CBSMAP-model. Optimization and control

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***Abstract*—** The present paper is devoted to the research of new controlled queuing model at control of Controlled Batch Semi-Markov Arrival Process (CBSMAP). Note that it is very reasonable to change the characteristics of arrival flows in various queuing models for optimization of its functioning. The control is based on the theory of controlled semi-markov processes for system optimization. The control is carried out using a type of the next batch, the moments of batch arrivals and the quantity of queries in the batch. CBSMAP-flow (new type flow) is constructed and research queuing model using the controlled flow is investigated. Several control measures for the model are used. Two theorems about the income functional for the model are formulated.

***Keywords*—** System control, queueing models, optimization, semi-markov process, queueing theory.

I. INTRODUCTION

THE functioning of different systems can be described using queuing models, for example [2], [4], [14]. Application of control is used to increase the efficiency of the system functioning. In the present paper the process of system functioning is investigated using control by the arrival flow.

The Controlled Batch Semi-Markov Arrival Process is a generalization of the BMAP-flow [2]. BMAP-flow is good for modeling of data-flows in telecommunication networks.

Define CBSMAP-flow [5]. After holding in the state comes to an end, the Controlled Semi-Markov process jumps to the other state and the batch of queries of CBSMAP-flow will be generated. Note that, CBSMAP-flow is also good for modeling of data-flows.

In the previous papers control for models was carried out using two types of control: choice of a batch type, and choice of a batch type and the moments of batch arrival [11], [12].

In the present paper control of the model is complicated. We define semi-markov kernel, construct control measure set and income functional on the trajectories of controlled semi-markov process. Use the control of three parameters at the same time in arrival flow: type of the next batch, the moments of batch arrivals and the number of queries in the batch. The control is carried out using theory of semi-markov processes that is used for different types of problems [1], [7], [9].

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II. DESCRIPTION OF THE MODEL. ALGORITHM

As in the paper the theory of controlled semi-markov processes is used, see Appendix for the main points of the theory used for the Model.

Consider, that at the moment of CSMP (Controlled Semi-Markov Process) transition in state k in queuing model the final batch of customers (queries) of k-th type arrives, a number of queries in group v_k is defined by a generating function.

$$\Phi_k(z) = \sum_{m=0}^{M_k} z^m p^{(k)}(m), \tag{1}$$

where $p^{(k)}(m)$ - is a probability that the number of k-th type queries in a batch is m, M_k – a maximal number of queries in a batch of k-th type.

Formulate the important assumptions at which the further researches will be carried out.

1. Queries (customers) of the same type arrives in the system (subsystem), each of which functions irrespective of other subsystems. The subsystem which is carrying out service of queries of k-th type we designate as System(k).

Notice, that the process of service in each system is realized irrespective of other systems states, however functioning of the systems is coordinated with the general arrival flow.

2. Between the next moments of the change of the CSMP states, queries do not arrive in system, only the process of service in subsystems is carried out. The process of service in k-th subsystem is characterized by number $v^{(k)}(t)$ of the customers which are being in the subsystem during the moment t.

The probability $p_{ms}^{(k)}(t)$ is defined as follows

$$P\{v^{(k)}(t) = s / v^{(k)}(0) = m\} = \begin{cases} p_{ms}^{(k)}(t) \geq 0, & m \geq s \geq 0; \\ p_{ms}^{(k)}(t) = 0, & m < s. \end{cases} \tag{2}$$

probability of that in k-th subsystem during time t (between the markov moments) *(m-s)* queries are operated, provided that during the initial moment in System there were m queries.

If System (k) is a system without queue, $N_k=0$, then

$$p_{ms}^{(k)}(t) = C_m^s (1 - e^{-\mu_k t})^{(m-s)} e^{-s\mu_k t}, \quad n_k \ge m \ge s \ge 0, \tag{3}$$

where n_k - a number of channels (service buffers), μ_k - a parameter of exponential distribution of the service duration. For other combinations of parameters n_k, m, s the probabilities are equal to zero.

If System (k) is a single-channel system with queue, $n_k=1$, $N_k>0$, then

$$p_{ms}^{(k)}(t) = \frac{(\mu_k t)^{m-s}}{(m-s)!} e^{-\mu_k t}, \quad N_k + 1 \ge m \ge s > 0,$$

$$p_{m0}^{(k)}(t) = 1 - \sum_{l=1}^{m} p_{ml}^{(k)}(t) = 1 - e^{-\mu_k t} \sum_{l=0}^{m-1} \frac{(\mu_k t)^l}{l!}, \tag{4}$$

$$N_k + 1 \ge m \ge s = 0.$$

For other combinations of parameters required probabilities are equal to zero.

In general case, if System *(k)* is a multichannel system with the queue $n_k \ge 1, N_k \ge 0$, then

$$p_{ms}^{(k)}(t) = \begin{cases} C_m^s (1 - e^{-\mu_k t})^{(m-s)} e^{-s\mu_k t}, & n_k \ge m \ge s \ge 0, \\ \frac{(n_k \mu_k t)^{m-s}}{(m-s)!} e^{-n_k \mu_k t}, & m \ge s \ge n_k, \\ \int_0^t C_{n_k}^s (1 - e^{-\mu_k (t-x)})^{(n_k - s)} e^{-s\mu_k x} n_k \mu_k \frac{(n_k \mu_k x)^{m - n_k - 1}}{(m - n_k - 1)!} e^{-n_k \mu_k x} dx, & \\ m > n_k > s \ge 0. & \end{cases} \tag{5}$$

For other combinations of parameters n_k, N_k, m, s required probabilities are equal to zero.

Note, that there are several types of admission discipline.

Three admission disciplines are known:

- partial admission, when only a part of the batch corresponding to the number of free places in the buffer is allowed to join the system;
- complete admission, when the whole batch is allowed to enter the system, if there is at least one free place in the buffer;
- complete rejection, when the whole batch is rejected.

Probabilities $\tilde{p}_m^{(k)}(s)$ - probability to accept m queries from batch of k-th type at presence of s empty seats (for Subsystem (k)) depending on the admission discipline.

In case of partial admission

$$\tilde{p}_m^{(k)}(s) = P\{\min(m, \nu_k) = s\} = \begin{cases} p^{(k)}(s), & s = 1, 2, \ldots, m-1, \quad m > 0; \\ \sum_{l=m}^{\infty} p^{(k)}(l), & s = m > 0; \\ 1, & s = m = 0. \end{cases} \tag{6}$$

In case of complete admission

$$\tilde{p}_m^{(k)}(s) = \begin{cases} p^{(k)}(s), & s > 0, \quad m > 0; \\ 1, & s = m = 0. \end{cases} \tag{7}$$

In case of complete rejection

$$\tilde{p}_m^{(k)}(s) = \begin{cases} p^{(k)}(s), & 0 < s \le m; \\ \sum_{l=m+1}^{\infty} p^{(k)}(l), & s = 0, \quad m > 0; \\ 1, & s = m = 0. \end{cases} \tag{8}$$

Further calculate $q_{mn}^{(k)}(s)$ conditional distribution for the number of the accepted queries (probability to accept queries) provided that n queries arrived and there was m empty seats.

- If queries are accepted only for empty seats,

$$q_{mn}^{(k)}(s) = \begin{cases} 1, & s = n, \quad m \ge n; \\ 0, & s \ne n, \quad m \ge n; \\ 1, & s = m, \quad 1 \le m < n; \\ 0, & s \ne m, \quad 1 \le m < n; \\ 1, & s = m = 0; \\ 0, & s > m = 0. \end{cases} \tag{9}$$

- If all batch of queries in the presence of at least one empty seat is accepted, then

$$q_{mn}^{(k)}(s) = \begin{cases} 1, & s = n, \ m \geq 1; \\ 0, & s \neq n, \ m \geq 1; \\ 1, & s = m = 0; \\ 0, & s > m = 0. \end{cases} \tag{10}$$

- If there is no at least one empty place for all queries in batch, the batch is lost. In this case

$$q_{mn}^{(k)}(s) = \begin{cases} 1, & s = 0, \ 0 \leq m < n; \\ 0, & s \neq 0, \ 0 \leq m < n; \\ 1, & s = n, \ m \geq n; \\ 0, & s \neq n, \ m \geq n. \end{cases} \tag{11}$$

Describe the model. The given system consists of N subsystems.

The subsystem of k-th type $k = \overline{1, N}$, in designations of Kendall's notation can be described as follows:

CBSMAP/M$_k$/n$_k$/N$_k$, where

- *CBSMAP* means, that the arrival flow is controlled flow, described earlier;

- Symbol M_k means, that service duration of customer in a subsystem is exponentionally distributed with parameter μ_k;

-Symbols n_k and N_k define the quantity of the service buffers and the number of places in the queue n_k and N_k accordingly.

In classification of queuing systems, the system can be considered as controlled semi-markov system as its evolution is defined with controlled semi-markov process.

For construction of CSMP, describing the evolution of the system, it is necessary to realize the following algorithm:

- Define Markov moments,
- Define the states of semi-markov process;
- Define control set and control strategy;
- Define semi-markov kernel and a matrix of transition probabilities for embedded Markov chain;
- Construct income functional on the trajectories of CSMP;
- Define optimum strategy of control.

In the given model the Markov moments are the moments of arrivals of any type queries in system. In case of *k-th* type customers arrival, the given queries are taken on service to a subsystem of *k-th* type, and in other subsystems the batch of zero quantity "arrives".

The system states are defined using a vector $(i, l_1, l_2, ..., l_N)$, where i - a state of an arrival flow (at Markov moment the batch of i-th type customers arrives), l_k - a quantity of queries in a subsystem of k-th type, $M_k + N_k + n_k > l_k \geq 0, \ k \neq i, \ M_i + N_i + n_i > l_i > 0,$ n_k and N_k - accordingly the quantity of service channels and the quantity of places in the queue in the System (k), $i \in E = \{1, 2, ..., N\}$. The quantity of customers in a subsystem is final and depends on the admission discipline and the structure of a queuing model. Therefore $l_k \in E_k = \{1, 2, ..., M_k + N_k + n_k - 1\}$

and $(i, l_1, l_2, ..., l_N) \in E \times E_1 \times ... \times E_N$.

Enter the following designations:

$$(l_1, l_2, ..., l_N) = \vec{l} \ ; \quad E \times E_1 \times ... \times E_N = \tilde{E} \ .$$

The transition from state $(i, l_1, l_2, ..., l_N)$ to state $(j, l'_1, l'_2, ..., l'_j, ..., l'_N)$ with positive probability occurs if $l'_k \leq l_k, \ k \neq j$. So in all subsystems, except for a subsystem of j-th type, there is only a service customers which can be presented as process of death process, accordingly the quantity of customers in these subsystems is not more than the quantity of customers in subsystems at previous Markov moment of batch arrivals.

Note, that the system control is carried out using a control of arrival-flow at the moments of SMP (semi-Markov process) states change, at the Markov moments.

Remind, that control Markov strategy $\vec{G} = (G_{(i,\vec{l})}(u), \ (i, \vec{l}) \in \tilde{E})$, depending only on a current state of controlled process, is a set of the probability measures, given for each state $(i, \vec{l}) \in \tilde{E}$ on σ-algebra of subsets of decisions' set $U_{(i,\vec{l})}$.

III. Control measures for the model

Remind, that control Markov strategy $\vec{G} = (G_{(i,\vec{l})}(u), \ (i, \vec{l}) \in \tilde{E})$, depending only on a current state of controlled process, is a set of the probability measures, given for each state $(i, \vec{l}) \in \tilde{E}$ on σ-algebra of subsets of decisions' set $U_{(i,\vec{l})}$.

As it was noted above, the problem of controlled arrival flow is investigated. The flow is set as semi-Markov process. The arrival flow is defined by three factors: the type of the next batch, the arrival time of the next batch and the number of queries in the batch.

Therefore, the following options for construction of control measures set are possible:

-control of the next batch type,

-control of the next batch type and the moment of the batch arrival,

-control of the next batch type, the moment of the batch arrival and the number of queries in the batch.

1. If depending on the state of semi-markov process $(i,\vec{l}) \in \tilde{E}$ the next batch type is chosen. Then the equation is fair $U_{(i,\vec{l})} = E = \{1,2,...,N\}$ for control measures, and probability measure for discrete set *E = {1,2 ..., N}* is define in the following way

$$G_{(i,\vec{l})}(j) = P\{u(t) = j / \xi(t) = (i,\vec{l})\} = p_{(i,\vec{l}),j},$$
$$p_{(i,\vec{l}),j} \geq 0, \ \sum_{j \in E} p_{(i,\vec{l}),j} = 1, \ E = \{1,2,...,N\}, \quad (12)$$

where *u(t)* – is a decision made in markov moment *t*;

2. If depending on the state of semi-markov process $(i,\vec{l}) \in \tilde{E}$ the next batch type, the moment of the batch arrival. T Then the equation is fair

$$U_{(i,\vec{l})} = E \times [0,\infty) = \{(j,u),$$
$$j \in E = \{1,2,...,N\}, \ u \in R^+ = [0,\infty)\}$$

i.e. the set of controls consists of final number of half-lines. The probability measure on the set can be given a set of probabilities (12) and conditional distributions continuous components. If to designate a casual interval of time through which the following group of queries comes to queuing system *θ*, it is possible to determine conditional distribution of continuous component by equality

$$F_{(i,\vec{l})}(j,u) = P\{\theta < u / \xi(t) = (i,\vec{l}), \ u(t) = j\},$$
$$j \in E = \{1,2,...,N\}, \ u \in R^+,$$

and strategy is defined by equalities

$$G_{(i,\vec{l})}(j,u) = P\{u(t) = j, \theta < u / \xi(t) = (i,\vec{l})\} =$$
$$= p_{(i,\vec{l}),j} F_{(i,\vec{l}),j}(u), \quad (13)$$
$$j \in E = \{1,2,...,N\}, \ u \in R^+.$$

3. If depending on the state of semi-markov process $(i,\vec{l}) \in \tilde{E}$ the next batch type, the moment of the batch arrival and the number of queries in the batch are chosen, then the equations are fair for control measures

$$U_{(i,\vec{l})} = E \times [0,\infty) \times \tilde{M}_k =$$
$$= \{(j,u,s),$$
$$j \in E = \{1,2,...,N\}, \ u \in R^+ = [0,\infty),$$
$$s \in \tilde{M}_k = \{1,...,M_k\},$$

And for conditional distributions

$$P\{u(t) = (j,s) / \xi(t) = (i,\vec{l})\} =$$
$$= p_{(i,\vec{l}),(j,s)} = p_{(i,\vec{l}),j} p^{(j)}(s),$$
$$p_{(i,\vec{l}),(j,s)} \geq 0, \quad (14)$$
$$\sum_{(j,s) \in E \times E_k} p_{(i,\vec{l}),(j,s)}, = 1,$$
$$j \in E = \{1,2,...,N\}, \ s \in \tilde{M}_k$$

Designate θ - an interval of time through which in system the following batch arrives, it is possible to define conditional distribution in the following way

$$F_{(i,\vec{l})}(j,u,n) = P\{\theta < u / \xi(t) = (i,\vec{l}),$$
$$u(t) = (j,n)\},$$
$$j \in E = \{1,2,...,N\}, \ u \in R^+, \ n \in \tilde{M}_k,$$

And strategy is defined

$$G_{(i,\vec{l})}(j,u,n) =$$
$$= P\{u(t) = (j,n), \theta < u / \xi(t) = (i,\vec{l})\} =$$
$$= p_{(i,\vec{l}),j} p^{(j)}(n) F_{(i,\vec{l})}(j,u,n), \quad (15)$$
$$j \in E = \{1,2,...,N\}, \ u \in R^+, \ n \in (1,...,M_i).$$

The semi-markov kernel is a probability of that the semi-markov process will pass in the state $(j,\vec{l})$ (at the Markov moment the batch of *j*-th type arrives and in subsystems will be $\vec{l}$ queries) and the time of the transition will not surpass *t*, provided that the process stays in state $(i,\vec{m})$ and in the state the decision $u \in U_{(i,\vec{m})}$ from the control set is accepted. Designate the probability as $Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t,u)$.

If $U_{(i,\vec{m})} = E = \{1,2,...,N\}$, then we choose the type of the queries, which will arrive to the queuing model. Then we have

$$u \in U_{(i,\vec{m})} = E = \{1,2,...,N\},$$
$$k \in E = \{1,2,...,N\}, \ (i,\vec{m}),(j,\vec{l}) \in \tilde{E}, \ t \geq 0$$

$$Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t,u) = \\ = \begin{cases} \int_0^t \sum_{s=0}^{m_j} p^{(j)}_{m_j,s}(x)\tilde{p}^{(j)}_{N_j+n_j-s}(l_j - s)\cdot \\ \cdot \prod_{v\neq j} p^{(v)}_{m_v,l_v}(x)dF_{i,j}(x) , & u = j; \\ 0, \quad u \neq j. \end{cases} \qquad (16)$$

In equality (16) designations are used:

- the probabilities $p^{(j)}_{m_j,s}(x)$ are defined by equalities (3)-(5) depending on structure of subsystems of the investigated queuing model,
- the probabilities $\tilde{p}^{(j)}_{m}(l)$ are defined by equalities (6)-(8) – probability to admit m queries from j-th batch in the presence of empty seats depending on a admission discipline,
- $F_{i,j}(x)$ - the probability of that the following batch of queries will come to system till the moment *x* provided that it is batch of *j*-th type, the previous batch was a batch of *i*-th type (the set characteristic of an arrival flow).

Integration of function (16) on a discrete measure (12) we receive a semi-Markov kernel

$$Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t) = \\ = \int_{U_{(i,\vec{m})}} Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t,u)G_{(i,\vec{m})}(du) = \\ = p_{(i,\vec{m}),j}\cdot \qquad (17)$$

$$\cdot\int_0^t \sum_{s=0}^{m_j} p^{(j)}_{m_j,s}(x)\tilde{p}^{(j)}_{N_j+n_j-s}(l_j - s)\prod_{v\neq j} p^{(v)}_{m_v,l_v}(x)dF_{i,j}(x).$$

Passing to a limit at $t\to\infty$, we receive a matrix of transitional probabilities of the embedded Markov's chain

$$p^{(k)}_{(i,\vec{m}),(j,\vec{l})} = \lim_{t\to\infty} Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t) = \\ = \int_{U_{(i,\vec{m})}} Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(\infty,u)G_{(i,\vec{m})}(du) = \qquad (18) \\ = p_{(i,\vec{m}),j}\int_0^\infty \begin{matrix} \sum_{s=0}^{m_j} p^{(j)}_{m_j,s}(x)\tilde{p}^{(j)}_{N_j+n_j-s}(l_j - s)\cdot \\ \cdot\prod_{v\neq j} p^{(v)}_{m_v,l_v}(x)dF_{i,j}(x) \end{matrix}.$$

If

$$U_{(i,\vec{l})} = E\times[0,\infty) = \{(j,u), \\ E = \{1,2,...,N\},\ u\in R^+ = [0,\infty)\},$$

that is we choose type of queries which will arrive to queuing model and the moment of their arrival. Then we have

$$u = (j,x)\in U_{(i,\vec{m})} = E\times[0,\infty)$$

$$Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t,u) = \sum_{s=0}^{m_j} p^{(j)}_{m_j,s}(x)\tilde{p}^{(j)}_{N_j+n_j-s}(l_j - s)\cdot \\ \cdot\prod_{v\neq j} p^{(v)}_{m_v,l_v}(x),\ u = (j,x)\ x < t, \qquad (19) \\ Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t,u) = 0, \text{ in other cases.}$$

(19) functions entering equality all are defined above.

Integration of function (19) in process of (13) we receive a semi-Markov kernel

$$Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t) = \\ = \int_{U_{(i,\vec{m})}} Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t,u)G_{(i,\vec{m})}(du) = \qquad (20) \\ = p_{(i,\vec{m}),j}\int_0^t \begin{matrix} \sum_{s=0}^{m_j} p^{(j)}_{m_j,s}(x)\tilde{p}^{(j)}_{N_j+n_j-s}(l_j - s)\cdot \\ \cdot\prod_{v\neq j} p^{(v)}_{m_v,l_v}(x)dF_{(i,\vec{m}),j}(x) \end{matrix}.$$

Passing to a limit at $t\to\infty$, we receive a matrix of transitional probabilities of the embedded Markov's chain

$$p^{(k)}_{(i,\vec{m}),(j,\vec{l})} = \\ = \int_{U_{(i,\vec{m})}} Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(\infty,u)G_{(i,\vec{m})}(du) = \qquad (21) \\ = p_{(i,\vec{m}),j}\int_0^\infty \begin{matrix} \sum_{s=0}^{m_j} p^{(j)}_{m_j,s}(x)\tilde{p}^{(j)}_{N_j+n_j-s}(l_j - s)\cdot \\ \cdot\prod_{v\neq j} p^{(v)}_{m_v,l_v}(x)dF_{(i,\vec{m}),j}(x) \end{matrix}.$$

At control $U_{(i,\vec{l})} = E\times[0,\infty)\times\tilde{M}_k$, and at strategy $u\in U_{(i,\vec{m})} = \tilde{E}\times[0,\infty)$

$$Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t,u) = \\ = \sum_{s=0}^{m_j} p^{(j)}_{m_j,s}(x)q^{(j)}_{(N_j+n_j-s),n}(l_j - s)\prod_{v\neq j} p^{(v)}_{m_v,l_v}(x), \qquad (22) \\ u = (j,x,n),\ \ x < t,$$

$Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t,u) = 0,$ in the other cases.

Then the semi-markov kernel is defined in the following way

$$
\begin{aligned}
&Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t) = \\
&= \int\limits_{U_{(i,\vec{m})}} Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(t,u) G_{(i,\vec{m})}(du) = \\
&= p_{(i,\vec{m}),j} \int_0^t \sum_{n=1}^{\infty} \begin{array}{l} \sum\limits_{s=0}^{m_j} p^{(j)}_{m_j,s}(x) q^{(j)}_{(N_j+n_j-s),n}(l_j-s) \times \\ \times p^{(j)}(n) \prod\limits_{v \neq j} p^{(v)}_{m_v,l_v}(x) dF_{(i,\vec{m}),(j,n)}(x) \end{array} = \\
&= p_{(i,\vec{m}),j} \int_0^t \sum_{s=0}^{l_j} \begin{array}{l} p^{(j)}_{m_j,l_j-s}(x) \prod\limits_{v \neq j} p^{(v)}_{m_v,l_v}(x) \times \\ \times \sum\limits_{n=1}^{\infty} q^{(j)}_{(N_j+n_j-l_j+s),n}(l_j-s) p^{(j)}(n) dF_{(i,\vec{m}),(j,n)}(x) \end{array} .
\end{aligned} \quad (23)
$$

At $t \to \infty$, there is a matrix of transition probabilities of embedded Markov chain

$$
\begin{aligned}
&p^{(k)}_{(i,\vec{m}),(j,\vec{l})} = \\
&= \int\limits_{U_{(i,\vec{m})}} Q^{(k)}_{(i,\vec{m}),(j,\vec{l})}(\infty,u) G_{(i,\vec{m})}(du) = \\
&= p_{(i,\vec{m}),j} \int_0^{\infty} \sum_{s=0}^{l_j} \begin{array}{l} p^{(j)}_{m_j,l_j-s}(x) \prod\limits_{v \neq j} p^{(v)}_{m_v,l_v}(x) \times \\ \times \sum\limits_{n=1}^{\infty} q^{(j)}_{(N_j+n_j-l_j+s),n}(l_j-s) p^{(j)}(n) dF_{(i,\vec{m}),(j,n)}(x) \end{array} .
\end{aligned} \quad (24)
$$

The equations for matrixes of transitional probabilities (24) allow to make two conclusions:

- Elements of these matrixes don't depend on subsystem number;
- Elements of a line of these matrixes corresponding to a state $(i,\vec{m})$ are linear functionalities only rather probability measure $G_{(i,\vec{m})}(u)$.

So stationary probabilities $\pi^{(k)}_{(i,\vec{m})} = \pi_{(i,\vec{m})}$, $(i,\vec{m}) \in \tilde{E}$ of the embedded Markov chain don't depend on the subsystem number k and are presented in the following way

$$
\pi_{(i,\vec{m})} = \frac{\Delta_{(i,\vec{m})}}{\Delta}, \quad (i,\vec{m}) \in \tilde{E}, \quad (25)
$$

Where determinator Δ is a linear functional concerning all measures $G_{(j,\vec{l})}(u)$, $(i,\vec{l}) \in \tilde{E}$, and determinator $\Delta_{(i,\vec{m})}$ is a linear functional concerning all measures $G_{(j,\vec{l})}(u)$, $(i,\vec{l}) \in \tilde{E} \setminus \{(i,\vec{m})\}$, except measure $G_{(j,\vec{m})}(u)$.

IV. Construction of the Income Functional

The point is devoted to the construction of the income functional on the trajectories of the controlled semi-markov process. Further it is necessary to define functions $R^{(k)}_{(i,\vec{m})(j,\vec{l})}(x,u)$ - a conditional mathematical expectation of the saved up income in System (k) provided that process is staying in state $(i,\vec{m})$, through time t, it will pass in state $(j,\vec{l})$ and the decision u is made.

The conditional mathematical expectation of the saved up income $R^{(k)}_{(i,\vec{m})(j,\vec{l})}(x,u)$ depends on incomes and charges received at a system work. Enter the constants describing the incomes and charges:

$c_1^{(k)}$ - an income received for the service of one query (customer);

$c_2^{(k)}$ - a payment for a time unit during the working of one channel during the service;

$c_3^{(k)}$ - a payment for a time unit of idle time of one channel;

$c_4^{(k)}$ - a payment for a time unit for staying in the queue for one query;

$c_5^{(k)}$ - a payment for one lost query of k-th type.

Then

$$
\begin{aligned}
&R^{(k)}_{(i,\vec{m})(j,\vec{l})}(x,u) = c_1^{(k)} C_1(x, m_k, l_k, u) + \\
&+ \sum_{s=2}^{4} C_s(x, n_k, m_k, l_k, u) + c_5^{(k)} C_5(x, m_k, l_k, u),
\end{aligned} \quad (26)
$$

where

$$
\begin{aligned}
&C_1(x, m_k, l_k, u) = \\
&= M(\zeta_k / \nu^{(k)}(0) = m_k, \xi^{(k)}(x) = l_k, u(0) = u)
\end{aligned}
$$

- provided that at the Markov moment there were m queries in queuing model, and at the next x Markov moment there is l_k queries in the model and the decision u is made, $\xi^{(k)}(0) = \nu^{(k)}(0) = m_k, \xi^{(k)}(x) = l_k, u(0) = u$;

similar to previous formula

$$C_2(x,n_k,m_k,l_k,u)=$$
$$=c_2^{(k)}M[\int_0^x \min(n_k,v^{(k)}(t))dt\,/\,/v^{(k)}(0)=m_k,\xi^{(k)}(x)=l_k,u(0)=u],$$
$$C_3(x,n_k,m_k,l_k,u)=$$
$$=c_3^{(k)}M[\int_0^x (n_k-\min(n_k,v^{(k)}(t))dt\,/\,v^{(k)}(0)==m_k,\xi^{(k)}(x)=l_k,u(0)=u],$$
$$C_4(x,n_k,m_k,l_k,u)=c_4^{(k)}\cdot$$
$$\cdot M[\int_0^x \max(0,v^{(k)}(t)-n_k)dt\,/\,/v^{(k)}(0)=m_k,\xi^{(k)}(x)=l_k,u(0)=u],$$
$$C_5(x,m_k,l_k,u)=$$
$$=M(\eta\,/\,\xi^{(k)}(0)=v^{(k)}(0)=m_k,\xi^{(k)}(x)=l_k,u(0)=u).$$

Notice, that $\xi^{(k)}(x)$ - the number of customers at the Markov moment x, $v^{(k)}(t)$ - a Markov process of destruction on the period $t\in(0,x)$.

Designate $S^{(k)}_{(j,\vec{l})}(t)$ - the mathematical expectation of the saved up income in the subsystem k during time $t>0$, provided that the process starts from the state $(j,\vec{l})$.

For the functional $S^{(k)}_{(j,\vec{l})}(t)$ the following equality is fair

$$S^{(k)}=\frac{\sum\limits_{(i,\vec{l})\in\tilde{E}} s^{(k)}_{(i,\vec{l})}\pi^{(k)}_{(i,\vec{l})}}{\sum\limits_{(i,\vec{l})\in\tilde{E}} m^{(k)}_{(i,\vec{l})}\pi^{(k)}_{(i,\vec{l})}},\tag{28}$$

Where

$$m^{(k)}_{(i,\vec{m})}=\int_0^\infty[1-\sum_{(j,\vec{l})\in\tilde{E}}Q^{(k)}_{(i,\vec{m})(j,\vec{l})}(t)]dt\tag{29}$$

mathematical expectation for the time of continuous staying of the process $\xi(t)$ in state $(i,\vec{m})$;

$$s^{(k)}_{(i,\vec{m})}=$$
$$=\int_{u\in U_{(i,\vec{m})}}\sum_{(j,\vec{l})}[\int_o^\infty R^{(k)}_{(i,\vec{m})(j,\vec{l})}(x,u)dQ^{(k)}_{(i,\vec{m})(j,\vec{l})}(x,u)]G_{(i,\vec{m})}(du)\tag{30}$$

mathematical expectation of the saved up income during the system (*k*) working during continuous staying of the process $\xi(t)$ in the state $(i,\vec{m})$.

For the model income functional the following theorems are proved.

Theorem 1. The income functional $S^{(k)}$ for the System (*k*) is a fractional-linear functional concerning the distributions $\vec{G}=\{G_{(i,\vec{m})}(u),\ (i,\vec{m})\in\tilde{E}\}$ defining the Markov homogeneous strategy.

Theorem 2. The income functional $S=\sum\limits_{k=1}^{N}S^{(k)}$ for the System (all system) is a fractional-linear functional concerning the distributions $\vec{G}=\{G_{(i,\vec{m})}(u),\ (i,\vec{m})\in\tilde{E}\}$ defining the Markov homogeneous strategy.

The final stage of the research is a construction of the optimum control strategy. For solving of the problem we use the known fact [6]: if a fractional-linear functional has an extremum (a maximum or a minimum), the extremum is reached in a class of the determined strategy, where fixed determined probability measure:

$$P(\zeta<x)=\begin{cases}0, & x\le c,\\ 1, & x>c.\end{cases}.$$

Describe the set of the determined strategies.

1. Control of batch type. Strategy is defined by equality (12). The fixed degenerate measure in state $(i,\vec{l})$ is defined by equality

$$\begin{aligned}G_{(i,\vec{l})}(j)&=P\{u(t)=j\,/\,\xi(t)=(i,\vec{l})\}=p_{(i,\vec{l}),j}=1,\\ G_{(i,\vec{l})}(n)&=P\{u(t)=n\,/\,\xi(t)=(i,\vec{l})\}=p_{(i,\vec{l}),n}=0.\end{aligned}\tag{31}$$

Therefore, the number of degenerate measures in state $(i,\vec{l})$ is equal N. If the quantity of states $(i,\vec{l})\in\tilde{E}$ is equal to $K=\sum\limits_{s=1}^{N}(M_s+N_s+n_s)\prod\limits_{\substack{l=1\\ l\ne s}}^{N}(M_l+N_l+n_l+1)$, that number of degenerate strategy is $L=N^K$.

Thus we receive the algorithm for searching of the optimum strategy:

- For the fixed strategy the matrix of transition probabilities is calculated;
- For the matrix the system of the algebraic equations is solved and the stationary distribution of the embedded Markov chain at the chosen fixed strategy is determined;
- At the chosen fixed strategy the characteristics (29) and (30) are calculated;
- The income functional (28) for the chosen strategy is calculated;
- Using all fixed strategies and the values of the income functional for the strategies, we define the maximal income and optimum strategy.

2. Control of a type of the next batch and the moments of arrivals. Strategy is defined by equality (12). The fixed degenerate measure in state $(i,\vec{l})$ is defined by equality for discrete component and by equality

$$F_{(i,\vec{m}),j}(x)=\begin{cases}1, \tau_{(i,\vec{m})}<x,\\ 0, \tau_{(i,\vec{m})}>x.\end{cases} \quad (32)$$

Thus we receive the algorithm for searching of the optimum strategy:

• For the fixed strategy the matrix of transition probabilities is calculated;

• For the matrix the system of the algebraic equations is solved and the stationary distribution of the embedded Markov chain at the chosen fixed strategy is determined;

• At the chosen fixed strategy the characteristics (29) and (30) are calculated;

• The income functional (28) corresponding to the chosen degenerate strategy of discrete components is calculated;

• The maximum of the specific income (2.29) is determined by variables $\tau_{(i,\vec{m})}\ge 0,\ (i,\vec{m})\in\tilde{E}$;

• Touching all degenerate strategy and the income sizes corresponding to them, we define the maximum income and optimum strategy

3. Control of a type of the next batch, the moments of batch arrivals and the quantity of queries in the batch. The strategy is defined by the equality (15). The fixed degenerate measure in state $(i,\vec{l})$ is defined by equality for discrete component

$$\begin{aligned}G_{(i,\vec{l})}(j,n)&=P\{u(t)=(j,n)/\xi(t)=(i,\vec{l})\}=\\&=p_{(i,\vec{l}),j}\,p^{(j)}(n)=1,\\G_{(i,\vec{l})}(s,m)&=P\{u(t)=(s,m)/\xi(t)=(i,\vec{l})\}=\\&=p_{(i,\vec{l}),s}=0,\ (s,m)\ne(j,n).\end{aligned} \quad (33)$$

and by equality

$$F_{(i,\vec{m})}(j,x,n)=\begin{cases}1, \tau_{(i,\vec{m})(j,n)}<x,\\ 0, \tau_{(i,\vec{m})(j,n)}>x.\end{cases} \quad (34)$$

In essence, for this case the previous algorithm remains, the number of options for search discrete components only increases.

Thus, the algorithms formulated above allow to construct compliance:

$$(i,\vec{m})\to j^{(0)}_{(i,\vec{m})},\ (i,\vec{m})\to (j^{(0)}_{(i,\vec{m})},\tau^{(0)}_{(i,\vec{m})}),\ (i,\vec{m})\to (j^{(0)}_{(i,\vec{m})},\tau^{(0)}_{(i,\vec{m})},n^{(0)}_{(i,\vec{m})}), \quad (35)$$

and every time being $(i,\vec{m})$ in a state to make the relevant decision with probability 1.

Remark. The main difficulty when obtaining dependences is large dimension of set of states and set of controls.

V. Conclusion

In the present paper control of the model is complicated. We use the control of three parameters at the same time in arrival flow: type of the next batch, the moments of batch arrivals and the number of queries in the batch.

Earlier in several papers [7], [8], [10] it was already shown that in various queuing systems (single-channel/multichannel, with exponential/randomly distributed duration of service, with queue/without queue) it is possible to control, changing at the same time some characteristics of system for maximizing the income. Algorithmization of work is constructed on application of the theory of controlled semi-Markov processes and maximizing of income functional.

Respectively, with control of arrival flow it is possible to operate in addition the service duration, number of serving channels, the quantity of places in the queue. It is possible in further researches, that is confirmed with the previous works.

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VI. Appendix

Controlled Semi-Markov process $X(t)=\{\xi(t),u(t)\}$ is defined using homogeneous three-dimensional Markov chain

$$(\xi_n,\theta_n,u_n),\ n\ge 0,\ \xi_n\in E,\ \theta_n\in R^+=[0,\infty),\ u_n\in U,$$

which is defined by transition probabilities of a special type.

$$\begin{aligned}&P\{\xi_{n+1}=j,\theta_{n+1}<t,u_{n+1}\in B/\xi_n=i,\theta_n=\tau,u_n=u\}=\\&=P\{\xi_{n+1}=j,\theta_{n+1}<t,u_{n+1}\in B/\xi_n=i\},\\&i,j\in E,\ t,\tau\in R^+,\ u\in U,\ B\in A\end{aligned} \quad (1A)$$

$$p_i=P\{\xi_0=i,\theta_0<\infty,u_0\in U\}.$$

In the further we shall use the following designations:

$$\begin{aligned}&P\{\xi_{n+1}=j,\theta_{n+1}<t,u_{n+1}\in B/\xi_n=i\}=\\&=\tilde{Q}_{ij}(t,B).\end{aligned} \quad (2A)$$

For each state *i* the set of controls U_i and σ-algebra A_i of subsets of this set U_i is given.

At $t\to\infty$ and $B=U_i$ we obtain the transition probability

$$p_{ij} = \tilde{Q}_{ij}(\infty, U_i) = P\{\xi_{n+1} = j / \xi_n = i\} \tag{3A}$$

for embedded Markov chain.

Use definition $\tilde{Q}_{ij}(t, B)$

$$\tilde{Q}_{ij}(\infty, B) = lim_{t\to\infty}\tilde{Q}_{ij}(t, B) = \\ = P\{\xi_{n+1} = j, u_{n+1} \in B / \xi_n = i\}. \tag{4A}$$

then

$$G_i(B) = \sum_{j\in E} \tilde{Q}_{ij}(\infty, B) = P\{u_{n+1} \in B / \xi_n = i\} \tag{5A}$$

and

$$\tilde{Q}_{ij}(t, B) = \int_B Q_{ij}(t, u) G_i(du), \tag{6A}$$

where

$$Q_{ij}(t, u) = P\{\xi_{n+1} = j, \theta_{n+1} < t / \xi_n = i, u_{n+1} = u\}. \tag{7A}$$

Thus, homogeneous Controlled Semi-Markov process can be set by family of matrixes, set of probability measures and initial distribution of probabilities p_i, $i,j\in E$, $t\in R^+$, $u\in U_i$, $B\in A_i$.

Family of matrixes $\{Q_{ij}(t, u)\}$ is a Semi-Markov kernel of controlled Semi-Markov process, and family of probability measures $\vec{G} = \{G_1(B), G_2(B), ..., G_N(B)\}$ is family of controlling measures.

The counting process $v(t)$ is defined in following way

$$v(t) = \sup\{n : \sum_{k\le n} \theta_k \le t\}, \ \theta_0 = 0.$$

The Controlled Semi-Markov process is defined as

$$X(t) = \{\xi(t), \ u(t)\},$$

where $\xi(t) = \xi_{v(t)}$, $u(t) = u_{v(t)+1}$.

Process $\xi(t)$ coincides with a standard Semi-Markov process. The second component of controlled Semi-Markov process $u(t)$ defines a trajectory of accepted decisions.

It is possible to define one more way to give controlled Semi-Markov process. It is necessary to set:

- Markov homogeneous control strategy $\vec{G} = \{G_1(B), G_2(B), ..., G_N(B)\}$,
- characteristics of controlled Markov chain - initial distribution $p_i = P\{\xi_0 = i\} \ge 0$, $i \in E$, $\sum_{i\in E} p_i = 1$ and a matrix of transition probabilities $p_{ij}(u) = P\{\xi_{n+1} = j / \xi_n = i, u_{n+1} = u\}$;
- conditional distributions of intervals $F_{ij}(t, u) = P\{\theta_{n+1} < t / \xi_{n+1} = j, \xi_n = i, u_{n+1} = u\}$.

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