Fitting Anisochronic Models by Method of Moments for Anisochronic Control of Time Delay Systems

Milan Hofreiter

Abstract**—** The paper introduces an original approach to parameter estimation of anisochronic models describing systems, which are conventionally described by a serial combination of a rational transfer function and a transportation delay. The method of moments is used for this purpose. Since the method of moments uses step responses, the paper also introduces an original formula for the derivation of step responses from relay feedback control. At the end of this paper, the anisochronic controller using the anisochronic model and the Desired Model Method will be derived. The applicability of the suggested methodology is presented examples in the Matlab/Simulink environment.

Keywords**—**Anisochronic model, control, parameter estimation, time delay, moments, step response, relay identification.

I. Introduction

TIME delay systems constitute an important category of dynamical systems in which time delays exist between the application of input or control to the system and their resulting effect on it. Time delays often occur in biological systems, metallurgical systems, chemical processes, thermal systems, transportation and communication systems, power systems, remote control systems, economic systems, etc. (See [11],[16]).

The time delay systems may contain delays in input-output relations, but also in internal feedback loops, i.e. state delays. Systems with delays, latencies and after effects can be described around operating point by a serial combination of a rational transfer function and a transfer function of a transport delay [10], [17] or by anisochronic models [4], [25]. These anisochronic models (as opposed to isochronic models) contain delays in both inputs and states. Transfer functions of linear anisochronic models include transcendental exponential terms in both the numerator and the denominator. Also, they have an infinite number of poles and zeros in general. Consequently the transfer functions are not rational. Although the systems with state delays are relatively common, practical applications of anisochronic models for control are rare. The main reasons are:

M. Hofreiter, Department of Instrumentation and Control Technology, Czech Technical University in Prague, Faculty of Mechanical Engineering, Technicka 4, 166 07 Prague 6, Czech Republic, http://fsinet.fsid.cvut.cz/~hofremil, (e-mail: milan.hofreiter@fs.cvut.cz).

a) Parameter estimate of anisochronic models is more difficult than parameter estimate of standard models.

b) Transfer functions cannot be factored into a product of a pure transportation delay and a rational transfer function. Neither Smith predictor nor any other control schemes based on this factorization are applicable.

For these reasons several approaches for parameter estimate of anisochronic models and for anisochronic control design were recently proposed, e.g. [25], [26]. Unfortunately these approaches are marked by the enormous complexity, which limits their application in common practice.

II. Linear Anisochronic Model of Second Order

Transfer function (1) offers a wide variability for description of continuous SISO time delay systems around operating points, see [24].

$$G_a(s) = \frac{K \cdot e^{-s\cdot\tau_u}}{(\tau_1 \cdot s + 1)\left(\tau_2 \cdot s + e^{-s\cdot\tau_y}\right)} \tag{1}$$

This linear anisochronic model is the second order and it contains only five parameters, where K is the plant static gain, τ_u is the pure input time delay, τ_1 and τ_2 are the time constants and τ_y is the feedback time delay (the internal delay in state). The variable s represents the complex argument defined by the Laplace transform.

The delay τ_y appears in the denominator of transfer function (1) and consequently characteristic equation (2) becomes transcendental in s.

$$(\tau_1 \cdot s + 1)\left(\tau_2 \cdot s + e^{-s\cdot\tau_y}\right) = 0 \tag{2}$$

The characteristic quasipolynomial has an infinite set of roots. It may also lead to an alternative interpretation of model (1) as an infinite dimensional model. Therefore it may serve as a good approximation for general higher-order delay free systems.

In the time domain, model (1) is described by equation (3).

$$\tau_1\tau_2 y''(t) + \tau_2 y'(t) + \tau_1 y'(t-\tau_y) + y(t-\tau_y) = Ku(t-\tau_u) \tag{3}$$

The parameters K, τ_u, τ_1, τ_2 and τ_y are depicted in Fig. 1 on the unit step response of model (1), where I represents the position of the inflection point, p is the tangent at the inflection point I, see [12], [24].

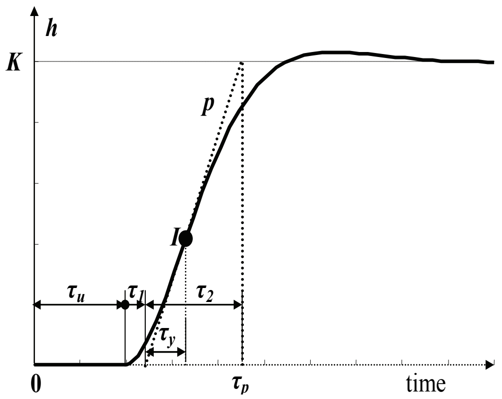

Fig. 1 The unit step response $h(t)$ of model (1)

Model (1) is stable if $\tau_y/\tau_2<\pi/2$, overdamped if $\tau_y/\tau_2<1/e$, critically damped if $\tau_y/\tau_2=1/e$ and underdamped if $\tau_y/\tau_2>1/e$, where e is Euler's number, see [4]. Therefore model (1) may be used both for nonoscillatory and oscillatory processes.

The parametrs K, τ_u, τ_1, τ_2 and τ_y can be estimated from the unit step response $h(t)$ by the graphical method [12], see Fig. 1. This approach is based on evaluation of the step response at single points. Such methods are quite sensitive to measurement noise and it may lead to large errors. For example, the position of the inflection point I is vague and therefore the estimate of the parameter τ_y may be erroneous. To improve the estimates, the method of moments is further applied.

III. Method of Moments

The method of moments [2] is based on the computation of the integrals $M_i, i=0,1,2,....$, where

$$M_i=\int_0^\infty t^i\cdot\left(h(\infty)-h(t)\right)dt, \tag{4}$$

and $h(t)$ is the unit step response of a stable process.

Integrals $M_i, i=0,1,2,...$ can be determined using the following relationship between the transfer function $G(s)$ and the impulse response function $g(t)$

$$G(s)=\int_0^\infty g(t)e^{-s\cdot t}dt\,. \tag{5}$$

Taking derivatives with respect to s we get

$$G^{(n)}(s)=\frac{d^nG(s)}{ds^n}=(-1)^n\int_0^\infty t^n g(t)e^{-st}dt, \quad \text{for } n=0,1,2,\cdots \tag{6}$$

Hence for stable processes with the static gain K

$$G(0)=h(\infty)=K\,, \tag{7}$$

and

$$\begin{aligned}G'(0)&=-\int_0^\infty t\cdot g(t)dt=-\int_0^\infty t\cdot h'(t)dt\\&=\int_0^\infty t\cdot m'(t)dt=\left[t\cdot m(t)\right]_0^\infty-\int_0^\infty m(t)dt\end{aligned}\,, \tag{8}$$

where

$$m(t)=h(\infty)-h(t)\,. \tag{9}$$

and

$$g(t)=\frac{dh(t)}{dt}=h'(t)\,. \tag{10}$$

$$m'(t)=-h'(t)\,. \tag{11}$$

If it holds

$$\lim_{t\to\infty} t\cdot m(t)=0 \tag{12}$$

then

$$G'(0)=-\int_0^\infty m(t)dt=-M_0\,. \tag{13}$$

In a similar way it can be derived

$$G^{(n)}(0)=(-1)^n\cdot n\cdot M_{n-1}\,, \tag{14}$$

if it holds

$$\lim_{t\to\infty} t^n\cdot m(t)=0\,. \tag{15}$$

For the next calculations of the moments it is advantageous to decompose the transfer function $G(s)$ into a product of factors $G_1(s)$ and $G_2(s)$, i.e.

$$G(s)=G_1(s)\cdot G_2(s)\,. \tag{16}$$

Then it holds

$$G(s)=G_1'(s)\cdot G_2(s)+G_1(s)\cdot G_2'(s) \tag{17}$$

hence with respect to (16) one can express

$$\frac{G'(s)}{G(s)}=\frac{G_1'(s)}{G_1(s)}+\frac{G_2'(s)}{G_2(s)}\,. \tag{18}$$

From it follows

$$\frac{G'(0)}{G(0)}=\frac{G_1'(0)}{G_1(0)}+\frac{G_2'(0)}{G_2(0)}\,. \tag{19}$$

Differentiating the fraction $G'(s)/G(s)$ we can obtain

$$\left(\frac{G'(s)}{G(s)}\right)'=\frac{G''(s)}{G(s)}-\left(\frac{G'(s)}{G(s)}\right)^2 \tag{20}$$

hence

$$\frac{G''(s)}{G(s)} = \left(\frac{G'(s)}{G(s)}\right)^2 + \left(\frac{G'(s)}{G(s)}\right)'. \tag{21}$$

From it follows

$$\frac{G''(0)}{G(0)} = \left(\frac{G'(0)}{G(0)}\right)^2 + \left(\frac{G'(s)}{G(s)}\right)'_{s=0}. \tag{22}$$

The derived mathematical relations can be used for estimating parameters of models describing linear stable systems.

IV. Fitting Anisochronic Models of Second Order by Moments

A. Determination of M_0 and M_1

Anisochronic model (1) is very universal mathematical model enabling to describe the behavior of many real processes. But this model contains both a transport delay and a state delay. Therefore parameter fitting for this model is more difficult than for conventional linear models. This is main reason why that up to now only a few methods for parameter estimation have been developed that can be applied in practice, e.g. [5], [21].

The introduced method shows how to estimate the parameters of model (1) from the step responses by means of moments.

The moment M_0 can be determined using (13).

$$M_0 = -G'(0). \tag{23}$$

With respect to (1), (19) and (18)

$$\frac{G'(s)}{G(s)} = -\tau_u - \frac{\tau_1}{\tau_1 s + 1} - \frac{\tau_2 + e^{-s\tau_y}(-\tau_y)}{\tau_2 s + e^{-s\tau_y}} \tag{24}$$

and so the moment M_0 can be expressed using relations (1), (7), (18), (19) and (24)

$$M_0 = -\frac{G'(0)}{G(0)} \cdot G(0) = \tau_{ar} \cdot K, \tag{25}$$

$$\tau_{ar} \triangleq -\frac{G'(0)}{G(0)} = \tau_u + \tau_1 + \tau_2 - \tau_y, \tag{26}$$

where τ_{ar} is the average residence time.

The moment M_1 can be expressed by relation (14)

$$M_1 = \frac{G''(0)}{2}, \tag{27}$$

where $G''(0)$ can be determined using (7), (22) and (26). From it follows

$$M_1 = \left(\left(\frac{G'(0)}{G(0)}\right)^2 + \left(\frac{G'(s)}{G(s)}\right)'_{s=0}\right) \cdot \frac{G(0)}{2}, \tag{28}$$

where according to (26)

$$\left(\frac{G'(0)}{G(0)}\right)^2 = \tau_{ar}^2 = \left(\tau_u + \tau_1 + \tau_2 - \tau_y\right)^2 \tag{29}$$

and

$$\begin{aligned} \left(\frac{G'(s)}{G(s)}\right)'_{s=0} &= \left\{ \frac{\tau_1^2}{(\tau_1 s + 1)^2} - \frac{\tau_y^2 \cdot e^{-s\tau_y}}{\tau_2 s + e^{-s\tau_y}} \right. \\ &\left. + \frac{\left(\tau_2 - \tau_y \cdot e^{-s\tau_y}\right)^2}{\left(\tau_2 s + e^{-s\tau_y}\right)^2} \right\}_{s=0} = \tau_1^2 - \tau_y^2 + \left(\tau_2 - \tau_y\right)^2 \end{aligned} \tag{30}$$

The moment M_1 can be then expressed

$$M_1 = K\left(\frac{\tau_1^2 + \tau_2^2 + \tau_{ar}^2}{2} - \tau_2 \cdot \tau_y\right). \tag{31}$$

B. Calculation of M_0 and M_1

Since digital computers are used for data processing, continuous measurements are converted into a digital form [6]. [The moments M_0 and M_1 can be obtained from a sampled data record $h(k\cdot\Delta t)$, $k=1,2,\ldots,N$ of the unit step response, where Δt represents the sampling period and N relates to the last sample, when the unit step response achieved the new steady state. The integrals

$$M_0 = \int_0^\infty \left(K - h(\tau)\right) d\tau \tag{32}$$

and

$$M_1 = \int_0^\infty \tau \cdot \left(K - h(\tau)\right) d\tau \tag{33}$$

can be computed by a numerical integration, e.g. a trapezoidal or rectangular integration algorithm, e.g. [7], [8]. For the calculation we can use the normalized step response $\overline{h}_k$, where

$$\overline{h}_k \triangleq \frac{h(k \cdot \Delta t)}{K} \tag{34}$$

The integrals M_0 and M_l can be then calculated (by means of rectangular approximation) as follows

$$\begin{aligned} M_l &= \int_0^\infty \tau^l \left(K - h(\tau)\right) d\tau = K \int_0^\infty \tau^l \left(1 - \frac{h(\tau)}{K}\right) d\tau \\ &\approx K \cdot \sum_{k=0}^{N} (k \cdot \Delta t)^l \left(1 - \overline{h}_k\right) \Delta t, \quad l = 0,1 \end{aligned} \tag{35}$$

C. Parameter Estimates

Without the knowledge of the exact position of the inflexion point I in the step response (see Fig.1), one can construct the tangent p and calculate integrals (32) and (33) by numerical integration. Then the parameters of anisochronic model (1) can be found in according to the following steps.

1. Determine the static gain K from a graphical construction or through static tests.
2. Determine the time constant τ_2 from a graphical construction, see Figure 1.
3. Compute the average residence time τ_{ar} from (25)

$$\tau_{ar} = \frac{M_0}{K}. \tag{36}$$

4. Determine the transient time τ_p from the graphical construction, see Fig. 1, where

$$\tau_p = \tau_1 + \tau_2 + \tau_u. \tag{37}$$

5. Compute the feedback time delay τ_y

$$\tau_y = \tau_p - \tau_{ar}. \tag{38}$$

6. Compute the time constant τ_1 from (31)

$$\tau_1 = \sqrt{2 \cdot \frac{M_1}{K} - \left(\tau_{ar}^2 + \left(\tau_2 - \tau_y\right)^2 - \tau_y^2\right)}. \tag{39}$$

7. Compute the pure input time delay τ_u

$$\tau_u = \tau_p - \tau_1 - \tau_2. \tag{40}$$

One will notice that in this method only the time constants τ_p and τ_2 are obtained directly from the graphical construction and they are easily determined by the tangent p. The other constants τ_1, τ_u and τ_y are calculated using the values M_0 and M_1. This procedure easily enables the estimation of the delays τ_u and τ_y without knowledge of the exact position of the inflection point I.

V. Determination M_0 and M_1 for Universal Linear Anisochronic Model

The previous results can be generalized for a stable system with transfer function (41)

$$G_a(s) = \frac{Ke^{-s\cdot\tau_u} \prod_{k=1}^{N_3}(T_k s + 1) \cdot \prod_{l=1}^{N_4}\left({}_aT_l s + e^{-s\cdot {}_yT_l}\right)}{\prod_{i=1}^{N_1}(\tau_i s + 1) \cdot \prod_{j=1}^{N_2}\left({}_a\tau_j s + e^{-s\cdot {}_y\tau_j}\right)}, \tag{41}$$

where τ_j, ${}_a\tau_k$, ${}_y\tau_k$, T_i, ${}_yT_l$, $\tau_u \in \mathbb{R}^+$ for $\forall i, j, k, l$; $N_1, N_2, N_3, N_4 \in \mathbb{N}$; $N_1+N_2 \leq N_3+N_4$; $\mathbb{R}^+$ is the set of positive real numbers.

With respect to (18)

$$\frac{G_a'(s)}{G_a(s)} = \frac{\left(e^{-s\cdot\tau_u}\right)'}{e^{-s\cdot\tau_u}} + \sum_{i=1}^{N_1} \frac{\left(\frac{1}{\tau_i s + 1}\right)'}{\frac{1}{\tau_i s + 1}}$$
$$+ \sum_{j=1}^{N_2} \frac{\left(\frac{1}{{}_a\tau_j s + e^{-s\cdot {}_y\tau_j}}\right)'}{\frac{1}{{}_a\tau_j s + e^{-s\cdot {}_y\tau_j}}} + \sum_{k=1}^{N_3} \frac{(T_k s + 1)'}{T_k s + 1} \tag{42}$$
$$+ \sum_{l=1}^{N_4} \frac{\left({}_aT_l s + e^{-s\cdot {}_yT_l}\right)'}{{}_aT_l s + e^{-s\cdot {}_yT_l}}$$

After calculation of the derivations we get

$$\frac{G_a'(s)}{G_a(s)} = -\tau_u - \sum_{i=1}^{N_1} \frac{\tau_i}{\tau_i s + 1}$$
$$- \sum_{j=1}^{N_2} \frac{{}_a\tau_j - e^{-s\cdot {}_y\tau_j} \cdot {}_y\tau_j}{{}_a\tau_j \cdot s + e^{-s\cdot {}_y\tau_j}} + \sum_{k=1}^{N_3} \frac{T_k}{T_k s + 1} \tag{43}$$
$$+ \sum_{l=1}^{N_4} \frac{{}_aT_l - {}_yT_l \cdot e^{-s\cdot {}_yT_l}}{{}_aT_l s + e^{-s\cdot {}_yT_l}}.$$

Hence, with respect to (7), (13) and (43)

$$M_0 = -\frac{G'(0)}{G(0)} \cdot G(0) = \tau_{ar} \cdot K \tag{44}$$

$$\tau_{ar} = -\frac{G'(0)}{G(0)} = \tau_u + \sum_{i=1}^{N_1} \tau_i + \sum_{j=1}^{N_2}\left({}_a\tau_j - {}_y\tau_j\right)$$
$$- \sum_{k=1}^{N_3} T_k + \sum_{l=1}^{N_4}\left({}_yT_l - {}_aT_l\right) \tag{45}$$

The moment M_1 can be determined by (28), (43) and (45), where

$$\left(\frac{G_a'(s)}{G_a(s)}\right)' = \sum_{i=1}^{N_1} \frac{\tau_i^2}{(\tau_i s + 1)^2}$$
$$- \sum_{j=1}^{N_2}\left(\frac{e^{-s\cdot {}_y\tau_j} \cdot {}_y\tau_j^2}{{}_a\tau_j \cdot s + e^{-s\cdot {}_y\tau_j}} - \left(\frac{{}_a\tau_j - {}_y\tau_j \cdot e^{-s\cdot {}_y\tau_j}}{{}_a\tau_j \cdot s + e^{-s\cdot {}_y\tau_j}}\right)^2\right)$$
$$- \sum_{k=1}^{N_3}\left(\frac{T_k}{T_k s + 1}\right)^2 \tag{46}$$
$$+ \sum_{l=1}^{N_4}\left(\frac{{}_yT_l^2 e^{-s\cdot {}_yT_l}}{\left({}_aT_l s + e^{-s\cdot {}_yT_l}\right)^2} - \left(\frac{{}_aT_l - {}_yT_l e^{-s\cdot {}_yT_l}}{{}_aT_l s + e^{-s\cdot {}_yT_l}}\right)^2\right)$$

and from it follows

$$\left(\frac{G'(s)}{G(s)}\right)'_{s=0}=\sum_{i=1}^{N_1}\tau_i^2-\sum_{j=1}^{N_2}\left({}_y\tau_j^2-\left({}_a\tau_j-{}_y\,\tau_j\right)^2\right)$$
$$-\sum_{k=1}^{N_3}T_k^2+\sum_{l=1}^{N_4}\left({}_yT_l^2-\left({}_aT_l-{}_y\,T_l\right)^2\right) \tag{47}$$

Hence with respect to (28), (45) and (7) it holds for the moment M_1

$$M_1=\left(\tau_{ar}^2+\sum_{i=1}^{N_1}\tau_i^2-\sum_{j=1}^{N_2}\left({}_y\tau_j^2-\left({}_a\tau_j-{}_y\,\tau_j\right)^2\right)\right.$$
$$\left.-\sum_{k=1}^{N_3}T_k^2+\sum_{l=1}^{N_4}\left({}_yT_l^2-\left({}_aT_l-{}_y\,T_l\right)^2\right)\right)\frac{K}{2} \tag{48}$$

VI. Derivation of Step Responses from Relay Feedback Control

The step response obtained from an open-loop experiment is a convenient way to characterize process dynamics. Many methods for determining parametric models from the step response are based on it, e.g. [2], [17]. The open-loop step test is vulnerable to load disturbances especially for systems with large time constants. As well there are many identification methods based on closed-loop transient response analysis, e.g. [15], [19]. The Ziegler-Nichols frequency response method [23] is one of the most popular and simple methods for characterization of the process dynamics. Probably the more successful part of the Ziegler-Nichols method is not the tuning rule but a way to find the ultimate gain and the ultimate frequency. However, the method is time consuming. The information obtained from the Ziegler-Nichols frequency response method can be also received from the relay feedback test proposed by Åström and Hägglund [1]. However, an important difference is that the relay feedback test is controlled. From one standard relay test, one point on the process frequency response is obtained. This point can be used to calculate controller parameters directly or it is possible use it for system identification, e.g. [13], [14], [18]. This approach was also generalized for a biased relay feedback, e.g. [3], [20].

But if a process model has more than two unknown parameters, it is necessary for system identification to find more points on a frequency response function or to add some other information [3], [9].

The relay feedback control can be also used for determination of the step response. For this purpose a two position symmetrical or asymmetrical (biased) relay, with or without a hysteresis, can be applied.

The block diagram of a process under relay feedback control is shown in Fig. 2. The time courses of the biased relay output u and the process output y are shown in Fig. 3 provided that the system was initially in a steady state and a biased (asymmetrical) relay with a hysteresis (see Fig. 4) was used for control. The variable u changes its values at the time moments t_i, $i\in\mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers.

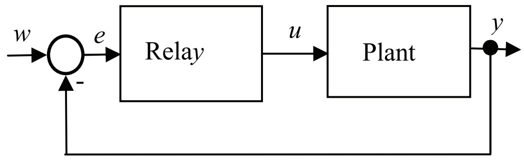

Fig. 2 Block diagram of a plant under relay feedback

The step response function $h(t)$ at the time t can be calculated recursively according to derived original formula (49).

$$h(t)=\begin{cases}\dfrac{y(t)}{u_1}\,,\ \text{for } t\in\langle 0,t_1)\\[2mm] \dfrac{1}{u_1}\left(y(t)+u_C\,\mathrm{sgn}(u_1)\displaystyle\sum_{i=1}^{t_n}(-1)^{i-1}\,h(t-t_i)\right),\\ \text{for } t\in\langle t_n,t_{n+1}),\, n=1,2,3,\cdots\end{cases} \tag{49}$$

where

$$u_1\triangleq u(\tau),\tau\in(0,t_1)\,, \tag{50}$$

$$\mathrm{sgn}(u_1)\triangleq\frac{u_1}{|u_1|}\,, \tag{51}$$

$$u_c=u_A+|u_B|\,. \tag{52}$$

Formula (49) can be also used for a two position symmetrical relay, with or without a hysteresis.

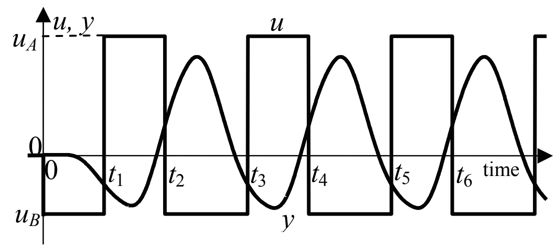

Fig. 3 The biased relay output u and the process output y for a process under relay feedback

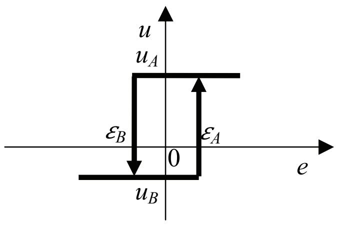

Fig. 4 The static characteristic of a biased relay with a hysteresis

It is obvious that this approach to system identification can be used even for more complicated dynamical systems, where the mentioned recursive calculation is acceptable. Therefore, this approach can be applied even for systems with delays, latencies and after-effects, which can be described by linear anisochronic models.

VII. Example for the Estimation of the Five Parameters of the Anisochronic Model

A plant with the transfer function

$$G_P(s) = \frac{e^{-8s}}{(5s+1)^6} \tag{53}$$

is connected with the relay controller in the closed-loop. The biased relay with hysteresis has following parameters, see Fig. 4:

$$u_A = 2, u_B = -1, \varepsilon_A = 0.5, \varepsilon_B = -0.5\,. \tag{54}$$

The time courses of the biased relay output u and the plant output y are shown in Fig. 5 provided that the system was initially in a steady state.

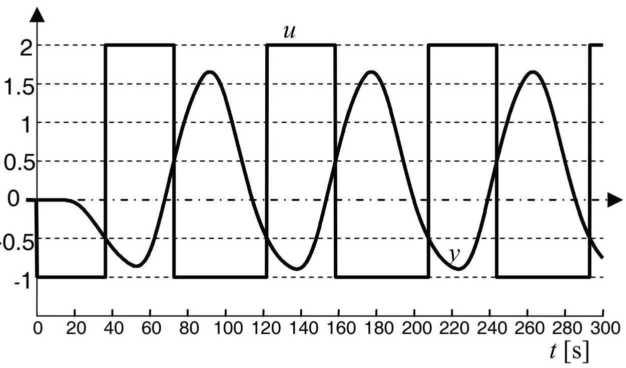

Fig. 5 The time courses of the biased relay output u and the plant output y

The task is to estimate parameters K, τ_u, τ_1, τ_2 and τ_y only from the observation data depicted in Fig. 5 provided that plant transfer function (53) is not known and anisochronic model (55) is used for plant description.

$$G_a(s) = \frac{K \cdot e^{-s\tau_u}}{(\tau_1 s + 1)\left(\tau_2 s + e^{-s\tau_y}\right)} \tag{55}$$

Solution:

The unit step response $h(t)$ can be determined recursively from formula (49) and the observation data depicted in Fig. 5. From Fig. 5 it follows that u_C=3 and the times t_i, i=1,2,3,... correspond to time moments when the manipulated variable changed its value. The obtained unit step response is shown in Fig. 6.

Therefore, the plant steady-state gain

$$K = 1\,. \tag{56}$$

Parameter estimate of anisochronic model (55), with respect to the unit step response $h(t)$, is done here using the method of moments.The method is based on the computation two integrals (32) and (33). The values of these integrals are computed by a numerical integration and for the given example

$$M_0 = 38 \text{ s}, \quad M_1 = 797 \text{ s}^2\,. \tag{57}$$

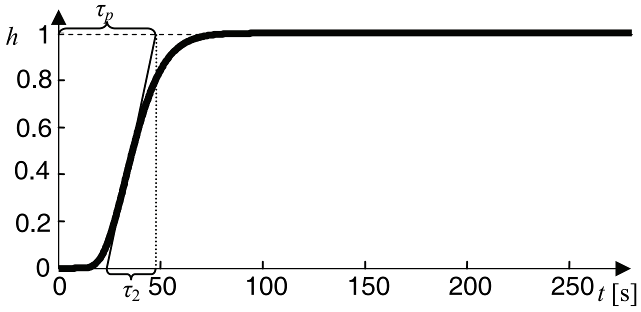

Fig. 6 The unit step response $h(t)$ determined by formula (49)

For anisochronic model (55) holds according to (25) and (31)

$$\frac{M_0}{K} = \tau_{ar}\,, \tag{58}$$

$$\frac{M_1}{K} = \frac{\tau_{ar}^2 + \tau_1^2 + \tau_2^2}{2} - \tau_2 \cdot \tau_y\,, \tag{59}$$

where the average residence time

$$\tau_{ar} = \tau_1 + \tau_2 - \tau_y + \tau_u = \tau_p - \tau_y\,. \tag{60}$$

The time constant τ_2 and the transient time τ_p can be determined from a graphical construction, see Fig. 6, where

$$\tau_p = 48 \text{ s}, \; \tau_2 = 24 \text{ s}\,. \tag{61}$$

Due to (56), (61) and (58) the average residence time

$$\tau_{ar} = 38 \text{ s} \tag{62}$$

and therefore with respect to (60) the delay

$$\tau_y = 10 \text{ s}\,. \tag{63}$$

The time constant τ_1 can be calculated from equation (59)

$$\tau_1 = \sqrt{2 \cdot \frac{M_1}{K} - \left(\tau_{ar}^2 + \left(\tau_2 - \tau_y\right)^2 - \tau_y^2\right)} \doteq 7.4 \text{ s} \tag{64}$$

and the apparent dead time τ_u follows from formula (60)

$$\tau_u = \tau_p - \tau_1 - \tau_2 = 16.6 \text{ s}\,. \tag{65}$$

The transfer function of anisochronic model is then

$$G_a(s) = \frac{e^{-16.6s}}{(7.4s+1)\left(24s + e^{-10s}\right)}\,. \tag{66}$$

The unit step response h_P of the plant with transfer function (53) and the unit step response h_a of model (66) are in Fig. 7.

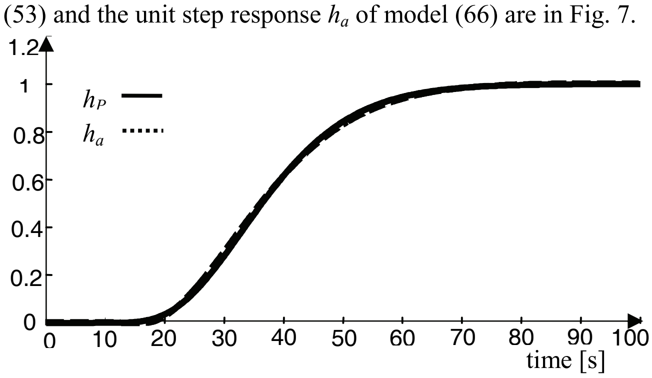

Fig. 7 The unit step response h_P of the plant with transfer function (53) and the unit step response h_a of model (66)

The frequency responses of plant (53) and model (66) are depicted in Fig. 8. Fig. 7 and Fig. 8 show very good conformity the step and frequency responses of identified plant (53) and anisochronic model (66) although the transfer functions $G_P(s)$ and $G_a(s)$ are different.

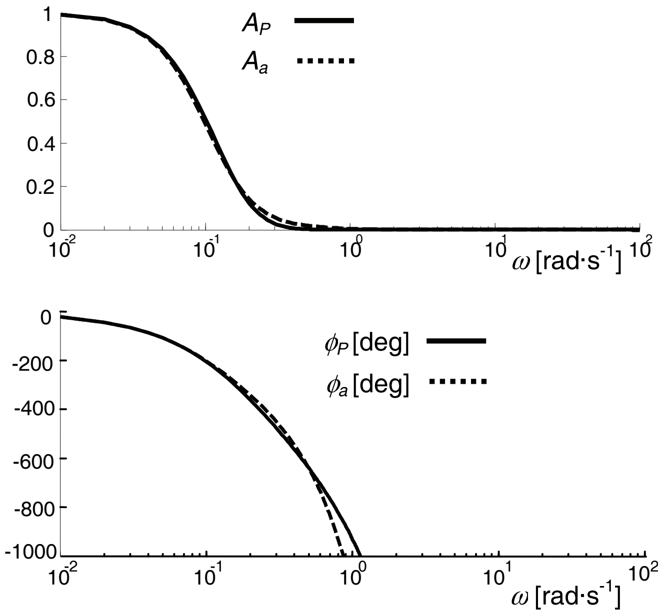

Fig. 8 The frequency responses of plant (53) and model (66), where $A_P(\omega)=|G_P(j\omega)|$, $\phi_P(\omega)=\angle G_P(j\omega)$ and $A_a(\omega)=|G_a(\omega)|$, $\phi_a(\omega)=\angle G_a(j\omega)$.

VIII. Anisochronic controller design

Anisochronic model (1) is able to describe a broad class of time delay systems [24]. In the previous part of this paper it was shown how to estimate the model parameters by experiments with the relay feedback. The following part is devoted to the anisochronic controller design where it is used the Desired Model Method (DMM) [19] and anisochronic model (1).

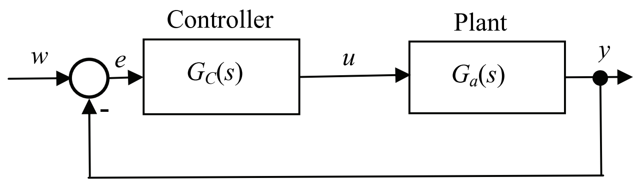

Fig. 9 Closed-loop system

The DMM uses the formula for direct synthesis (see Fig. 9)

$$G_C(s)=\frac{G_{wy}(s)}{G_a(s)\cdot\left(1-G_{wy}(s)\right)}, \quad (67)$$

where $G_C(s)$ is the controller transfer function, $G_a(s)$ is anisochronic model (1) used for plant description, $G_{wy}(s)$ is the desired control system transfer function and it is selected in the form

$$G_{wy}(s)=\frac{k_0}{s+k_0\cdot e^{-\tau_u\cdot s}}e^{-\tau_u s}, \quad (68)$$

k_0 is the open-loop gain. The open-loop transfer function

$$G_0(s)=G_C(s)\cdot G_a(s)=\frac{k_0}{s}e^{-\tau_u s} \quad (69)$$

corresponds to desired control system transfer function (68).

After substitution of anisochronic model (1) to relationship (69) one obtains

$$G_0(s)=G_C(s)\cdot\frac{K\cdot e^{-s\tau_u}}{\left(\tau_1 s+1\right)\left(\tau_2 s+e^{-s\tau_y}\right)}=\frac{k_0}{s}e^{-\tau_u s} \quad (70)$$

hence

$$G_C(s)=\frac{k_0}{K}\cdot\frac{\left(\tau_1 s+1\right)\left(\tau_2 s+e^{-s\tau_y}\right)}{s}. \quad (71)$$

With respect to (68) the characteristic equation of the closed-loop system is

$$s\cdot e^{\tau_u\cdot s}+k_0=0. \quad (72)$$

From it follows (see [4], [24]) that the closed loop system is stable for

$$k_0<\frac{\pi}{2\tau_u}, \quad (73)$$

and over-damped if

$$k_0<\frac{1}{\tau_u\cdot e}, \quad (74)$$

critically damped if

$$k_0=\frac{1}{\tau_u\cdot e}, \quad (75)$$

and under-damped if

$$k_0>\frac{1}{\tau_u\cdot e} \quad (76)$$

where e is Euler's number.

The open-loop gain k_0 can be easily determined analytically [19] assuming that the non-dominant poles and zeros of the control system have a negligible influence on its behaviour. The value of the open-loop gain k_0 can be decided according to

$$k_0=\frac{1}{\beta\cdot\tau_d} \quad (77)$$

where β is the coefficient depending on the relative overshoot κ, see Table I (copy from [19]).

Table I Values of coefficients β for given relative overshoot κ

κ	0	0.05	0.1	0.2	0.3	0.4	0.5
β	2.718	1.944	1.720	1.437	1.248	1.104	0.992

Transfer function (71) is completed by a low-pass filter with a steady-state gain of one to guarantee the physical realizable controller. The transfer function of the controller is then

$$G_C(s)=\frac{k_0}{K}\cdot\frac{\left(\tau_1 s+1\right)\left(\tau_2 s+e^{-s\tau_y}\right)}{\left(\tau_f s+1\right)^r s}, \quad (78)$$

where τ_f is the time constant of the filter and the value of $r\in\mathbb{N}$ can be chosen so that the order of the denominator is at least the same order as the numerator. The value of the time constant τ_f also allows restricting actions of the manipulated variable u.

IX. An Example for Anisochronic Controller Design

In accordance with Fig. 10 a plant with transfer function (53) is described by model (66). In Fig. 10 the disturbance variables d and d_1 are also marked. Design an anisochronic controller according to the DMM

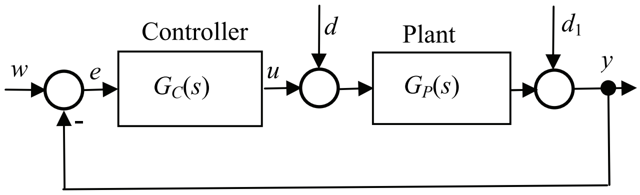

Fig. 10 Block diagram of the control system – Example #2

Solution

The transfer function G_C(s) of the controller can be selected using the DMM with respect to (66) and (78) in the form

$$G_C(s) = k_0 \cdot \frac{(7.4s+1)\cdot\left(24s+e^{-10s}\right)}{(0.5s+1)\cdot s}, \qquad (79)$$

where

$$K=1, \tau_1 = 7.4 \text{ s}, \tau_2 = 24 \text{ s}, \tau_y = 10 \text{ s}, \tau_f = 0.5 \text{ s}, r = 1 . \quad (80)$$

Therefore with respect to Tab. 1, the value of the open-loop gain is

$$k_0 = \frac{1}{\beta\cdot\tau_u} = \frac{1}{1.720\cdot 16.6} = 0.035 \text{ for } \kappa{=}0.1 \qquad (81)$$

$$k_0 = \frac{1}{\beta\cdot\tau_u} = \frac{1}{e\cdot 16.6} = \frac{1}{2.718\cdot 16.6} = 0.022 \text{ for } \kappa{=}0. \quad (82)$$

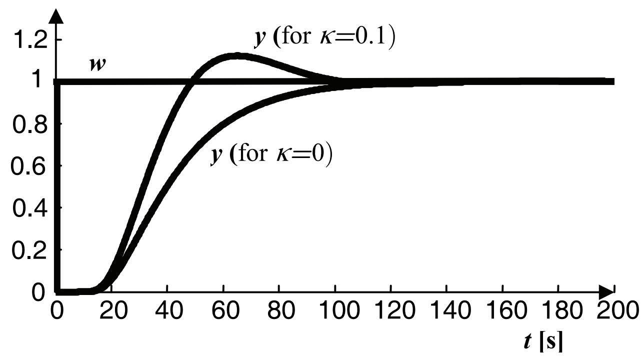

Fig. 11 Set point response for anisochronic control

Fig. 11 shows the closed loop control for the unit step of the desired variable w and the required relative overshoots κ=0.1 and κ=0. The disturbance variables d and d_1 have zero value.

Fig. 12 shows the closed loop control for the unit step of the disturbance variable d and the required relative overshoots κ=0.1 and κ=0. The other variables w and d_1 have zero value.

Fig. 13 shows the closed loop control for the unit step of the disturbance variable d_1 and the required relative overshoots κ=0.1 and κ=0. The other variables w and d have zero value.

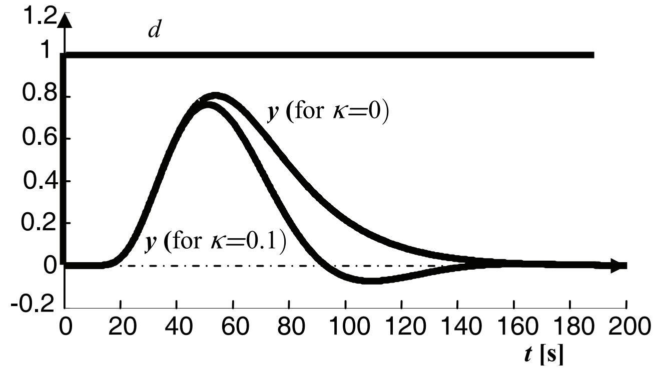

Fig. 12 Response of the control system to unit step change in the disturbance variable d

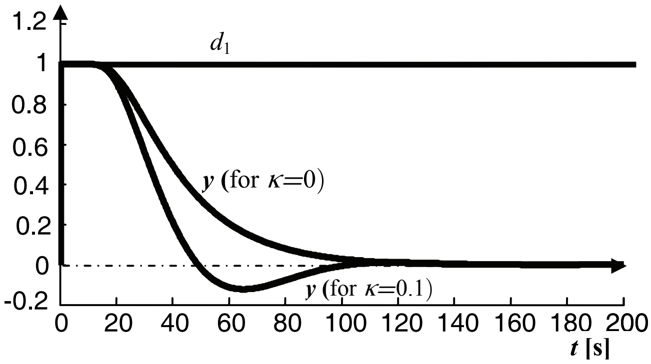

Fig. 13 Response of the control system to unit step change in the disturbance variable d_1

X. Conclusion

This paper presents an original approach to parameter estimation of anisochronic model (1) that is suitable for the description of most industrial processes. For this purpose using the moment method, the simple computational formulas were derived for parameter estimation of anisochronic model (1) from a step response. The practical applicability of this approach is demonstrated by the example. The formulas were also derived for determining the integrals M_0 and M_1 of more general anisochronic model (41).

The presented method for plant step response identification has been developed in the context of a relay feedback test. The method has several unique features. It can estimate the whole step response of a plant with one single relay experiment. For this purpose no approximation is made. The involved computations are simple so that it can be easily implemented on micro-processors. The method allows estimating more parameters of mathematical models with various structures. It can therefore be used for parameter tuning both isochronic and anisochronic models. It was demonstrated on one anisochronic model in estimating five parameters. The presented identification method requires zero initial conditions and due to recursive calculation, it is sensitive to noise which can be corrected by filtration.

As the aim of the identification was to design a practicably applicable controller, therefore, the anisochronic controller was derived to broad class systems describable by anisochronic model (1) using the Desired Model Method.

REFERENCES

[1] K., J. Åström and T. Hägglund, "Automatic tuning of simple regulators with specifications on phase and amplitude margins,"*Automatica,* vol. 20, issue 5, 1984, pp. 645-651.

[2] K., J. Åström and B. Wittenmark, *Adaptive Control.* Addison-Wesley Publishing Company, Inc., 1995, ch. 8.

[3] Q. Bi, Q. G. Wang a nd C. C. Hang, "Relay-based estimation of multiple points on process frequency response," *Automatica*, vol. 33, no. 9, 1997, pp. 1753-1757.

[4] J. Hlava, Anisochronic Internal Model Control of Time Delay Systems,Ph. D. Thesis, CTU FME, Prague, 1998.

[5] M. Hofreiter, "Parameter Identification of Anisochronic Models*", DAAAM International Scientific Book 2002*. Vienna: DAAAM International, Editor B. Katalanic, 2002, pp. 247-252.

[6] M. Hofreiter, "Discretization of continuous linear anisochronic models," *Studies in Informatics and Control*, vol. 12, no. 1, 2003, pp. 69-76.

[7] M. Hofreiter, "Parameter Fitting for Anisochronic Models by Means of Moments," *Int. J.. of Mathematical Models and Methods in Applied Sciences,* vol. 8, no. 4, 2014, pp. 394-400. Available: http://www.naun.org/main/NAUN/ijmmas/2014/a262001-041.pdf

[8] P. Klán and R. Gorez, *Process Control*, FSS Public s.r.o., Prague, 2011.

[9] T. H. Lee, Q. G. Wang and K. K. Tan, "Knowledge-based process identification from relay feedback," *J. Proc.Control,* vol. 5, no. 6, 1995, pp. 387-397.

[10] L. Ljung, System Identification-Theory for the User, 2nd ed., PTR Prentice Hall, Upper Sadle River, N. J., 1999.

[11] M. Malek-Zavarei and M. Jamshidi, *Time Delay Systems.Analysis, Optimization and Applications*. North-Holland Systems and Control Series vol. 9. Amsterdam: North Holland, 1987.

[12] M. Martinásková, "Control of hereditary DIDO systems using anisochronic models, " Thesis in Czech, CTU in Prague, FME, 2000.

[13] R. Prokop, J. Korbel and O. Líška, "A novel principle for relay-based autotuning," *Int. Journal of Mathematical Models and Methods in Applied Science,*" vol. 5, no. 7, 2011, pp. 1180-1188. Available: http://www.wseas.us/e-library/conferences/2011/Lanzarote/ACMOS/ACMOS-30.pdf

[14] R. Prokop, L. Pekař, R. Matušů and J. Korbel, "Autotuning principles for delayed systems", *Int. J. of Mathematical Models and Methods in Applied Sciences,* vol. 6, no. 2, 2012, pp. 273-280. Available: http://naun.org/main/NAUN/ijmmas/17-738.pdf

[15] R. Prokop, J. Korbel and R. Matušů,"Relay feedback identification of dynamical SISO systems-analysis and settings," in *Proc. of the 18th Int. Conf. on Systems - Latest Trends on Systems (part of CSCC'14).* Santorini Island, 2014*,* pp. 450-454.

[16] J. P. Richard, "Time-delay systems: An overview of some recent advances and open problems," *Automatica*, vol. 39, no. 10, pp. 1667-1694, 2003.

[17] D. E. Seborg, T.F. Edgar, and D. A. Mellichamp, *Process Dynamics and Control*. John Wiley & Sons, Inc., 2011.

[18] M. H. Schwarz, C. S. Cox and J. Börcsök, "A controller design strategy for closed Loop Identification," in *Proc. of the 12th WSEAS Int. Conf. CSECS '13,* Budapest, 2013, pp. 42-48. Available: http://www.wseas.us/e-library/conferences/2013/Budapest/CSECS/CSECS-04.pdf

[19] A. Víteček and M. Vítečková, *Closed-Loop Control of Mechatronic Systems*. Ostrava: VŠB TU Ostrava, 2013, pp. 131-141.

[20] M. Vítečková, and A. Víteček, "Plant identification by relay methods," *Engineering the future* (edited by L. Dudas). Sciyo, Rijeka, 2010, pp. 242-256. Available: http://www.intechopen.com/books/engineering-the-future/plant-identification-by-relay-method

[21] T. Vyhlídal, P. Zitek, "Control System Design Based on Universal First Order Model with Time Dealys," *Acta Polytechnica*, vol. 41, no. 4-5, 2001, pp. 49-53.

[22] C. C. Yu, *Autotuning of PID Controllers.* London: Springer-Verlag, 1999, ch. 2 and 3.

[23] J. G. Ziegler and N. B. Nichols, "Optimum settings for automatic controllers," Trans. ASME, vol. 65, pp. 433-444

[24] P. Zítek, "Time Delay Control System Design Using Functional State Models," *CTU Reports*, CTU Prague, no. 1, 1998.

[25] P. Zítek and J. Hlava, "Anisochronic Internal Model Control of Time Delay Systems," *Control Engineering Practice*, vol. 9*,* no. 5, 2001, pp. 501-516.

[26] P. Zítek and V. Kučera, "Algebraic Design of Anisochronic Controllers for Time Delay Systems," *International Journal of Control*, vol. 76, no. 16, 2003, pp. 1654-1665.