# Some new findings on the mathematical structure of the cell method

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***Abstract*—**In the classification diagram of the Cell Method (CM), which is the truly algebraic numerical method, the global variables are stored in two columns: the column of the configuration variables, with their topological equations, and the column of the source variables, with their topological equations. The structure of the classification diagram is the same for both the global and the field variables of every physical theory of the macrocosm. The importance of this diagram stands just in its ability of providing a concise description of physical variables, without distinguishing between the physical theories. Recently, we have shown that we can provide the classification diagram of the CM with a mathematical meaning, in addition to a physical meaning. Actually, we can recognize in the classification diagram of the CM a structure of bialgebra. In this paper, we give a further insight into the mathematical foundations of the CM by comparing the structure of the algebraic formulation with the structure of the differential formulation. Particular attention is devoted to the computation of limits, by highlighting how the numerical techniques used for performing limits may imply a loss of information on the length scales associated with the solution. Since the algebraic formulation does not make use of the limit process, this means that the algebraic formulation preserves the information on the length scales associated with the solution. Conversely, the differential formulation is forced to introduce a proper enrichment of the equations and/or the space of reals for taking into account the length scales associated with the solution.

***Keywords*—**Cell Method, Non-locality, Non-Standard Calculus, Numerical Stability.

## I. Introduction

THE association of physical variables to elements of a cell-complex and its dual cell-complex was introduced by [1] and [2]. In particular, Branin [2] treated the duality between two cell-complexes as an extension of the duality that exists in graph theory between a graph and its dual graph. The physical variables were subsequently arranged in a classification diagram, denoted as the classification diagram of the Cell Method (CM) [3], on the basis of physical considerations on the associations between global variables in space and geometry, where a global variable in space is a space variable that is not the line, surface or volume density of another space variable. Accordingly, a global variable in time is defined as a time variable that is not the rate of another time variable.

Specifically, the classification diagram of the CM associates the global variables in space with the geometrical and topological features usually neglected by the differential formulation, by highlighting how any global variable in space is related to one of the four space elements in three-dimensional space (Fig. 1):

- the point ( **P** ),
- the line ( **L** ),
- the surface ( **S** ),
- the volume ( **V** ).

The global variables in space are then stored in two columns:

- The column of the configuration variables includes the variables describing the field configuration: displacements (solid mechanics), velocity (fluid dynamics), electric potential (electrostatics), temperature (thermal conduction).
- The column of the source variables includes the variables describing the field sources: forces (solid mechanics and fluid dynamics), masses (geodesy), electric charges (electrostatics), electric currents (magnetostatics), heat (thermal conduction).

This classification allows us to point out an important feature of the global variables: by associating the global configuration variables with the space elements of a first cell-complex, we say the primal cell-complex, the global source variables turn out to be automatically associated with the elements of the dual cell-complex. A point, a line, a surface and a volume of the primal cell-complex are denoted by putting bars over their symbols, whereas a point, a line, a surface and a volume of the dual cell-complex are denoted by putting tildes over their symbols (Fig. 2).

In [4] and [5], we showed that we may start from mathematical considerations for obtaining the same associations established by the classification diagram of the CM, associations that also extend to the inner and outer orientations of the global variables (Fig. 2). In effect, the

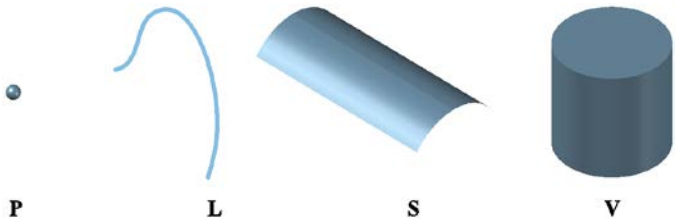


Fig. 1 notations for the four space elements in three-dimensional space

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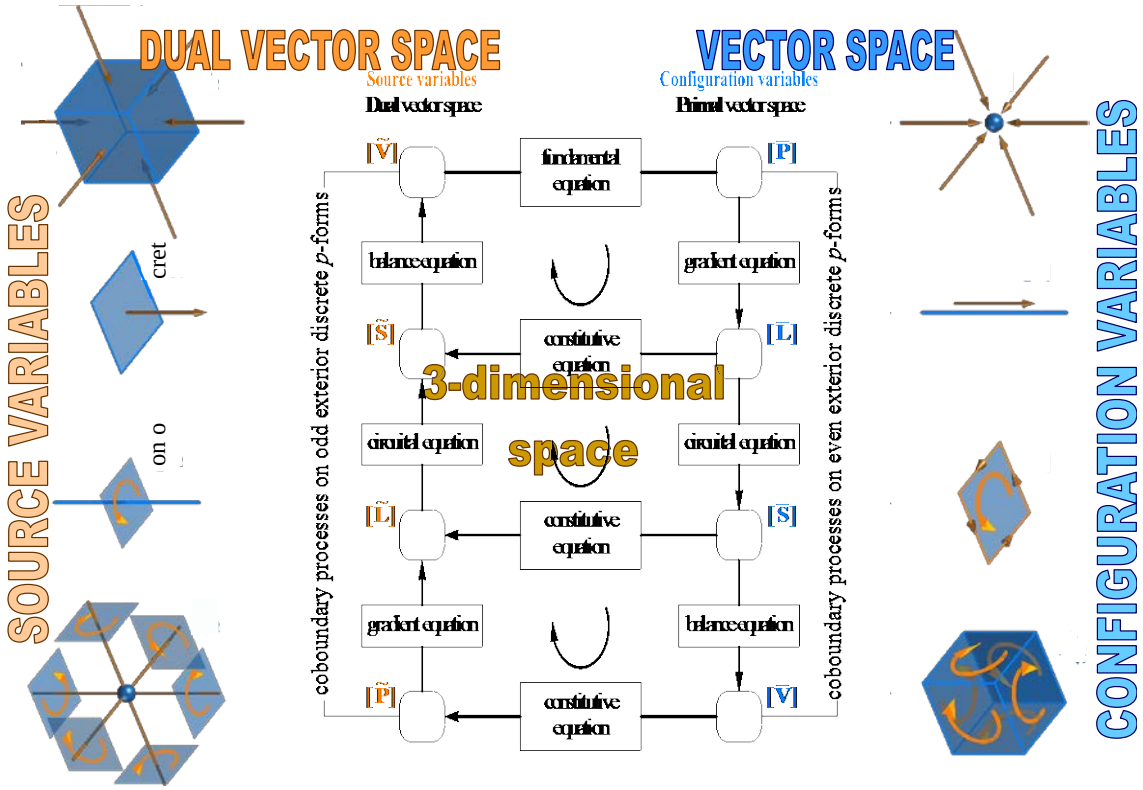


Fig. 2 the columns of the configuration and source global variables in the classification diagram of the Cell Method: analogies with a vector space and its dual vector space in a bialgebra

geometric algebra [6]–[12] provides the wider algebraic setting in which to answer geometric questions. Specifically, as we discussed in [13], we can recognize in the classification diagram of the CM a structure of bialgebra (Fig. 2), where:

- The set of topological equations between global configuration variables defines a geometric algebra on the space of global configuration variables, provided with a geometric product.
- The operators of the topological equations between global configuration variables are generated by the outer product of the geometric algebra, which is equal to the exterior product of the enclosed exterior algebra.
- The dual algebra of the enclosed exterior algebra is the space of global source variables, associated with the dual $p$-cells , and is provided with a dual product that is compatible with the exterior product of the exterior algebra.
- The pairing between the exterior algebra and its dual gives rise to the energetic variables, by the interior product.

The CM could seem very similar to the direct or physical approach, initially used in the Finite Element Method (FEM), and the vertex-based scheme of the Finite Volume Method (FVM) or the Finite Difference Method (FDM). Nevertheless, the similarity is apparent. In fact, in a truly algebraic formulation [14] – as the CM is [15] – all operators must be discrete and the limit process must be avoided at each level of the formulation [16], [17]. The physical approach is not suitable to this aim, since it starts from point-wise and instant-wise conservation equations, the field functions of the differential formulation. If the field functions are not described in terms of point position and instants directly, they are obtained by performing densities and rates of the global variables, thus using the limit process (Fig. 3).

Even FVM [18] and FDM [19], [20] are based on a differential formulation (Fig. 3). In fact, in all the so-called discrete methods the algebraic formulation is actually induced by the differential formulation, by means of an integration process (Fig. 3). This integration process is needed since, whereas the elimination of geometry from the physical laws is essential for the differential formulation, geometry is essential for the numerical solution.

Moreover, the space distribution of the point-wise field functions of the differential formulation requires the introduction of coordinate systems (Fig. 3), whose purpose is that to create a correspondence between points of the space and numbers, their coordinates. This allows us to describe geometry by means of mathematics.

Conversely, the CM uses global variables and balance equations in global form (Fig. 3). As a consequence, the governing equations are expressed in algebraic form directly, thus distinguishing the CM from any other so-called discrete method. Even the cell-complexes assume in the CM a different meaning. In fact, the two cell-complexes of the CM not simply are the result of a domain discretization, needed by the numerical analysis, as in the FEM [21]–[49]. They are the generalization of the coordinate systems to the algebraic formulation. Indeed, they are required by the algebraic formulation since the space global variables are associated not

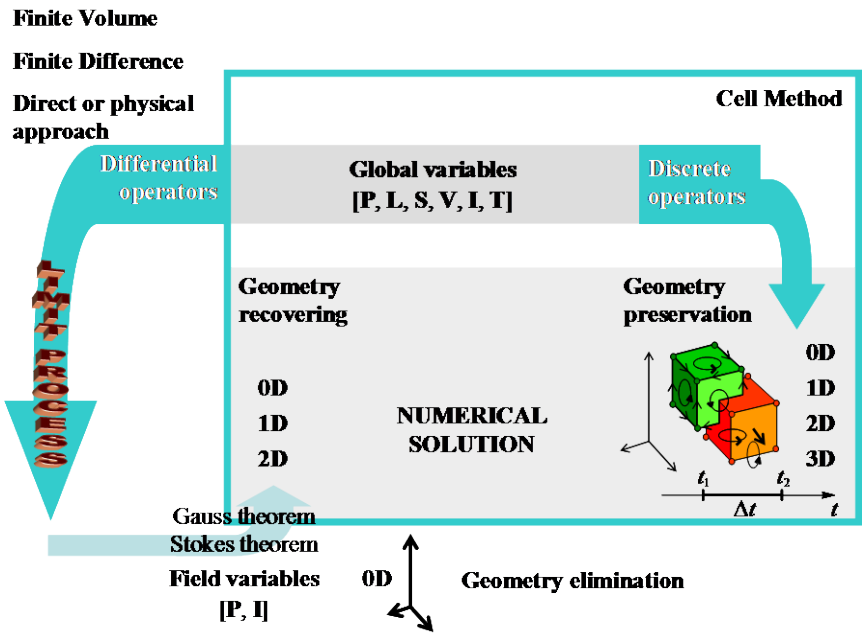


Fig. 3 building an algebraic formulation through the Direct or Physical Approach, the Finite Volume Method, the Finite Difference Method, and the Cell Method

only with points, as in the differential formulation, but also with lines, surfaces, and volumes. By using cell-complexes, we can describe the space global variables directly, by associating them with the related space elements of the cell-complexes. Consequently, in the algebraic formulation of physics the cell-complexes play the same role that coordinate systems play in the differential formulation. Thus, the physical notions are translated into mathematical notions by means of the intermediation of topology and geometry.

Finally, if we consider a time axis (Fig. 3) and perform a first subdivision of a given time interval into many adjacent small time intervals, we have a primal cell-complex in time. For building the dual cell-complex in time, we can consider the middle instant of each primal time interval. The result is that, similarly for the space elements (Fig. 2), also the primal and dual time elements are provided with inner and outer orientation, respectively.

Viewed in these terms, it might seem that the main difference between the two formulations is that the limit process is used in the differential formulation whereas it is not used in the algebraic formulation. In effect, the difference does not lie in performing the limit in itself, but rather in the technique used for finding the limit. The aim of this paper is just to point out how some of the most known mathematical tools – which we are accustomed to use systematically, without pondering on the implications and/or limitations for the result – may actually have a determinant influence on the quality of the numerical solution.

## II. A Discussion on How to Compute Derivatives

As is well known, the first derivative $f'(x)$ of a continuous function $f(x)$ is defined as either of the two limits (if they exist)

$$f'(x) \triangleq \lim_{s \to x} \frac{f(s) - f(x)}{s - x}, \tag{1}$$

and

$$f'(x) \triangleq \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \; h > 0. \tag{2}$$

The ratio in (2), which is denoted as the difference quotient, is not a continuous function at $h = 0$ because it is not defined there. In fact, the limit (2) has the indeterminate form $(\to 0)/(\to 0)$ as $h \to 0$, since both the numerator and the denominator approach 0 as $h \to 0$.

We can compute the limiting value (2) both in an approximated way, as per the $\varepsilon - \delta$ definition of a limit, or in an exact way, by making use of the Cancelation Rule for limits.

The $\varepsilon - \delta$ definition of a limit is the formal mathematical definition of a limit: let $f$ be a real-valued function defined everywhere on an open interval containing the real number $c$ (except possibly at $c$) and let $L$ be a real number, then the statement

$$\lim_{x \to c} f(x) = L \tag{3}$$

means that, for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that, for all real $x$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$. Symbolically:

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \; \forall x \left( 0 < \left| x - c \right| < \delta \Rightarrow \left| f\left( x \right) - L \right| < \varepsilon \right), \tag{4}$$

where:

- $\varepsilon$ is a small positive number. It represents the error in the measurement of the value at the limit.
- $\delta$ is a positive number. It represents the distance to the limit point, $c$.

The absolute value $\left| x - c \right|$ in (4) means that $x$ is taken sufficiently close to $c$ from either side (but different from $c$). The limit value of $f\left( x \right)$ as $x$ approaches $c$ from the left, $x \to c^{-}$, is denoted as left-hand limit, and the limit value of $f\left( x \right)$ as $x$ approaches $c$ from the right, $x \to c^{+}$, is denoted as right-hand limit. Left-handed and right-handed limits are called one-sided limits. A limit exists only if the limit from the left and the limit from the right are equal. Consequently, the limit notion requires a smooth function.

By using the $\varepsilon - \delta$ definition of a limit, we compute the limiting value by reducing the error with subsequent iterations.

In the perspective of a computational analysis, it is obvious that the choice falls on the exact, rather than the approximated, computation of limits. In effect, by using the Cancelation Rule for limits, one can obtain an exact solution of the physical phenomenon under consideration only in few elementary cases, with simple geometric shapes of the domain and under particular boundary conditions.

Anyway, the most important aspect when using the Cancelation Rule for limits is not that the exact numerical solution is hardly ever attained in real cases, but rather, that the choice itself of the term "exact" for the limit promised by the Cancelation Rule for limits is not entirely appropriate [50]. Indeed, in order to provide the solution of the limit directly, the Cancelation Rule for limits reduces the order of zero both in the numerator and the denominator by one. Under the numerical point of view, this reduction is made by canceling a quantity with the order of a length, both in the numerator and in the denominator. Under the topological point of view, we could say that the reduction degrades the solution, in the sense that, being deprived of one length scale, the solution given by the Cancelation Rule for limits provides us with a lower degree of detail in describing the physical phenomenon under consideration.

Specifically, with the Cancelation Rule for limits, we factor $h$ out of the numerator in (2)

$$\left[ f\left( x + h \right) - f\left( x \right) \right]\Big|_{x = \bar{x}} = h \cdot g\left( h \right), \tag{5}$$

and cancel this common factor in numerator and denominator. Then, we find the limit by evaluating the new expression at $h = 0$, that is, by plugging in 0 for $h$, because the new expression is continuous at $h = 0$:

$$\lim_{h \to 0} \frac{h \cdot g\left( h \right)}{h} = g\left( h \right)\Big|_{h = 0}, \tag{6}$$

where the result is a real number.

The equality in (6), established by the Cancelation Rule for limits, is undoubtedly numerically correct, in the sense that the results of the left- and right-hand-side expressions are actually numerically equal, but the way in which these results are achieved is radically different in the two cases. As a matter of fact, the limit on the left side is defined on the open interval of length $h$, whereas the function $g\left( h \right)$ is evaluated for a given value of the variable, $h = 0$. This difference, negligible from the purely numerical viewpoint, is instead essential from the topological viewpoint.

In effect, this difference is so much essential that the opportunity of using an algebraic rather than a differential formulation could be discussed just on the basis of the numerical equality between the left- and right-hand-side terms in (6). Actually, the $\varepsilon - \delta$ definition of a limit implies choosing an (open) interval, containing the point in which we want to estimate a function, with the aim of making the distance between the points in which we compute the function and the point in which we want to estimate the function as small as we want. In other words, the limit on the left side in (6) is strictly bonded to the idea of neighborhood of a point and cannot be separated from it. Moreover, since the amplitude of the neighborhood is fixed by the distance $\delta$ between computation and estimation points, the notion of limit is also bound to the idea of distance or, in other words, to the idea of length scale. Consequently, since the amplitude of the neighborhood is never reduced to zero, the iterative technique preserves information on the length scale, exactly as the algebraic formulation of the CM does.

The result of the limit is the value to which the function output appears to approach as the computation point approaches the estimation point. For evaluating this result, we must enough carefully choose the computation points, in order to derive the trend of the output to a specific degree of approximation. That is, the result we obtain by choosing increasingly close points is only an estimation of the actual result and the approximation of the estimation is as much better (the degree of approximation is as much low) as the computation point is close to the estimation point. In conclusion, the $\varepsilon - \delta$ definition of a limit also bounds the limit to the notions of approximation and degree of approximation, or accuracy.

Completely different is the discussion on the right-hand-side function of (6). Actually, the new function $g\left( h \right)$ is computed at a point, the point $h = 0$, without any need of evaluating its trend on an interval. The consequence is that the result we obtain is exact and we do not need to prefix any desired accuracy for the result itself.

This is very useful from the numerical point of view, but, from the topological point of view, we lose information on

what happens approaching the evaluation point. Specifically, we lose the same type of information that is lost in passing from the description of a phenomenon in a space to the description of the same phenomenon in the tangent space at the evaluation point. This is also known as the principle of the local linearity and leads to the differential formulation.

In order to stress that the solution given by the Cancelation Rule provides us with a lower degree of detail, we will say that its solution is exact in a broad sense, and not in a narrow sense.

The idea underlying this paper is that the Cancelation Rule can actually be employed only in those cases where the specific phenomenon uniquely depends on what happens at the point under consideration. In effect, this happens in few physical problems, whereas, in most cases, the physical phenomenon under consideration also depends on what happens in a neighborhood centered at the point.

By extension of (6) to functions of more than one variable, studying the physical phenomenon as if it were a point-wise function means that we are using the right-hand side of (6), whereas studying the physical phenomenon as a function of all the points contained in a neighborhood means that we are using the left-hand side, with $h$ approaching zero but never equal to zero. In the first case, we are facing a differential formulation, whereas, in the second case, we are facing an algebraic formulation.

Operatively, we are using an algebraic formulation whenever we choose increasingly close points (to both the right and the left) of the estimation point, until the outputs remain constant to one decimal place beyond the desired accuracy for two or three calculations. How much the computation points must be close to the estimation point depends on how fast the result of the limit is approached as we approach the point in which the limit is estimated. Therefore, the dimension of the neighborhood is fixed by the trend of the phenomenon around the point under consideration, or, in other words, the distance $\delta$ for the evaluation of $f'(c)$ depends both on the error $\varepsilon$ and on $f''(c)$. The information we lose by using the Cancelation Rule lies just in the trend of the phenomenon, that is, in the curvature, since the curvature cannot be accounted for in passing from a space to its tangent space at the evaluation point.

In the differential formulation, the notion of limit is used for defining not only derivatives, but also densities. In this second case, the denominator that tends to zero has the dimensions of a length raised to the power of 1, 2, or 3. The Cancelation Rule for limits can be employed also in this second case, by factorizing and canceling length scales in dimension 1, 2, or 3, respectively. This leads to point-wise variables in any cases, the line, surface, and volume densities.

Finally, the Cancelation Rule for limits is used for finding also rates, by factorizing and canceling time scales in dimension 1. This last time, the limit, which is a time derivative, provides an instant-wise variable.

In conclusion, with reference to the space of the physical phenomena, the differential formulation provides the numerical solution in the tangent space of degree 0, where we can describe each physical phenomenon in terms of the space elements of degree 0, the points, and the time elements of degree 0, the time instants. Conversely, the algebraic formulation allows us to take account of, we could say, the curvatures in space and time at a point, where a point of the space of the physical phenomena is a given physical phenomenon, in a given configuration, at a given time instant.

Now, the question is if canceling length and time scales is an acceptable loss. We can find many examples of how this is not an acceptable loss in several numerical and mathematical strategies, which share the common need of recovering length and time scales. Although these techniques are widely used, unfortunately the scientific literature does not stress enough that their common origin is having lost one or more length scales, by performing a limit process. Neither it is stressed that we can avoid to use these techniques if we do not previously apply the Cancelation Rule for limits.

By way of example, in the following two Sections we will discuss how the non-local approaches used for modeling heterogeneous material and the non-standard analysis are, in reality, two different attempts of recovering the length scales in engineering and mathematics, respectively.

## III. A Numerical Attempt of Recovering the Length Scales: Non-Local Formulation

It is now a common opinion that the classical local continuum concept, with the stress at a given point uniquely depending on the history of deformation and temperature at that point only, is not adequate for describing damagement of heterogeneous materials with the differential formulation, particularly when the size-effect is involved. Actually, modeling the size-effect is impossible in the context of classical plasticity, both in problems with strain-softening and in those with no strain-softening at all.

The first critiques of the local approach date back to the ‘60s [51]–[55] and are based on the micro-structure of matter. In effect, all materials are characterized by micro-structural details, the size of which ranges over several order of magnitude. Therefore they cannot be broken down into a set of infinitesimal volumes, each of which could be described independently. Consequently, the idea was promulgated that heterogeneous materials should properly be modeled by some kind of non-local continuum, in which the stress at a certain point is a function of the strain distribution over a certain representative volume centered at that point.

According to the previous discussion on when it is appropriate to use the Cancelation Rule for limits (Section II), we are thus facing with a case where we should abandon the differential formulation, which is based on the Cancelation Rule for limits, and reformulate the numerical approach from its foundations, by using the algebraic formulation of the CM. Unfortunately, the consolidated custom of using the Cancelation Rule for limits without taking into account the

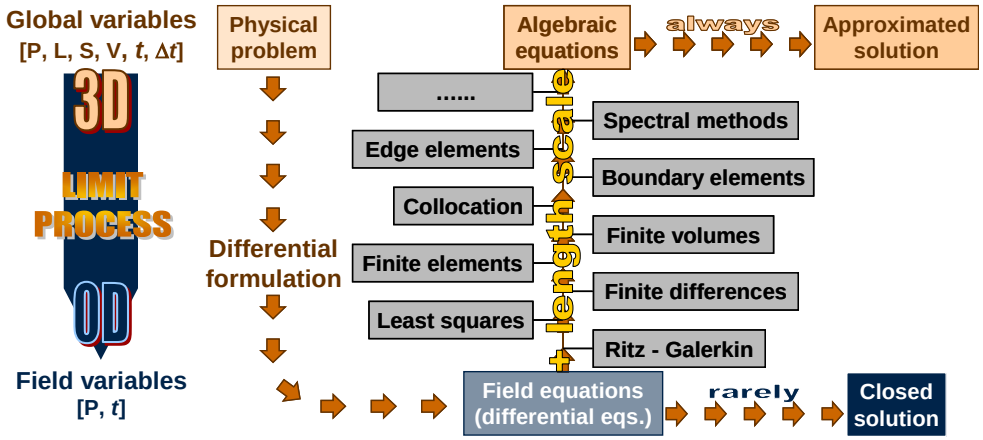


Fig. 4 losing and reintroducing metrics in nonlocal models

topological implications of this choice did not leave room for the idea of abandoning the differential formulation. Consequently, the differential formulation was used in any case and the problem was addressed by improving the classical continuum description with an internal length parameter, introduced into the constitutive laws. This idea led to models denoted as the non-local models.

The point we would like to stress in this paper is that the terms "improved continuum" and "enriched continuum," used by the non-local formulation, are not completely appropriate. It would be more correct speaking of recovering of the length scale in a continuum that has been deprived of its metrics by the systematic use of the Cancelation Rule for limits. Obviously, since the recovering is performed empirically, and not after having identified and solved the cause for the loss of the length scale, the recovering can only be partial.

We may further prove the incapability of the differential formulation to model non-locality on the basis of the definition itself of non-locality. Indeed, according to the mathematical definition of non-locality given in [56], the operator $A$ in the abstract form of the fundamental equations of any physical theory,

$$Au = f \ , \tag{7}$$

is called local when, if

$$u(\boldsymbol{x}) = v(\boldsymbol{x}) \tag{8}$$

for all $\boldsymbol{x}$ in a neighborhood of point $\boldsymbol{x}_0$, then

$$Au(\boldsymbol{x}) = Av(\boldsymbol{x}). \tag{9}$$

Well, the differential operators satisfy the conditions (8) and (9), because the derivatives of any arbitrary order do not change if the differentiated function changes only outside a small neighborhood of the point at which the derivatives are taken [57]. Consequently, the differential operators are local [58] and any formulation using differential operators is intrinsically local (left side of Fig. 3). This is ultimately a consequence of having built the differential operators by using the Cancelation Rule for limits.

In conclusion, even in those few cases where the differential formulation provides us with a solution in a closed form (Fig. 4), by reducing the global variables to point and instant variables we lose the possibility of describing more than 0-dimensional effects, that is, the non-local effects. Consequently, metrics must be reintroduced a-posteriori, in most cases in the discretization process (Fig. 4), if we want to model non-locality.

One may ask, now, where the length scale must be reintroduced. In differential non-local approaches, a length scale is incorporated into the constitutive laws, but there is no evidence that this is the only possible choice, or the physically most appealing one.

Due to the multi-dimensional geometrical content of global variables, emphasized by the classification diagram of the CM (Fig. 2), we can provide an alternative interpretation of non-locality. In fact, dimensional scales and non-local effects are associated with global variables directly and non-locality is a property of the global variables, not a prerogative of just the constitutive laws. Consequently, preserving non-locality into governing equations by means of the algebraic formulation of the CM is physically more correct than recovering non-locality into constitutive equations.

In effect, the CM is a non-local numerical method due to three reasons:

- It replaces the field variables of the differential formulation with the global variables, which are associated with geometrical objects provided with an extent (we could say, with internal length scales in dimension 1, 2, and 3).
- It uses two staggered meshes, also in plane domains (Fig. 5). Since the configuration variables are associated with the space elements of the primal mesh and the source variables are associated with the space elements of the dual (staggered) mesh, the algebraic constitutive relations are not established in the point, but within a volume surrounding the point. The non-degenerate dimension of this volume is also the reason why the CM does not present problems of

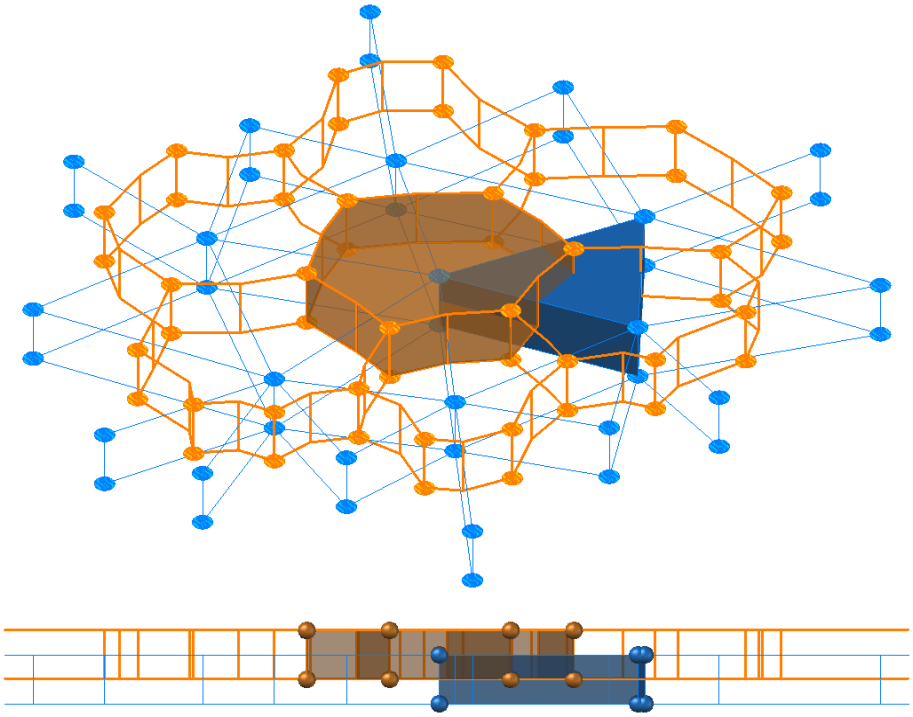

Fig. 5 building staggered cell-complexes in two-dimensional domains with barycentric dual cells

localization with zero dissipated energy.

- It obtains the algebraic topological equations (that is, both balance and kinematic equations) by means of coboundary processes [59]. The typical two-step procedure of the coboundary process establishes a relationship between the $p$-cells and their cofaces, which also takes into account the extent of the $p$-cells. Therefore, the equations are not established in a point but within a volume, once again.

This three-fold motivation provides a non-local nature both to the algebraic variables and the governing equations of the CM.

## IV. A Mathematical Attempt of Recovering the Length Scales: Non-Standard Analysis

Concerns about the soundness of arguments involving infinitesimals date back to ancient Greek mathematics. Actually, the notion of infinitely small quantities was discussed by the Eleatic School. Specifically, Democritus (c. 460–c. 370 BC) is the first person recorded to consider seriously the division of objects into an infinite number of cross- sections, but his inability to rationalize discrete cross-sections with a cone's smooth slope prevented him from accepting the idea. Antiphon the Sophist (probably, the last two decades of the 5th century BC) and Eudoxus of Cnidus (408 BC–355 BC) are generally credited with implementing the method of exhaustion, which is seen as a precursor to the methods of calculus. Archimedes (c. 287-c. 212 BC) was the first to propose a logically rigorous definition of infinitesimals. He replaced the proofs on the soundness of infinitesimals with ones using other techniques, such as the method of exhaustion.

An important early intermediate step toward calculus was Cavalieri's principle, named after the Italian mathematician Bonaventura Francesco Cavalieri (in Latin, Cavalerius) (1598–November 30, 1647), also termed the "method of indivisibles," which was a bridge between the method of exhaustion and full-fledged integral calculus.

More recently, the discussion on the soundness of infinitesimals permeate the history of calculus, which is fraught with philosophical debates about the meaning and logical validity of fluxions, Newton's term for differential calculus [60] (fluents was his term for integral calculus). The standard way to resolve these debates is to define the operations of calculus using limits, rather than infinitesimals. Non-standard analysis, instead, reformulates the calculus using a logically rigorous notion of infinitesimal number.

Non-standard analysis is a rigorous formalization of calculations with infinitesimals. It was introduced in the early 1960s by the mathematician Abraham Robinson (born Robinsohn; October 6, 1918–April 11, 1974). By using the language of non-standard analysis, the infinite and infinitesimal quantities can be treated by the system of hyperreal numbers, or hyperreals, or non-standard reals.

Denoted by $^*\mathbb{R}$, the hyperreal numbers are an extension of the real numbers, $\mathbb{R}$, that contains numbers greater than anything of the form:

$$1+1+...+1. \tag{10}$$

Such a number is infinite, and its reciprocal is infinitesimal.

The hyperreal numbers satisfy the transfer principle, a rigorous version of Leibniz's heuristic Law of Continuity. The transfer principle states that true first order statements about $\mathbb{R}$ are also valid in $^*\mathbb{R}$. Therefore, the hyperreals were

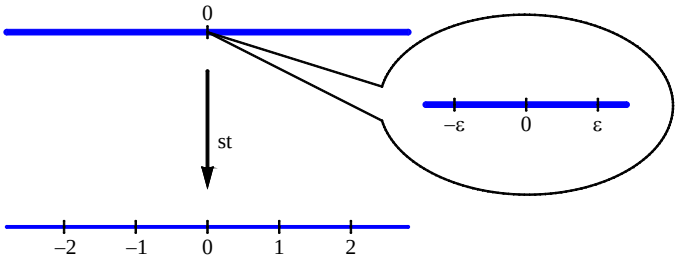


Fig. 6 the bottom line represents the "thin" real continuum. The line at top represents the "thick" hyperreal continuum. The "infinitesimal microscope" is used to view an infinitesimal neighborhood of 0

logically consistent if and only if the reals were. This put to rest the fear that any proof involving infinitesimals might be unsound, provided that they were manipulated according to the logical rules which Robinson delineated.

Non-standard analysis deals primarily with the hyperreal line, which is an extension of the real line, containing infinitesimals in addition to the reals (Fig. 6). In the hyperreal line every real number has a collection of numbers (called a monad, or a halo) of hyperreals infinitely close to it.

The standard part function is a function from the limited (finite) hyperreal to the reals. It associates with a finite hyperreal $x$, the unique standard real number $x_0$ which is infinitely close to it (Fig. 6):

$$\mathrm{st}(x) = x_0 . \tag{11}$$

As such, the standard part function is a mathematical implementation of the historical concept of adequality introduced by Pierre de Fermat. It can also be thought of as a mathematical implementation of Leibniz's Transcendental Law of Homogeneity.

The standard part of any infinitesimal is 0. Thus, if $N$ is an infinite hypernatural, then $1/N$ is infinitesimal and

$$\mathrm{st}\left(\frac{1}{N}\right) = 0 . \tag{12}$$

The standard part function was first defined by Abraham Robinson as a key ingredient in defining the concepts of calculus, such as the derivative and the integral, in non-standard analysis. Specifically, it allows the definition of derivative and integral in a direct fashion.

The derivative of $f$ at a standard real number $x$ becomes

$$f'(x) = \mathrm{st}\left(\frac{{}^*f(x+\Delta x) - {}^*f(x)}{\Delta x}\right), \tag{13}$$

where $\Delta x$ is an infinitesimal, smaller than any standard positive real, yet greater than zero, and ${}^*f$ is the natural extension of $f$ to the hyperreals ($^*$ is the transfer operator applied to $f$). Similarly, the integral is defined as the standard part of a suitable infinite sum.

In this approach, $f'(x)$ is the real number infinitely close to the hyperreal argument of st. For example, the non-standard computation of the derivative of the function $f(x) = x^2$ provides

$$f'(x) = \mathrm{st}\left(\frac{(x+\Delta x)^2 - x^2}{\Delta x}\right) = \mathrm{st}(2x + \Delta x) = 2x , \tag{14}$$

since

$$2x + \Delta x \approx 2x , \tag{15}$$

where the symbol "$\approx$" is used for indicating the relation "is infinitely close to." In order to make $f'(x)$ a real-valued function, we must dispense with the final term, $\Delta x$, which is the error term. In the standard approach using only real numbers, that is done by taking the limit as $\Delta x$ tends to zero. In the non-standard approach using hyperreal numbers, the quantity $\Delta x$ is taken to be an infinitesimal, a non-zero number that is closer to 0 than to any non-zero real, which is discarded by the standard part function.

The notion of limit can easily be recaptured in terms of the standard part function, st, namely

$$\lim_{x \to c} f(x) = L , \tag{16}$$

if and only if, whenever the difference $|x - c|$ is infinitesimal, the difference $|f(x) - L|$ is infinitesimal, as well. In formulas:

$$\mathrm{st}(x) = c \Rightarrow \mathrm{st}(f(x)) = L . \tag{17}$$

The standard part of $x$ is sometimes referred to as its shadow. Therefore, the derivative of $f(x)$ is the shadow of the hyperreal difference quotient.

We can thus conclude that the standard part function is a form of projection from hyperreals to reals. Since also the Cancelation Rule for limits is a form of projection, that is, the projection of the actual solution from the multi-dimensional space to the tangent space of degree 0 (Section II), we can state that the solution of the differential formulation is the shadow of the actual solution in the tangent space of degree 0. On the contrary, by avoiding the projection process, the algebraic formulation provides us with a higher degree solution, approximated in any case, which is more adherent to the physical nature of the phenomenon under consideration.

As a consequence, using the algebraic formulation is somehow similar to performing non-standard calculus, the modern application of infinitesimals, in the sense of non-standard analysis, to differential and integral calculus. In

effect, the extension of the real numbers, $\mathbb{R}$, is equivalent to providing the space of reals with a supplementary structure of infinitesimal lengths. We could say that this is an attempt to recover the loss of length scales due to the use of the Cancelation Rule for limits, in differential formulation. This configures the hyperreal number system as an infinitesimal-enriched continuum, and the algebraic approach can be viewed as the algebraic version of non-standard calculus.

The great advantage of the infinitesimal-enrichment is that of successfully incorporating a large part of the technical difficulties at the foundational level of non-standard calculus. Similarly, in the algebraic formulation many numerical problems, mainly instability or convergence problems, are avoided by the presence of a supplementary structure of (finite) lengths both in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$ [16], [59], [61]–[66].

## V. Some Implications on Numerical Stability

In differential formulation, the fundamental equations of any physical problem are expressed by partial differential equations (PDEs, [67]–[73]) of second order. Depending on the physical theory involved, the particular path followed for putting in relationship the configuration with the source variables may result in either elliptic [74] (Fig. 7), or parabolic (Fig. 8), and hyperbolic equations (Fig. 9). The reason for the terms "elliptic," "parabolic," and "hyperbolic" is the general

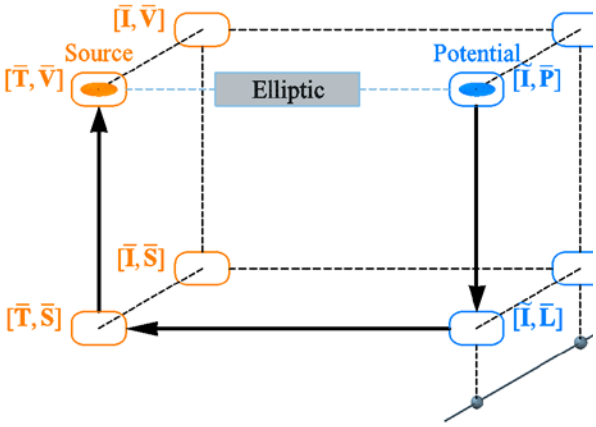


Fig. 7 elliptic equations in the classification diagram

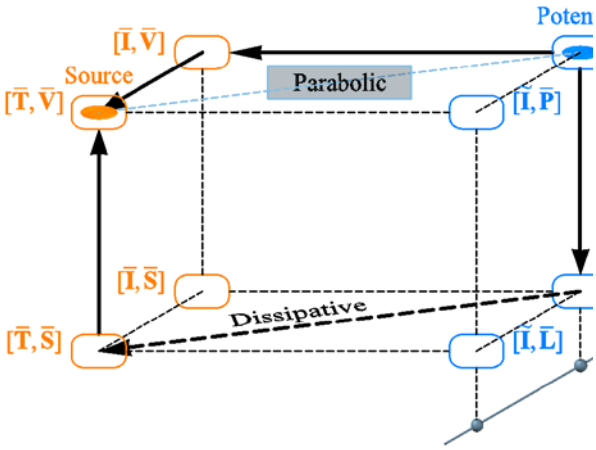


Fig. 8 parabolic equations in the classification diagram

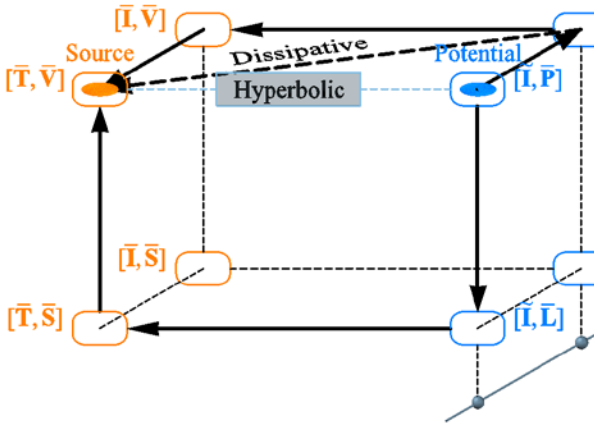


Fig. 9 hyperbolic equations in the classification diagram

form assumed by the second order PDE in two independent variables.

As we have previously discussed, in several real-world problems, it is not possible to derive closed form solutions of the fundamental equations, for the multitude of irregular geometries, various constitutive relations of media, and boundary conditions. Computational numerical techniques can overcome this inability, providing us with important tools for design and modeling. To achieve this, time and space are divided into a discrete grid and the continuous differential equations are discretized. In general, the simulated system behaves differently than the intended physical system. The amount and character of the difference depends on the system being simulated and the type of discretization that is used.

Choosing the right numerical technique for solving a problem is important, as choosing the wrong one can either result in incorrect results, or results which take excessively long time to compute. In particular, the equation which approximates the equation to be studied is probable to become unstable, meaning that errors in the input data and intermediate calculations can be magnified in the limit, instead of damped, causing the error to grow exponentially.

Form the numerical point of view, an unstable solution occurs in differential formulation whenever the algebraic system of discretized equations derived from an elliptic equation ceases to be elliptic. The same occurs when the algebraic systems of a parabolic or hyperbolic equation are not parabolic or hyperbolic, respectively. The causes for this are several. In some cases, they consist in the integration method adopted. In particular, it is important to use a stable method whenever we want to solve a stiff equation, that is, a differential equation for which certain numerical methods for solving the equation are numerically unstable, unless the step size is taken to be extremely small. A problem is stiff when the step size is forced down to an unacceptably small level in a region where the solution curve is very smooth, whereas one would expect the requisite step size to be relatively small in a region where the solution curve displays much variation and to be relatively large where the solution curve straightens out to approach a line with slope nearly zero. A method that is stable on stiff problems is called an A-stable method [75].

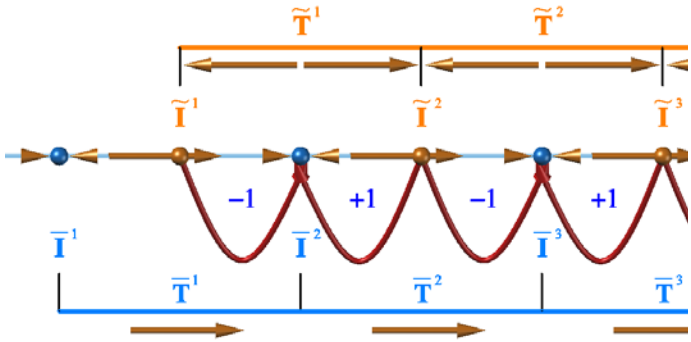


Fig. 10 Orientations of the primal and dual time elements in the CM for the case of four primal time instants

Leapfrog integration is a second-order method, which means that the error is roughly proportional to the square of the step size. Unlike Euler integration, it is stable for oscillatory motion, as long as the time-step $\Delta t$ is constant, and

$$\Delta t \leq \frac{2}{\omega}, \tag{18}$$

where $\omega$ is the angular frequency (measured in radians per second).

Leapfrog integration is equivalent to updating positions $x(t)$ and velocities $v(t) = \dot{x}(t)$ at interleaved time points, staggered in such a way that they "leapfrog" over each other. For example, the position is updated at integer time steps and the velocity is updated at integer-plus-a-half time steps:

$$x_i = x_{i-1} + v_{i-1/2}\Delta t, \tag{19}$$

$$v_{i+1/2} = v_{i-1/2} + a_i \Delta t, \tag{20}$$

where $x_i$ is the position at step $i$, $v_{i+1/2}$ is the velocity, or first derivative of $x$, at step $i+1/2$, $a_i = F(x_i)$ is the acceleration, or second derivative of $x$, at step $i$, and $\Delta t$ is the size of each time step.

This means that the leapfrog integration uses the same explicit time-marching scheme of the CM, where the primal time instants, $\overline{\mathbf{I}}$, and the primal time intervals, $\overline{\mathbf{T}}$, are represented by the nodes and lines of a one-dimensional primal cell-complex, respectively (Fig. 10), and primal and dual time instants (or time steps) are staggered for a-half time step. Moreover, the position $\overline{x}_1$ is associated with the first primal time instant, $\overline{\mathbf{I}}^1$ (first primal step), the velocity $\overline{v}_{x1}$ is associated with the first primal time interval, $\overline{\mathbf{T}}^1$, the velocity $\tilde{v}_{x1}$ is associated with the first dual time instant, $\tilde{\mathbf{I}}^1$ (first dual step), the acceleration $\tilde{a}_{x1}$ is associated with the first dual time interval, $\tilde{\mathbf{T}}^1$, and the acceleration $\overline{a}_{x2}$ is associated with the second primal time instant, $\overline{\mathbf{I}}^2$ (second primal step):

$$\overline{x}_2 = \overline{x}_1 + \overline{v}_{x1}\Delta t = \overline{x}_1 + \tilde{v}_{x1}\Delta t, \tag{21}$$

$$\tilde{v}_{x2} = \tilde{v}_{x1} + \tilde{a}_{x1}\Delta t = \tilde{v}_{x1} + \overline{a}_{x2}\Delta t. \tag{22}$$

Consequently, the time-marching scheme of the CM can be viewed as the algebraic version of the leapfrog integration in the differential formulation.

Leapfrog integration is used in the Finite-difference time-domain method (FDTD), which is a numerical analysis technique for modeling computational electromagnetics (CEM). CEM typically solves the problem of computing the $E$ (electric), and $H$ (magnetic) fields across the problem domain. The equations are solved in a cyclic manner: the electric field vector components in a volume of space are solved at a given instant in time, then the magnetic field vector components in the same spatial volume are solved at the next instant in time, and the process is repeated over and over again until the desired transient or steady-state electromagnetic field behavior is fully evolved. Since the change in the $E$-field in time (the time derivative) is dependent on the change in the $H$-field across space (the curl), at any point in space, the updated value of the $E$-field in time is dependent on the stored value of the $E$-field and the numerical curl of the local distribution of the $H$-field in space. Analogously, at any point in space, the updated value of the $H$-field in time is dependent on the stored value of the $H$-field and the numerical curl of the local distribution of the $E$-field in space. Iterating the $E$-field and $H$-field updates the results in a marching-in-time process wherein sampled-data analogs of the continuous electromagnetic waves under consideration propagate in a numerical grid stored in the computer memory.

This description holds true for 1-D, 2-D, and 3-D FDTD techniques. When multiple dimensions are considered, calculating the numerical curl can become complicated. Kane Yee's seminal paper [76] proposed spatially staggering the vector components of the $E$-field and $H$-field about rectangular unit cells of a Cartesian computational grid so that each $E$-field vector component is located midway between a pair of $H$-field vector components, and conversely. This scheme, now known as a Yee lattice, has proven to be very robust and remains at the core of many current FDTD software

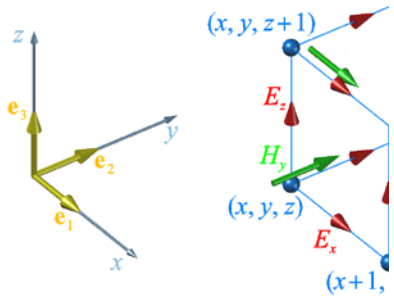


Fig. 11 illustration of a standard Cartesian Yee cell used for FDTD, about which electric and magnetic field vector components are distributed

constructs. Furthermore, Yee proposed a leapfrog scheme for marching in time wherein the $E$-field and $H$-field updates are staggered so that $E$-field updates are conducted midway during each time-step between successive $H$-field updates, and conversely. On the plus side, this explicit time-stepping scheme [77] avoids the need to solve simultaneous equations and furthermore yields dissipation-free numerical wave propagation. On the minus side, this scheme mandates an upper bound on the time-step to ensure numerical stability. This allows us to avoid spurious solutions, that is, to avoid a numerical drawback.

If visualized as a cubic voxel, the electric field components form the edges of the cube, and the staggered magnetic field components form the normals to the faces of the cube (Fig. 11). A three-dimensional space lattice consists of a multiplicity of such Yee cells, leading to a scheme analogous to the CM scheme with primal and dual cells (Fig. 3). Therefore, the Yee lattice can be considered the particularization of the primal and dual cell complexes of the CM, when a differential formulation is derived from the algebraic formulation. Moreover, we may generalize the notion of inherited association to the stability of the numerical solution, by assuming that the numerical stability is inherited by the Yee lattice from the CM cell complexes, when the field variables are derived from the global variables by performing densities and rates.

As far as continuum mechanics is concerned, a typical case where the numerical solution provided by the differential formulation becomes unstable because the governing differential equations may lose ellipticity is the boundary value problem with strain-softening constitutive model. As we have discussed in Section III, the non-local approach is an attempt to avoid the ill-posedness of these problems without abandoning the differential formulation. Nevertheless, a non-local differential formulation applicable to any inelastic constitutive model with strain-softening is not available, at present, and the non-local parameters need to be calibrated on the single physical phenomenon.

Conversely, the algebraic formulation of the CM gives a unified approach and does not need any parameter for providing a non-local description of physics. In fact, the intrinsic non-locality of the CM – due to the three-fold motivation discussed in Section III – allows us to employ any local law for describing the material behavior. In doing so, we can take into account both the local and the non-local effects and, according to [57] and [78], this is sufficient for avoiding numerical instabilities in strain-softening modeling. Therefore, the CM used together with a local material law provides a non-local description without abandoning the principle of the local action altogether. In particular, see [79]–[85] for a list of papers where a new local monotone non-decreasing material law, the effective law, was identified and successfully employed for modeling size-effect in so-called strain-softening materials.

The effective law is confirmed by experimental [80] and analytical considerations [79]. Its main contribution to the understanding of the failure mechanics of quasi-brittle materials, in general, and concrete, in particular, is having reopened the question of strain-softening [16], whose existence and mathematical well-posedness seemed to be no longer under discussion after the outcomes of the displacement controlled compression tests [86] and the numerical successes of non-local differential approaches. The effective law is based on the idea that strain-softening is not a real material property, such as argued in several theoretical papers of last century, particularly of the 1980s [87]–[93]. The identification procedure of the effective law does not consist of a mere scale factor: the material is separated from the structure scale and the constitutive behavior is no more the mirror image of a structural problem at a lower scale. This results in a size-effect insensitive effective law between effective strain and effective stress.

## VI. Conclusion

In this paper we focused on how to perform the limit process, not for introducing some new technique but rather for investigating the topological meaning of those techniques that we use more frequently. In effect, the extensive use of the most known techniques accustomed us to apply them without taking time to understand whether the choice itself of the technique could affect the quality of the result.

We found that the two most used techniques for finding the limiting value of the first derivative $f'(x)$ of a continuous function $f(x)$ are indeed very different from the topological point of view: the $\varepsilon-\delta$ definition of a limit preserves information on the curvatures in space and time at a given configuration, in a given time instant, whereas the Cancelation Rule for limits does not.

We also found that the $\varepsilon-\delta$ definition of a limit gives rise to the algebraic formulation, whereas the Cancelation Rule for limits gives rise to the differential formulation. Consequently, the topological differences between the two numerical techniques are inherited by the two numerical formulations. This is the reason why the algebraic formulation is provided with a structure of (finite) lengths both in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$, which allow us to account for the trend of the phenomenon around the point under consideration. Conversely, the

differential formulation provides the numerical solution in the tangent space of degree 0, where we can describe each physical phenomenon in terms of the space elements of degree 0, the points, and the time elements of degree 0, the time instants.

Many are the signals showing that a description in the tangent space of degree 0 is inadequate in most cases. In effect, most limitations of the differential formulation are originated by its inability of taking into account the curvatures in space and time, though this is not properly stressed in scientific literature. In Sections III and IV, we showed how non-local approaches and non-standard analysis – two strategies in very far fields, one engineeristic and the other purely mathematical – are actually two of these signals, since they are two different attempts of recovering the length scales that the Cancelation Rule for limits has discarded from the numerical solution. Consequently, we can avoid to use both these strategies and other similar strategies if we do not apply the Cancelation Rule for limits, that is, if we reformulate the numerical problem in algebraic manner, by means of the CM.

Finally, we have discussed how the supplementary structure of lengths of the algebraic formulation eliminates many numerical problems, mainly instability and convergence problems. Specifically, the time-marching scheme of the CM was found to be the algebraic version of the leapfrog integration, which is a second-order method for solving dynamical systems of classical mechanics. This similarity establishes a strict relationship between the CM and the Finite-difference time-domain method (FDTD), a differential time-domain numerical modeling method that uses the leapfrog integration, together with grid staggering (Yee lattice), for applications in computational electromagnetics. Therefore, the CM can be considered a generalization of the FDTD to space/time-domain numerical modeling.

The most important consequence of the similarity between the leapfrog integration and the time-marching scheme of the CM is that also the CM is stable for oscillatory motion. The similarity then extends to the convergence order, since even the CM, in its original formulation with barycentric or circumcentric dual polygons in space, is a second-order method, both in space and in time. Nevertheless, by modifying the shape of the dual polygons in space, it is possible to achieve higher convergence orders for the CM. In particular, we attain a fourth-order convergence in space by choosing Gauss points, besides the primal barycenters and the midpoints of the dual sides, for building the dual polygons around the primal nodes.

Attaining a fourth-order convergence with the CM is all the more relevant as it was not possible to attain convergence greater than second-order for any of the methods which are similar to the CM, such as the direct or physical approach of the FEM, the vertex-based scheme of the FVM, and the FDM.

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