to a kinematic approach. Both the methods are valuable: due to their different bases, the first one underestimates the structure strength, while the second overestimates it.

Another method frequently used to describe the structural behavior of the masonry arch bridges is the Finite Element Method. The Finite Element Method represents the most versatile tool for the numerical analysis of structural problems. However in the case of historic masonry, the peculiar nature of materials leads to pay particular attention to the application of this method (see e.g. [11]).

A particular closed-form approach has been recently developed in [12,13] .This method is based on the fundamental theorems of limit analysis and is used to determine the critical points with a relatively small modeling effort. To assure the stability of the masonry arch bridges, a model based on equilibrium equations and compatibility conditions is first developed. Next, the material properties are added to determine the formation of the hinges.

A.1 Thrust Line Analysis Method

This general method analyzes the arch stability, evaluating the location of the thrust line inside the cross section. The thrust line represents the locus of points along the arch through which the resultant forces pass. If all the arch voussoirs have the same size, the line of thrust has almost the shape of an inverted catenary.

"*Ut pendet continuum flexile, sic stabit contiguum rigidum inversum*"("*As hangs the flexible line, so but inverted will stand the rigid arch*") wrote Robert Hooke in [14]. "*None but the catenaria is the figure of a true and legitimate arch*." completed Gregory twenty years later in [15]. These quotes describe the mechanics of the arch in a brief, but precise way.

The thrust line method analyzes the location and the slope of the thrust line inside the cross section through two parameters. The first one is the eccentricity of the forces resultant that describes the location of the thrust line in the cross section. Calculation of thrust line location can be performed using the equilibrium equation or by solving a linear programming problem. So every thrust line is a possible equilibrium solution. Unfortunately masonry arch is not always a statically determinate structure and thus the solution is not unique. There are infinite possible lines of thrust. The equilibrium equations are not sufficient to obtain the inner forces.

The thrust line analysis method defines the load carrying capacity by limiting the zone where the resultant force can be positioned. This method presents some variants which differ from each other by the size of the limits. The limits depend on the theory and the material model assumed. The main approaches will be described below.

The first variant of this method is also the most ancient. The Middle Third Rule was anticipated by Thomas Young[16] in 1817, worked out by Claude-Louis Navier [17] in 1826 and applied to masonry arch by William Rankine [18] in 1858. This rule states that the thrust line must lay within the middle third of the cross section that is it must lie within the kern to avoid any tensile stresses.

This criterion is based on the elastic theory. Until the forces resultant remains within the kern, there are only compressive stresses. When the force passes the middle third, the section undergoes also tensile stresses. However it is assumed that the masonry has not tensile strength. Cracks may occur and this is wanted to avoid.

The middle third rule is an extremely safe approach for the determination of the collapse load. It is very difficult to satisfy because of this rigorous limit. It can be reach only: i) if it is considered in the design phase; ii) if the dead loads dominate considerably over live loads.

The difficulty to satisfy the previous criteria has led to apply a less conservative version of this method that is the middle half rule. This approach increases the limits for the thrust line. In this case, the thrust line should lie within the central half of the arch section.

Another variant of the thrust analysis method is proposed by Jacques Heyman. By employing the safe theorem, he assumes that an arch is safe simply if a thrust line can be drawn inside his thickness. An arch will collapse only if the thrust line reaches the arch edge at least in four points, converting the arch into a mechanism. This rule is surely the less conservative than the others because the whole cross section becomes the allowed zone for the thrust line.

This approach includes an important assumption concerning the masonry behavior. Infinite compression strength is attributed to the masonry material. This enables the thrust line to stay at the edge of the cross section. The assumption is not realistic, but this method can be considered a good method to use because in the majority of the masonry arch bridges the stress level is quite low respect to the masonry compressive strength.

All the variant of the thrust analysis method can be summarized by the Heyman's concept of "geometric safety factor". For example the masonry arches that satisfy the middle third rule have a geometric safety factor equal to three.

A.2 Computer Based Application: Archie-M

Thrust line analysis together with Heyman's safe theorem can be used to elaborate computational strategies for the structural analysis of masonry arch bridges. For example, Philip Block [19] developed an interactive computational procedure that uses the thrust lines to clearly visualize the forces within the masonry and to predict possible collapse modes.

The program lets the user to change the arch geometry, analyzing the different locations that can be assumed by the thrust line.

Between the specialized analysis programs based on this method, there is also Archie-M developed by Harvey and OBVIS Ltd12 in 2001 [20].

Archie-M is a computer program that analyzes multi-span arch bridges together with supports and backfill. It carries out a form of equilibrium analysis. That is to say it determines whether an arch will remain stable, without first considering how it will deform under load. In fact the software uses the thrust line analysis combined with a thrust zone to model the masonry finite crushing strength. In practice the program is based on the thrust zone analysis method [21]. Calculations are carried out on a static scheme of a three hinges arch. The hinge positions are chosen as the most likely for the given load pattern. The program is easy to use because it shows graphically the position of a potential thrust-line and the formed hinges for any given loading regime. Until the thrust zone is within the cross section of the arch at every point, the structure is safe. When the thrust zone begins to touch the arch edge in a fourth point, a mechanism is created and the collapse state is reached.

Although the aim of Archie-M is to demonstrate whether an arch bridge can withstand a given load or not, the collapse load can be estimate by varying the load value until a sufficient number of hinges is formed. The program provides also the internal forces and the thrust zone position for each arch segment. The live load is distributed through the fill with a sine shape. The backfill is modeled as a continuous body that spreads the load and provides both active and passive soil pressure.

B.1 Mechanism Method

The Mechanism Method is a kinematical method, based on the upper bound approach. This method belongs to the plasticity theory and was firstly used for steel structures. Later Heyman has applied it to masonry arch. The term mechanism refers to the possibility of structure to move in accordance to internal and external constraints. This Method assumes that a masonry arch becomes a mechanism when at least four plastic hinges open. Many experimental tests confirm this hypothesis. However position of hinges is unknown.

The first step is to assume the possible position of four hinges. In a simplified analysis with only a concentrated force on the arch, the first three hinges can be assumed to be located under the load and at the springing. The concentrated force is applied on the arch with no dispersion through the fill. Self weights include the weights of the backfill blocks and of the corresponding arch segment. The four unknowns are the reaction forces of the two abutments and the failure load.

The problem can be solved with the moment equilibrium equations at the hinges or with the equations of virtual works. In the first case, four equilibrium equations can be derived around the hinges and solved, giving the four unknowns. In the second case, the structure collapses if the total virtual work for at least one of the mechanisms allowable is positive. In order to find the best mechanism, it is necessary to repeat the analysis for each possible load position and adopt the lowest result.

There are some variants of this methods depending on the geometry and material model. The following two variants will be exposed.

According to the first variant the masonry arch is modeled as an assemblage of plane blocks that are infinitely rigid and have an infinite strength [22].The discretization is regular, but doesn't respect necessarily the actual number of units of the original arch. Usually the blocks are slightly larger than the physical ones because the mortar joints are not explicitly modeled. The blocks can be also extremely larger than the actual ones in order to reduce the computational effort. In this case it must be careful that the discretization does not affect the expected mode of response. As checked experimentally, the number of blocks to obtain a sufficiently exact solution is about forty.

At the collapse, the blocks can either slide or rotate. The blocks movement can be calculated using the minimal energy for global deformation.

An important extension of rigid block analysis has been made by Gilbert [23]. Since no real material can sustain infinite compressive stresses, this second variant of the mechanism method assumes a finite compressive strength, redefining the failure domain of normal stress and moment . Once again the failures are modeled in the contacts between the blocks, but the explained assumption constrains the hinges not to stay on the arch edges. In this way, the rotation point is brought back inside the arch that behaves as it would have a lower thickness. In the proximity of the hinges, the compressive force is carried by a rectangular stress block lying at the edge of masonry.

The assumption of a finite compressive strength complicates the computation. In fact it transforms a linear problem to a nonlinear one. Gilbert solves the question applying an iterative solution that uses a Linear Programming solver. In this way it is possible to obtain a solution to the global problem and to approximate the constraints as a series of linear constraints. The rigid-plastic block analysis can be considered the basic model for understanding the fundamental behavior of the masonry arches.

B.2 Computer Based Application: Ring

The two-dimensional rigid-plastic analysis has been inserted by Gilbert and Melbourne into a software called RING[24] , developed by the University of Sheffield spin-off company, LimitState Ltd. The program is able to analyze multi-span masonry arch bridges, built of arch barrels, supports and backfill. A particular feature of this software is the capacity to analyze multi-ring arches enabling separations between the various rings [25].

The program employs an efficient linear programming technique for the solution of virtual works equations. This mathematical optimization allows identifying the ultimate limit state, determining the percentage of live load that will lead to the collapse.

As a result of the analysis, the minimum adequacy factor for live load is obtained, together with a graphic representation of the thrust line and the failure mode. Exact location of hinges is indicated. The live load is distributed through a Boussinesq distribution with a maximum spread angle. The passive pressure is the only lateral pressure used.

III. COMPARISONS

In order to give a general overview on the use of these methods and different material models, the structural analysis of a generic fictitious arch bridge is performed. The material properties are reasonably hypothesized. The structure is statically determinate to the third degree and will collapse as soon the four hinges occur. A vertical concentrated point load P, applied at 0.75 (42.97°), and backfill load are imposed on the bridge.

Archie-M software related to the Thrust line method and Ring software related to the Mechanism model are used. The different methods and models are compared with each other in terms of collapse load and the position of the four hinges.

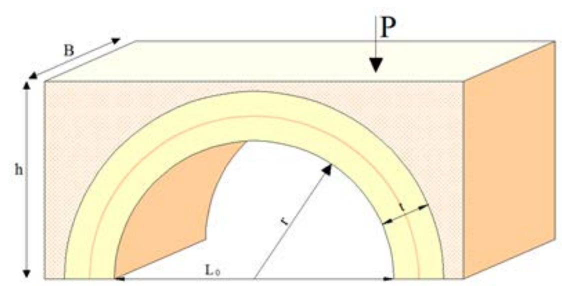

Fig. 2 Geometry of the Masonry arch bridge

The geometrical data are :
- Span = 2,80 m;
- Radius =1,4 m;
- Thickness of the Arch Barrel = 0,5 m;
- Height of the Backfill= 2 m;
- Width = 1 m.

The Masonry data are :
- Specific weight of the masonry arch = 21000 N/m3;
- Young's Modulus = 5000 MPa;
- Poisson's ratio = 0,3;
- Compressive Strenght of Masonry = 8 MPa.

The Backfill data are:
- Specific weight of the backfill = 21600 N/m3;
- Young's Modulus = 15000 MPa;
- Poisson's ratio = 0.3;
- Angle of friction = 350;
- Cohesion = 0.001;
- Angle of dilatancy = 350.

According to Figs. 3 and 4, the hinges are located alternatively in the intrados and in the extrados, following a pattern comparable to that described by Heyman for the point load case (Fig.5).

In addition to the hinge positions, Ring also gives the failure mode as graphic output. Concerning the collapse load, Archie-M estimates a load smaller than Ring: the first one is equal to 165.2 KN, the second one is equal to 558 KN. It can be concluded by comparing the results that the Limit Analysis Method is the most suitable to be applied to the arch masonry structures.

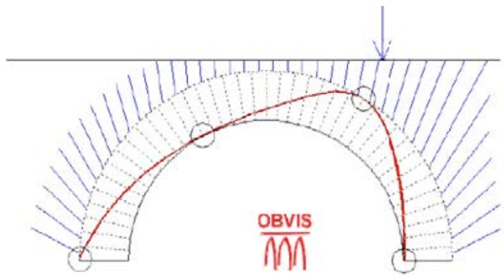

Fig. 3 Archie-M Output

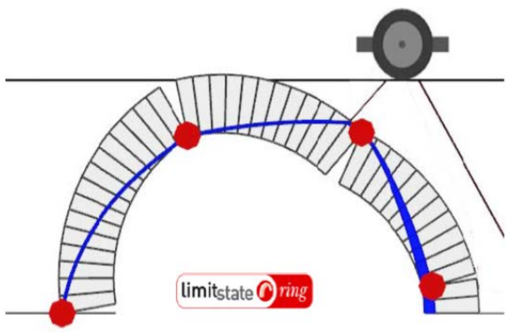

Fig. 4 Ring Output

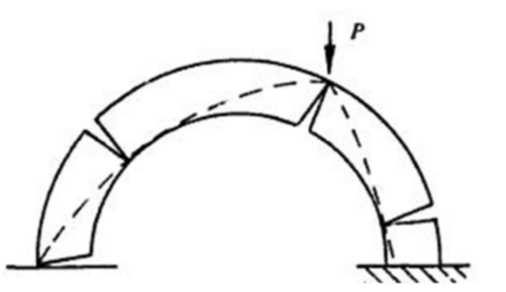

Fig. 5 Heyman pattern for the point load case

In this study, Ring is also used to perform a sensitivity analysis of the masonry arch bridge under consideration. Different parameters have been varied to evaluate their influence on the bridge behavior. The geometry parameters, such as span, rise and thickness of the arch, have not been investigated in the analysis. The reason is that the geometric parameters are more easy to measure than the material parameters.

The considered parameters are:

• number of segments in the arch;
• angle of internal friction;
• unit weight of masonry;
• unit weight of backfill;
• height of the backfill.

The sensitivity of each single parameter on the ultimate load is reported in Figs. 6-10.

It can be observed that:

• the number of arch segments has a limited influence on the collapse load (Fig.6). A sufficient number of segments is equal to forty. This may lead to a very small overestimate of the load capacity, but allows to save computational effort;

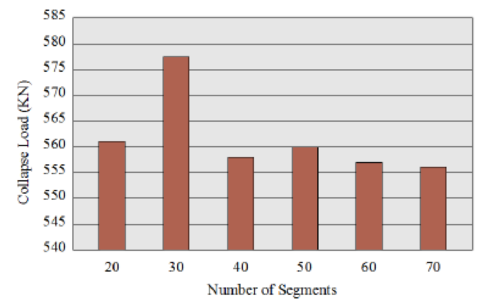

Fig.6 Influence of the Number of Segments in the Arch

• both unit weights of masonry and of backfill have a stabilizing effect on the arch behavior (Figs. 7 and 8). Their increase provides higher values of the collapse load;

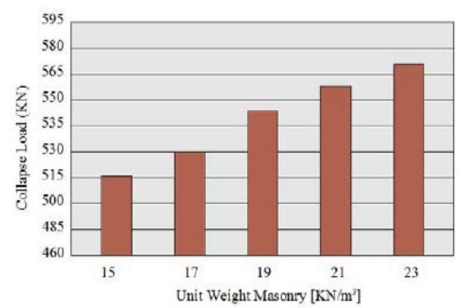

Fig. 7 Influence of the Unit Weight of Masonry

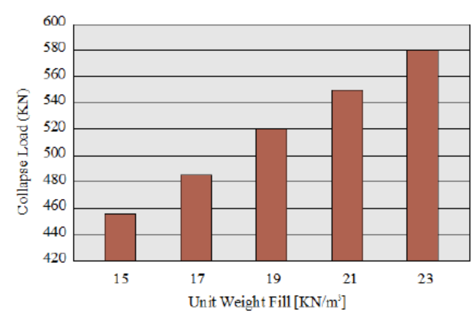

Fig. 8 Influence of the Unit Weight of Fill

• the increase of internal friction angle of the backfill gives higher values of the collapse load (Fig.9);

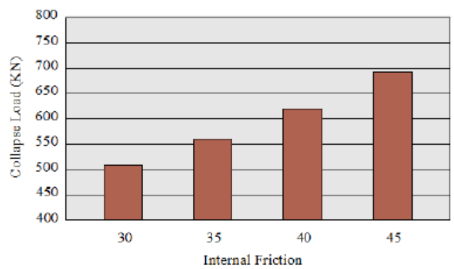

Fig. 9 Influence of Internal Friction

• the presence of backfill over the arch has a crucial influence on the ultimate load.

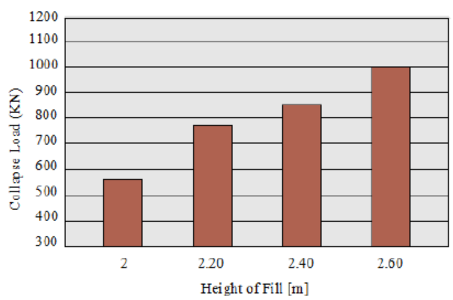

Fig. 10 Influence of the Height of Fill

IV. Conclusions

The methods for assessing historical masonry arches are mainly three: i) the Thrust Line Analysis Method; ii) the Mechanism Method; iii) the Finite Element Methods. The Thrust Line Analysis Method and the Mechanism Method are analytical methods and derive from two of the fundamental theorems of the Plastic Analysis, while the Finite Element Method is a numerical method, that uses different strategies of discretization to analyze the structure. The first two usually concern bi-dimensional arch bridges, while the third one refers to three dimensional model.

A comparison between the first two methods has been made. The two models lead almost to the same collapse pattern even if the Limit Analysis Method is the most suitable to be applied to the arch masonry structures.

In the future, the next analysis step will be the comparison of the results obtained by all the three methods applied to a real case study.

Acknowledgment

This research has been supported by the University of Bologna, Italy.

References

[1] L. Nobile, V. Bartolomeo, "Methods for the assessment of historical masonry arches", *in Recent Advances in Civil Engineering and Mechanics, Proc. of the 5th European Conference of Civil Engineering (ECCIE '14)*,Florence, Italy November 22-24, 2014, pp.160-167.

[2] E. Benvenuto, *La scienza delle costruzioni e il suo sviluppo storico*, Sansoni, Firenze,1981.

[3] P. De La Hire, *Traitè de Mècanique*, Acts of Acadèmie des Sciences, Paris,1730.

[4] A.J.S. Pippard, R.J. Ashby, "An experimental study of the voussoir arch", *Journal of the Institution of Civil Engineering*,10,1939, pp. 383–404

[5] A.J.S. Pippard , *The approximate estimation of safe loads on masonry bridges*, Civil Engineer in War: Institution of Civil Engineers 1948;1:365.

[6] A. Kooharian, *Limit analysis of voussoir (segmental) and concrete arches*, Proc. Am. Concr. Inst., Vol. 49,1953,pp.317-328.

[7] E.T. Onat W. Prager, *Limit Analysis of Arches*, Journal of Mechanics and Physics of Solids, Vol. 1,1953,pp.77-89.

[8] J. Heyman, *The stone skeleton. Structural Engineering of Masonry Architecture*, University of Cambridge Press,Cambridge, 1966.

[9] A.H. Hendry, S.R. Davies, R. Royles, *Test on a Stone, Masonry Arch at Bridgemill-Girvan*, Transport and Road Research Lab, Contractor Report 7, UK,1985.

[10] M.R. Horne, *Plastic theory of structures*, 2nd edition, Pergamon Press, Oxford,1979.

[11] G. Milani, *A simple equilibrated homogenization model for the limit analysis of masonry structures*. Wseas Transactions on Applied and Theoretical Mechanics, Issue 5, Vol. 2, 2007, pp. 119-125.

[12] A. Audeneart, W. Dullaert, G. Reniers , H. Peremans, *Evaluation of the limit load capacity of masonry arch bridges*, Wseas Transactions on Applied and Theoretical Mechanics, Issue 4, Vol. 4, 2009, pp. 137-146.

[13] A. Audeneart A., J. Beke, *Applicability analysis of 2D-models for masonry arch bridge assessment: Ring, Archie-M and the elasto-plastic model*, Wseas Transactions on Applied and Theoretical Mechanics, Issue 4, Vol. 5,2010, pp. 221-230.

[14] R. Hooke, *A description of Helioscopes, and some others instruments*, John & Martin Printer to the Royal Society, London, 1676.

[15] D. Gregory D. (1698), *Catenaria*, Philosophical Transactions of the Royal Society, vol. 19,1698, pp. 637-652.

[16] T. Young, Article in the Supplement to the fourth edition of the Encyclopaedia Britannica (1817).

[17] Navier 1826. Résumé des Leçons donées à L'École des Ponts et Chaussées, sur l'application de la mécanique à l'établissement des constructions et des machines . Paris.

[18] W.J.M. Rankine, W. J. M. 1858. A Manual of Applied Mechanics. London: Charles Griffin.

[19] P. Block, T. Ciblac, J. Ochsendorf, *Real-time limit analysis of vaulted masonry buildings*, Computers & Structures, 84 (29-30), 2006,pp. 1841-1852.

[20] R.K. Livesley, *Limit analysis of structures formed from rigid blocks*, International Journal for Numerical Method in Engineering, 12,1978, pp. 1853-1871.

[21] Obvis Ltd. Home page, http://www.obvis.com

[22] M. Gilbert, "On the analysis of multi-ring brickwork arch bridges". Proceedings of 2nd International Arch Bridges Conference, Venice, 1998, pp. 109-118.

[23] M. Gilbert , "Limit Analysis Applied to Masonry Arch Bridges: State of the art and Recent Developments". Proc. of ARCH'07 - 5th International Conference on Arch Bridges, Madeira, 2007, pp.13-28.

[24] RING home page, http://www.shef.ac.uk/ring

[25] L. Nobile, V. Bartolomeo, M. Bonagura, *Structural Analysis of Historic Masonry Arch Bridges: Case Study of Clemente Bridge on Savio River*, Key Engineering Materials, 488-489, 2012, pp 674-677.