On hamiltonian decomposition of direct graph bundle

Irena Hrastnik Ladinek and Janez Žerovnik

***Abstract*—**Hamiltonian decomposition of direct graph bundles is studied. Based on the recent proof of hamiltonicity of all connected direct graph bundles over hamiltonian base and hamiltonian fibres, we conjecture that all direct graph bundles with fibres and base graphs being hamiltonian decomposable also admit a hamiltonian decomposition. The conjecture is proved for direct bundles over cycles when the nontrivial automorphism is any reflection. We also prove that direct graph bundles with α a cyclic shift when the base cycle is even admit a hamiltonian decomposition. In the case of odd base cycle we look at one partical situation where we can construct a hamiltonian decomposition.

***Keywords*—**circulant 2-digraph, cyclic ℓ-shift, direct graph product, direct graph bundle, hamiltonian graph, hamiltonian decomposition, reflection.

I. Introduction

STUDIES of hamiltonian properties of graphs are among the fundamental topics in graph theory [10],[24]. Besides being related to some famous historical problems (Icosian game, chessboard puzzles, etc.) it has important practical applications. For example, in computer science, hamiltonicity and existence of hamiltonian decomposition are important properties of computer and communication network topologies. Furthermore, the traveling salesman problem [32] which is the most studied problem in combinatorial optimization asks for a minimal hamiltonian cycle in edge weighted graph. There is no efficient algorithm for deciding whether a graph is hamiltonian or not. (More precisely, as the problem is NP-complete, it is believed that there is no polynomial algorithm.) Therefore it is interesting to ask, given a subclass of graphs, whether the problem may be solved efficiently by designing a polynomial algorithm or by providing a characterization of hamiltonian graphs within the subclass. Graph products are one of the natural constructions giving more complex graphs from simple ones. Graph bundles, sometimes also called twisted products, are a generalization of product graphs, which have been (under various names) frequently used as computer topologies or communication networks, see for example [6]. A famous example is the ILIAC IV supercomputer [8]. While hamiltonian properties of the cartesian products are well studied, there is much less known on hamiltonian properties of direct products and bundles. The reason may be that the direct product has some, on the first sight not convenient properties. For example, the direct product of connected graphs is not necessarily connected.

The direct product is one of the (four) most important graph products. It is in some sense the most natural graph product as it can be viewed as the product in the category of graphs. The product was used by Greenwell and Lovasz [21] to demonstrate that for all $n \leq 3$, there is a uniquely n-colorable graph without odd cycles shorter than a given number s. Whether a product of hamiltonian decomposable graphs is itself hamiltonian decomposable has been an object of study for a long time. For example, Barayani and Szasz [7] showed that this problem admits of an affirmative answer with respect to the lexicographic product. Jha [22] proved that if the number of factor graphs of even order is at most one, then the direct product admits a hamiltonian decomposition and, if the number of factor graphs which are bipartite is at least two and the remaining factor graphs are all of odd order, then the direct product consists of isomorphic components each of which admits a hamiltonian decomposition.

Our less general motivation for this research is the following. It is well-known that the Cartesian product of two hamiltonian graphs is hamiltonian, and therefore it is interesting to investigate conditions under which the product is hamiltonian if at least one of the factors is not hamiltonian. In 1982, Batagelj and Pisanski [9] proved that the Cartesian product of a tree T and a cycle C_n has a hamiltonian cycle if and only if $n \geq \Delta(T)$, where $\Delta(T)$ denotes the maximum vertex degree of T. They introduced the cyclic hamiltonicity $cH(G)$ of graph G as the smallest integer n for which the Cartesian product of cycle C_n and G is hamiltonian. More than twenty years later, Dimakopoulos, Palios and Paulakidas [14] proved that $cH(G) \leq \mathscr{D}(G) \leq cH(G)+1$, as conjectured already in [9]. (Here $\mathscr{D}(G)$ denotes the minimum

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of $\Delta(T)$ over all spanning trees T of G.) These results can be extended in a certain way to Cartesian graph bundles, see [34] and [28].

It is natural to ask whether similar theory may be developed for other graph products. In the case of direct product, the question of hamiltonicity seems to be much more complicated, because even the direct product of two cycles is not necessarily hamiltonian ([23] gives a characterization which direct products of hamiltonian graphs are hamiltonian). For example, the direct product of two even cycles is not connected so it can not be hamiltonian. Furthermore, the product of a tree (on at least 3 vertices) with any graph is not hamiltonian, However, the direct graph bundle with even cycles as base and as fiber can be connected. Given two hamiltonian graphs F and B, a complete characterization of cases in which a direct graph bundle with fibre F over base B is hamiltonian is given in [29].

In this paper, we study hamiltonian decomposition of direct graph bundles of cycles over cycles. Based on the recent proof of hamiltonicity of all connected direct graph bundles over hamiltonian base and hamiltonian fibres, we conjecture a characterization of direct graph bundles that admit a hamiltonian decomposition. The conjecture is proved for some special cases including direct bundles over cycles when the nontrivial automorphism is any reflection.

The rest of the paper is organized as follows. In Section 2, terminology and notation is introduced. Some basic observation regarding bundles over K_2 are given in Section 3. In Section 4, results on connectedness of direct graph bundles are recalled. Based on a known result on hamiltonicity of graph bundles, two conjectures are stated in Section 5. In Section 6, Conjecture 12 is proved for a special case when the nontrivial automorphism of the bundle is any reflection. In Section 7, Conjecture 12 is considered in the special case when the nontrivial automorphism of the bundle is a cyclic shift. A summary of results and open questions is given in the last section. This paper is an extended version of the conference version that appears in [1].

II. Terminology and notation

A finite, simple and undirected graph $G = (V(G), E(G))$ is given by a set of vertices $V(G)$ and a set of edges $E(G)$. As usual, the edges $\{i, j\} \in E(G)$ are shortly denoted by ij. Although here we are interested in undirected graphs, the order of the vertices will sometimes be important, for example when we will assign automorphisms to edges of the base graph. In such case we assign two opposite arcs $\{(i, j), (j, i)\}$ to edge $\{i, j\}$.

Two graphs G and H are called *isomorphic*, in symbols $G \simeq H$, if there exists a bijection φ from $V(G)$ onto $V(H)$ that preserves adjacency and nonadjacency. In other words, a bijection $\varphi : V(G) \to v(H)$ is an *isomorphism* when: $\varphi(i)\varphi(j) \in E(H)$ if and only if $ij \in E(G)$. An isomorphism of a graph G onto itself is called an *automorphism*. The identity automorphism on G will be denoted by id_G or shortly id.

The *cycle* C_n on n vertices is defined by $V(C_n) = \{0, 1, \ldots, n-1\}$ and $ij \in E(C_n)$ if $i = j \pm 1 \bmod n$. Denote by P_n the *path* on $n \geq 1$ distinct vertices $0, 1, 2, \ldots, n-1$ with edges $ij \in E(P_n)$ if $j = i+1, 0 \leq i < n-1$. (Note that a subgraph isomorphic to the path graph is also called path.)

An arbitrary connected graph G is said to be *hamiltonian* if it contains a spanning cycle (called a hamiltonian cycle).

Let G and H be connected graphs. The *direct product* of graphs G and H is the graph $G \times H$ with vertex set $V(G \times H) = V(G) \times V(H)$. Edges of $G \times H$ are all pairs $(g_1, h_1)(g_2, h_2)$ with $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. Other names for the direct product are [27]: Kronecker product, categorical product, tensor product, cardinal product, relational product, conjunction, weak direct product or just product and even Cartesian product. The direct product of graphs is commutative and associative in a natural way. For more facts on the direct product of graphs and other graph products we refer to [27].

Let B and G be graphs and $\mathrm{Aut}(G)$ be the set of automorphisms of G. To any ordered pair of adjacent vertices $u, v \in V(B)$ we will assign an automorphism of G. Formally, let $\varphi : V(B) \times V(B) \to \mathrm{Aut}(G)$. For brevity, we will write $\varphi(u, v) = \varphi_{u,v}$ and assume that $\varphi_{v,u} = \varphi_{u,v}^{-1}$ for any $u, v \in V(B)$. Now we construct the graph X as follows. The vertex set of X is the Cartesian product of vertex sets, $V(X) = V(B) \times V(G)$. The edges of X are given by the rule: for any $b_1b_2 \in E(B)$ and any $g_1g_2 \in E(G)$, the vertices (b_1, g_1) and $(b_2, \varphi_{b_1,b_2}(g_2))$ are adjacent in X. We call X a *direct graph bundle* with base B and fibre G and write $X = B \times^{\varphi} G$.

Clearly, if all $\varphi_{u,v}$ are identity automorphisms, the graph bundle is isomorphic to the direct product $X = B \times^{\varphi} G = B \times G$. Furthermore, it is well-known that if the base graph is a tree, then the graph bundle is always isomorphic to a product, i.e. $X = T \times^{\varphi} G \simeq T \times G$ for any graph G, any tree T and any assignment of automorphisms φ [35],[36]. Furthemore, a graph bundle over a cycle can always be constructed in a way that all but at most one automorphism are identities. Fixing $V(C_n) = \{0, 1, 2, \ldots, n-1\}$, we denote $\varphi_{n-1,0} = \alpha$, $\varphi_{i-1,i} = id$ for $i = 1, 2, \ldots, n-1$, and write $C_n \times^{\alpha} G = C_n \times^{\varphi} G$. In this article we will frequently use this fact.

Let $F = K_2$ and $B = C_3$. On Figure 1 we see two nonisomorphic bundles with fibre F over the base graph B. Informally, one can say that bundles are "twisted products". The right figure shows cartesian graph bundle, which can be interpreted as a discrete analogy of Moebius strip, which is topological bundle [38],[45].

It is less known that graph bundles also appear as computer topologies. A well known example is the twisted torus

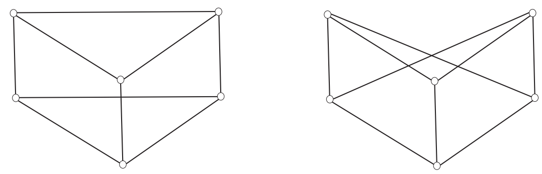


Figure 1: Nonisomorphic bundles with fibre K_2 over base C_3.

on Fig. 2. Cartesian graph bundle with fibre C_4 over base C_4 is the ILLIAC IV architecture [8], a famous supercomputer that inspired some modern multicomputer architectures. It may be interesting to note that the original design was a graph bundle with fibre C_8 over base C_8, but due to high cost a smaller version was build [49].

In fact, graph products and bundles are among frequently studied interconnection network topologies. For example the meshes, tori, hypercubes and some of their generalizations are Cartesian products. It is less known that some well-known topologies are Cartesian graph bundles, i.e. some twisted hypercubes [13],[15] and multiplicative circulant graphs [39]. Other graph products, sometimes under different names, have been studied as interesting network topologies [12],[33],[39]. Among the interesting models for the design of large reliable networks is the product graph [11],[47] which generalizes several other well known graph constructions including Cartesian graph products and bundles, but also for example the permutation graphs. Graph invariants of interest in computer science, including connectivity, wide diameter, the (vertex) fault diameter and the edge fault diameter and mixed fault diameter of graph products and graph bundles were studied recently [2],[3],[4],[5],[6],[16],[17],[18],[19],[20],[25],[30], [31], [41],[37],[42],[43],[46]. For more references see survey [48].

III. Bundles over K_2

Automorphisms of a cycle are of two types. A cyclic shift of the cycle by ℓ elements or briefly *cyclic ℓ-shift*, $0 \le \ell < n$, maps u_i to $u_{i+\ell}$ (index modulo n). As a special case we have the identity ($\ell = 0$). Other automorphisms of cycles are *reflections*. If C_n is a cycle on odd number of vertices, then all the reflections have exactly one fixed point. If the number n is even, then we have reflections without fixed points and reflections with two fixed points.

More formally, we define:

- **cyclic ℓ-shift** σ_ℓ, defined as $\sigma_\ell(i) = i + \ell$ for $i = 0, 1, \ldots, n-1$. (Recall the convention $i + \ell = (i + \ell) \bmod n$.)
- **reflection with no fixed points** ρ_0, defined as $\rho_0(i) = n - i - 1$ for $i = 0, 1, \ldots, n-1$. (For n even there is no fixed points.)
- **reflection with one fixed point** ρ_1, defined as $\rho_1(i) = n - i - 1$ for $i = 0, 1, \ldots, n-1$. (For n odd, there is exactly one fixed point, $\rho_1(\frac{n-1}{2}) = n - \frac{n-1}{2} - 1 = \frac{n-1}{2}$.)
- **reflection with two fixed points** ρ_2, defined as $\rho_2(0) = 0$ and $\rho_2(i) = n - i$ for $i = 1, 2, \ldots, n-1$. (For n even, there is the second fixed point $\rho_2(\frac{n}{2}) = n - \frac{n}{2} = \frac{n}{2}$.)

We first recall that the graph bundle $P_2 \times^\alpha C_t$ is either isomorphic to one or to two cycles. (See also Figures 3 and 4.)

Lemma 1 *[29] The direct graph bundle $P_2 \times^\alpha C_t$ for odd t is isomorphic to the cycle C_{2t} for every automorphism α of C_t. If t is even, then for every automorphism α of C_t the graph bundle $P_2 \times^\alpha C_t$ has two connected components that are isomorphic to C_t.*

Let us note that clearly the lemma also applies to the product (case $\alpha = id$).

Remark 2 *$P_2 \times C_t \simeq C_{2t}$ if t is odd and $P_2 \times C_t \simeq 2C_t$ if t is even.*

For a later reference define the two cycles of $P_2 \times^\alpha C_t$ for even t as follows:

Definition 3 *Let t be even. Let $C_t^{(0)}$ be the component of $P_2 \times^\alpha C_t$ containing the vertex $(0,0) \in V(P_2 \times^\alpha C_t)$ and $C_t^{(1)}$ be the component of $P_2 \times^\alpha C_t$ containing the vertex $(0,1) \in V(P_2 \times^\alpha C_t)$.*

Let us write explicitly the vertex sets that induce the cycles $C_t^{(0)}$ and $C_t^{(1)}$. Denote the subsets of odd and even vertices of C_t by $W_1 = \{1, 3, \ldots, 2\lceil \frac{t-1}{2} \rceil - 1\}$ and $W_0 = \{0, 2, 4, \ldots, 2\lfloor \frac{t-1}{2} \rfloor\}$, respectively. Hence $V(C_t) = W_0 \cup W_1$, and recall that $V(P_2) = \{0, 1\}$. From the proof of Lemma 1 the next two remarks directly follow.

Remark 4 *Let t be even and α be identity, an even shift or reflection ρ_2. Then $V(C_t^{(0)}) = Z_0 = (\{0\} \times W_0) \cup (\{1\} \times W_1)$ and $V(C_t^{(1)}) = Z_1 = (\{0\} \times W_1) \cup (\{1\} \times W_0)$.*

Remark 5 *Let t be even and α be an odd shift or reflection ρ_0. Then $V(C_t^{(0)}) = \{0, 1\} \times W_0$ and $V(C_t^{(1)}) = \{0, 1\} \times W_1$.*

IV. Connectdness of direct graph bundles

The fact that the direct product $G \times H$ of connected and bipartite factors G and H has exactly two components was first proved by Weichsel [40]. Hence if G and H are bipartite, then $G \times H$ can not be hamiltonian. In particular, the direct product $C_s \times C_t$, where C_s and C_t are even cycles, is not connected and hence not hamiltonian.

Below we summarize the necessary and sufficient conditions for connectedness of a direct graph bundle $C_s \times^\alpha C_t$ and

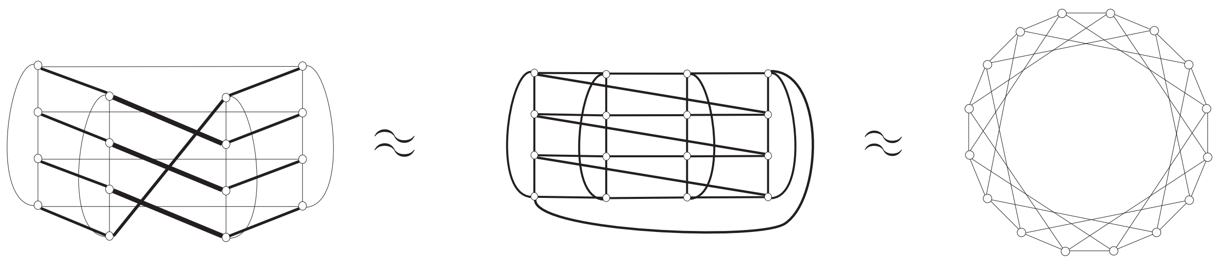


Figure 2: Twisted torus: Cartesian graph bundle with fibre C_4 over base C_4.

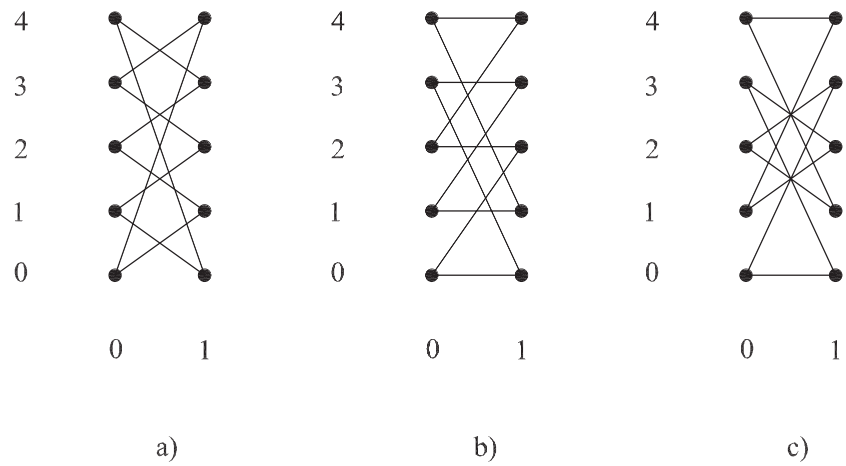


Figure 3: The direct graph bundles $P_2 \times^{\alpha} C_5$: a) $\alpha = id$, b) $\alpha = \sigma_1$ and c) $\alpha = \rho_1$

for graph bundles with fibre C_t over arbitrary connected base graph B. The case when t is odd is easier and is considered first.

Lemma 6 *Let C_t be a cycle on t vertices, where t is odd. Then $B \times^{\alpha} C_t$ is connected for every connected base graph B.*

Proof: Follows directly from Lemma 1. ■

As $B = C_s$ is just a special case of interest, we can write

Corollary 7 *Let t be odd. The direct graph bundle $C_s \times^{\alpha} C_t$ is connected for every automorphism $\alpha \in \mathrm{Aut}(C_t)$.*

The case when C_t for even t is more interesting. The results from [29] are summarized in Theorem 8 and in Table 1.

Theorem 8 *The direct graph bundle $C_s \times^{\alpha} C_t$ with fiber C_t and base C_s, $s,t \geq 3$, is connected:*

1. *when t is odd, for any automorphism $\alpha \in \mathrm{Aut}(C_t)$.*
2. *when t is even and s is odd, if and only if α is identity, even cyclic ℓ-shift or reflection with two fixed points ρ_2.*
3. *when t is even and s is even, if and only if α is odd cyclic ℓ-shift or reflection without fixed points ρ_0.*

Otherwise, $C_s \times^{\alpha} C_t$ is not connected.

Table 1: Connected direct graph bundles $C_s \times^{\alpha} C_t$

t odd	for any automorphism α of C_t	
t even	s odd	**List 1, $\mathscr{L}_1$:** $\alpha = id$ $\alpha = \sigma_\ell$, ℓ is even $\alpha = \rho_2$
	s even	**List 2, $\mathscr{L}_2$:** $\alpha = \sigma_\ell$, ℓ is odd $\alpha = \rho_0$

We conclude the section by recalling a necessary and sufficient condition for connectedness of a graph bundle with connected base B and fibre C_t for even t.

Theorem 9 *[29] Let X be a direct graph bundle with fiber C_t and connected base. If C_t is an odd cycle, then X is con-*

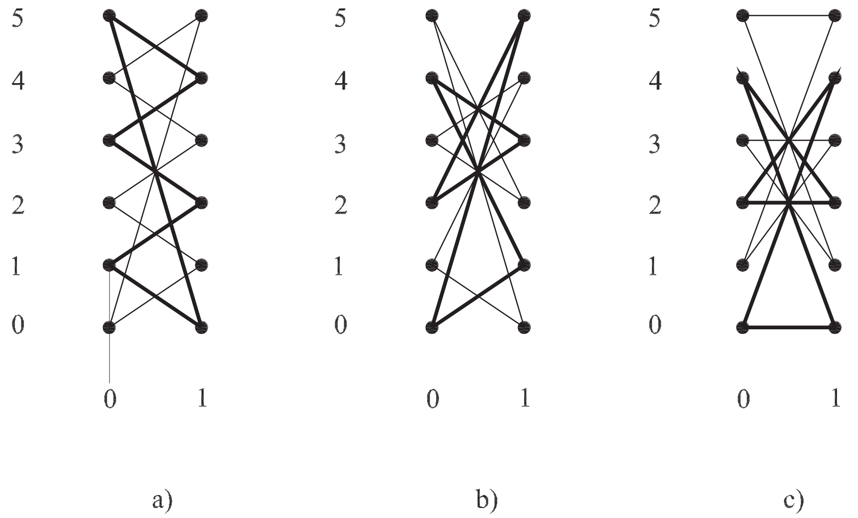


Figure 4: The direct graph bundles $P_2 \times^{\alpha} C_6$: a) $\alpha = id$, b) $\alpha = \rho_2$ and c) $\alpha = \rho_0$

nected. If C_t is an even cycle, then X is connected if and only if there is a cycle $C = v_1 v_2 \dots v_k$ in B such that either

- *$|V(C)| = k$ is odd and $\alpha = \sigma_{v_k,v_1} \circ \sigma_{v_{k-1},v_k} \circ \dots \circ \sigma_{v_2 v_3} \circ \sigma_{v_1 v_2}$ is an automorphism from $\mathscr{L}_1$, or*
- *$|V(C)| = k$ is even and $\alpha = \sigma_{v_k,v_1} \circ \sigma_{v_{k-1},v_k} \circ \dots \circ \sigma_{v_2 v_3} \circ \sigma_{v_1 v_2}$ is an automorphism from $\mathscr{L}_2$.*

V. Hamiltonicity and a conjecture on hamiltonian decomposition of direct graph bundles

In [29] it is shown that

Theorem 10 *Let B and F be hamiltonian graphs, with $t = |V(F)|$ odd. Then any direct graph bundle X with fiber F and base graph B is hamiltonian.*

The proof is based on

Theorem 11 *Let $X = C_s \times^{\alpha} C_t$ be a direct graph bundle with fibre C_t and base C_s. Then X is hamiltonian if and only if X is connected.*

This indicates that it is crucial to study graph bundles with base graphs and fibres being cycles. We conjecture that all such graphs have hamiltonian decomposition:

Conjecture 12 *Let $X = C_s \times^{\alpha} C_t$ be a direct graph bundle with fibre C_t and base C_s. Then X has a hamiltonian decomposition if and only if X is connected.*

The proof would imply correctness of a more general statement:

Conjecture 13 *Let B and F be hamiltonian graphs that have a hamiltonian decomposition, and $t = |V(F)|$ odd. Then any direct graph bundle X with fiber F and base graph B admits a hamiltonian decoposition if and only if X is connected.*

Conjecture 13 follows from Conjecture 12 easily. The idea of argument is as follows. Denote by $B_1, B_2, \dots, B_a$ the subgraphs of B corresponding to a hamiltonian cycles of a decomposition of B and by $F_1, F_2, \dots, F_b$ the subgraphs of F corresponding to b hamiltonian cycles of a decomposition of F. Each pair B_i and F_j ($i \in \{1,2,\dots,a\}$, $j \in \{1,2,\dots,b\}$) gives rise to a spanning subgraph X_{ij} of X. It can be shown that $E(X)$ is the disjoint union of all X_{ij}. X_{ij} are graph bundles with both base and fibre being cycles and X_{ij} are connected because $t = |V(F_j)| = |V(F)|$ is odd. Thus the hamiltonian decompositions of X_{ij} together form a hamiltonian decomposition of X.

The next two sections we provide proofs (constructions) of several special cases including all possible reflections thus providing a partial proof of Conjecture 12.

VI. Hamiltonian decomposition of direct graph bundles - reflections

In this section we give hamiltonian decompositions for connected graph bundles of cycles over cycles where the nontrivial automorphism is a reflection. The four propositions treat cases according to parity of the lengths of cycles, s and t. In all cases we denote hamiltonian cycles by $\mathscr{H}_1$ and $\mathscr{H}_2$.

Proposition 14 *Let C_s, C_t be two cycles, where $s,t \geq 3$ and s is odd and t even. Let $\alpha = \rho_2$ be reflection with two fixed points. Then $C_s \times^{\alpha} C_t$ admits a hamiltonian decomposition.*

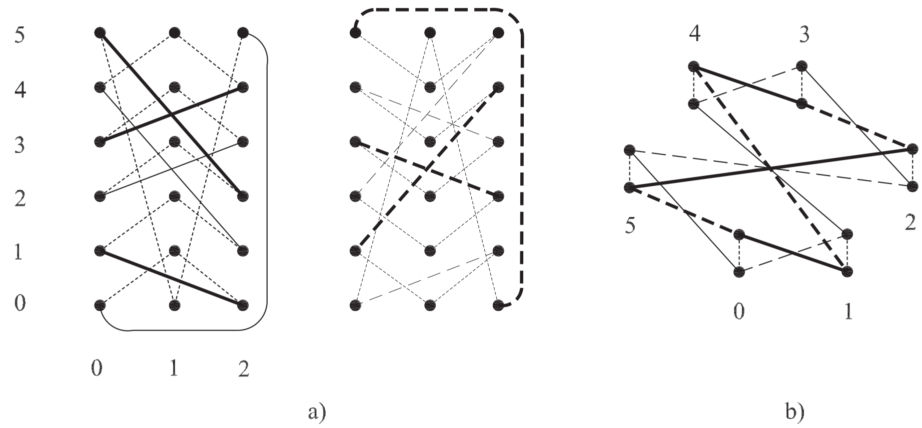


Figure 5: Hamiltonian cycles in the direct graph bundle $C_3 \times^{\rho_2} C_6$ (a), and the cycles $C_6^{(0)}$, $C_6^{(1)}$ (b).

Proof: The hamiltonian cycle $\mathscr{H}_1$ is constructed as follows. Form t disjoint paths of length $s-1$ from $(0,j)$ to $(s-1,j)$, $j=0,1,\dots,t-1$, by taking (for example) edges $(i,j)(i+1,(j+1)\bmod t))$ for even i and edges $(i,j)(i+1,(j-1)\bmod t)$ for odd i (and $j=0,1,\dots,t-1$). The edges between fibres $s-1$ and 0 are chosen from $C_t^{(0)}$: $(0,i)(1,\rho_2(i+1)), i \in W_0$, and from $C_t^{(1)}$: $(0,i)(1,\rho_2(i-1)), i \in W_1$, or, equivalently, from $C_t^{(0)}$: $(0,i)(1,\rho_2(i)-1), i \in W_0$, and from $C_t^{(1)}$: $(0,i)(1,\rho_2(i)+1), i \in W_1$.

Recall the partition of edges of $P_2 \times^{\rho_2} C_t$ from Remark 4), see Figure 5.) The claim that these edges form a hamiltonian cycle is easy to check, for example by observing that the edges $(0,i)(1,\rho_2(i)-1), i \in W_0$, and $(0,i)(1,\rho_2(i)+1), i \in W_1$, give rise to a permutation of the set $\{0,1,\dots,t-1\}$ with one cycle. We omit the details.

Observe that the edges not on the cycle just constructed form another hamiltonian cycle. It can be constructed by analogous construction: form t disjoint paths of length $s-1$ from $(0,j)$ to $(s-1,j)$, $j=0,1,\dots,t-1$, by taking edges $(i,j)(i+1,(j-1)\bmod t))$ for even i and edges $(i,j)(i+1,(j+1)\bmod t)$ for odd i and $j=0,1,\dots,t-1$. Observe that these edges are disjoint with edges of $\mathscr{H}_1$.

To obtain the second hamiltonian cycle $\mathscr{H}_2$ we must connect these paths by using edges from $C_t^{(0)}$ and $C_t^{(1)}$ which were not used in the construction of the first cycle. These are: from $C_t^{(0)}$: $(0,i)(1,\rho_2(i)+1), i \in W_0$, and from $C_t^{(1)}$: $(0,i)(1,\rho_2(i)-1), i \in W_1$.

By construction, cycles $\mathscr{H}_1$ and $\mathscr{H}_2$ are disjoint, and thus form a hamiltonian decomposition of $C_s \times^{\alpha} C_t$. ■

Proposition 15 *Let C_s, C_t be two cycles, where $s,t \geq 3$ and both s and t are even. Let $\alpha = \rho_0$ be reflection without fixed points. Then $C_s \times^{\alpha} C_t$ admits a hamiltonian decomposition.*

Proof: The subgraph induced on two consecutive fibres i and $i+1$ (for $i=0,1,\dots,s-2$) has two connected components (the first on the vertices from Z_0 and the second on the vertices from Z_1) that are isomorphic to C_t. One of this cycles contains the edge $(i,\frac{t}{2})((i+1)\bmod s, \frac{t}{2}-1)$, the other the edge $(i,\frac{t}{2}-1)((i+1)\bmod s, \frac{t}{2})$.

Deleting edges $(i,\frac{t}{2})((i+1)\bmod s, \frac{t}{2}-1)$ and $(i,\frac{t}{2}-1)((i+1)\bmod s, \frac{t}{2})$ thus gives two disjoint paths, that span all vertices (and all edges except the two deleted) of fibres i and $i+1$.

Furthermore, the subgraph induced on fibres $s-1$ and 0 has two connected components that are isomorphic to C_t, by Lemma 1. The first is induced by the vertices of $\{s-1,0\} \times W_0$, the second by the vertices of $\{s-1,0\} \times W_1$, by Remark 5. Two disjoint paths that span all vertices (and all edges but two) of fibres $s-1$ and 0 can be constructed by deleting the edges $(s-1,\frac{t}{2})(0,\frac{t}{2})$ and $(s-1,\frac{t}{2}-1)(0,\frac{t}{2}-1)$ (because $\rho_0(\frac{t}{2}-1)=\frac{t}{2}$ and $\rho_0(\frac{t}{2})=\frac{t}{2}-1$).

Hamiltonian cycles on $C_s \times^{\alpha} C_t$ are constructed as follows. In first case we take the two spanning paths on each of the pairs of fibres 0 and 1, 2 and 3,$\dots$, $s-2$ and $s-1$. In second case we take the two spanning paths on each of the pairs of fibres 1 and 2, 3 and 4,$\dots$, $s-1$ and 0. In the first case add the edges $(i,\frac{t}{2}-1)((i+1)\bmod s, \frac{t}{2})$ and $(i,\frac{t}{2})((i+1)\bmod s, \frac{t}{2}-1)$ for $i=1,3,\dots,s-3$ and two edges between the fiber $s-1$ and 0, which are $(s-1,\frac{t}{2})(0,\frac{t}{2})$, $(s-1,\frac{t}{2}-1)(0,\frac{t}{2}-1)$. In the second case add the edges$(i,\frac{t}{2}-1)((i+1)\bmod s, \frac{t}{2})$ and $(i,\frac{t}{2})((i+1)\bmod s, \frac{t}{2}-1)$ for $i=0,2,4,\dots,s-2$.

Observe that two deleted edges on each pair of fibres 0 and 1, 2 and 3,$\dots$, $s-2$ and $s-1$ are the same as the two edges that were added on the same pair of fibres from the second case and that two deleted edges on the pair of fibres 1 and 2, 3 and 4,$\dots$, $s-1$ and 0 are two added edges on the same pair of fibres from the first case.

Further, the edges between fibers 0 and 1, 1 and 2,$\dots$ $s-2$ and $s-1$ in both cases connect vertices from Z_0 with vertices from Z_0 and vertices from Z_1 with vertices from Z_1.

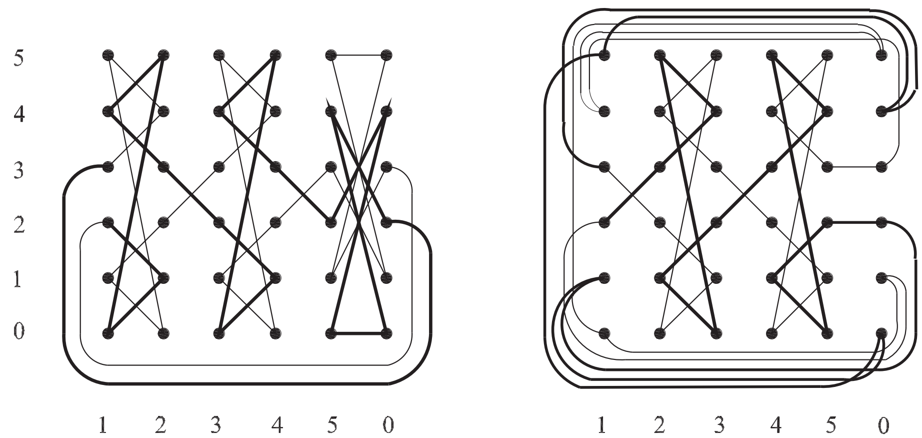


Figure 6: Hamiltonian cycles in the direct graph bundle $C_6 \times^{\rho_0} C_6$

Observation that the edges between fibres $s-1$ and 0 in first case connect Z_0 to Z_1 and Z_1 to Z_0 implies that hamiltonian cycle $\mathscr{H}_1$ is constructed.

In the second case we have to be more careful. Recall that on fibres $s-1$ and 0 we have two spanning paths. One begins in $(s-1,\frac{t}{2})$ and ends in $(0,\frac{t}{2})$ and the other begins in $(s-1,\frac{t}{2}-1)$ and ends in $(0,\frac{t}{2}-1)$. These two paths connect Z_0 to Z_1 and Z_1 to Z_0 and hamiltonian cycle $\mathscr{H}_2$ is constructed. (see Figure 6). ■

Proposition 16 *Let C_s, C_t be two cycles, where $s,t \geq 3$ and s is even and t odd. Let $\alpha = \rho_1$ be reflection with one fixed point. Then $C_s \times^{\alpha} C_t$ admits a hamiltonian decomposition.*

Proof: Note that the edges between two consecutive fibres i and $i+1$ (for $i = 0,1,\ldots,s-2$) form a cycle of length $2t$, because the subgraph induced on two consecutive fibres is isomorphic to $P_2 \times C_t$. Also the subgraph induced on fibres $s-1$ and 0 is isomorphic to $P_2 \times^{\rho_1} C_t \simeq C_{2t}$, by Lemma 1.

Each of these subgraphs contains the two edges $(i,\frac{t-1}{2})((i+1) \bmod s, \frac{t+1}{2})$ and $(i,\frac{t+1}{2})((i+1) \bmod s, \frac{t-1}{2})$. We have assumed here, withouf loss of generality, that the fixed point of ρ_1 is $\frac{t-1}{2}$.

By deleting one of these two edges we construct a path that spans all vertices (and all edges except the deleted) of fibres i and $i+1$.

Now we can construct a hamiltonian decomposition on $C_s \times^{\alpha} C_t$. Recall that each pair of consecutive fibres induces a cycle by Lemma 1. We will use these cycles to get spanning paths. The remaining edges (one for each cycle) will be used to connect the paths into hamiltonian cycles. First take the spanning paths on pairs of fibres 0 and 1, 2 and 3, $\ldots, s-2$ and $s-1$, where the edge $(i,\frac{t+1}{2})((i+1) \bmod s, \frac{t-1}{2})$ is deleted. Then take the spanning paths on pairs of fibres 1 and 2, 3 and 4,$\ldots, s-1$ and 0, where the edge $(i,\frac{t-1}{2})((i+1) \bmod s, \frac{t+1}{2})$ is deleted. Use the edges $(i,\frac{t-1}{2})((i+1) \bmod s, \frac{t+1}{2})$ for $i = 1,3,\ldots,s-1$ to connect the paths from the first case (fibres 0 and 1, 2 and 3, $\ldots, s-2$ and $s-1$) and similarly, paths from second case are connected with edges $(i,\frac{t+1}{2})(i+1,\frac{t-1}{2})$ for $i = 0,2,4,\ldots,s-2$. (see Figure 7.) It is clear that the resulting cycles are two disjoint hamiltonian cycles. ■

Proposition 17 *Let C_s, C_t be two cycles, where $s,t \geq 3$ and both s and t are odd. Let $\alpha = \rho_1$ be reflection with one fixed point. Then $C_s \times^{\alpha} C_t$ admits a hamiltonian decomposition.*

Proof: Consider two subsets of edges of direct graph bundle $C_s \times^{\alpha} C_t$ (all additions in the second coordinate are modulo t). In first subset A are edges

(a) $(i,j)(i+1,j+1)$ for $i = 0,1,2,\ldots,\frac{s-1}{2}$ and $j = 0,1,\ldots,t-1$,

(b) $(i,j)(i+1,j-1)$ for $i = \frac{s-1}{2}+1, \frac{s-1}{2}+2, \frac{s-1}{2}+3,\ldots,s-2$ and $j = 0,1,\ldots,t-1$, and

(c) $(s-1,j)(0,\rho_1(j-1))$ for $j = 0,1,\ldots,t-1$.

and in second subset B are edges

(a) $(i,j)(i+1,j-1)$ for $i = 0,1,2,\ldots,\frac{s-1}{2}$ and $j = 0,1,\ldots,t-1$,

(b) $(i,j)(i+1,j+1)$ for $i = \frac{s-1}{2}+1, \frac{s-1}{2}+2, \frac{s-1}{2}+3,\ldots,s-2$ and $j = 0,1,\ldots,t-1$, and

(c) $(s-1,j)(0,\rho_1(j+1))$ for $j = 0,1,\ldots,t-1$.

Observe that edges from (a) and (b) in subsets A form t parallel paths that join $(0,j)$ with $(s-1,(j+2) \bmod t)$. Similarly, edges from (a) and (b) in subset B form t parallel paths that join $(0,j)$ with $(s-1,(j-2) \bmod t)$. As $\rho_1(j-1) = t-(j-1)-1 = t-j$ and $\rho_1(j+1) = t-(j+1)-1 = t-j-2$, the edges from A defined in (c) can be written simpler

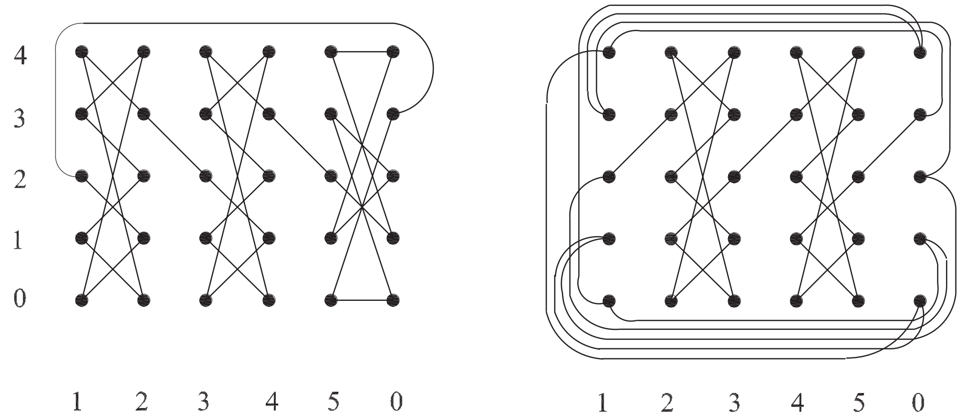


Figure 7: Hamiltonian cycles in the direct graph bundle $C_6 \times^{\rho_1} C_5$

as $(s-1,j)(0,t-j)$ and the edges from B defined in (c) as $(s-1,j)(0,t-j-2)$. Clearly, subsets A and B are disjoint and their union covers all edges of $C_s \times^{\alpha} C_t$.

Moreover, the edges from A and B meet each vertex exactly twice, so they form a union of cycles. More precisely, the edges from A form one (short) cycle

$$(s-1,1) \to (0,t-1) \to (1,0) \to \cdots \to (s-1,1)$$

and $\frac{t-1}{2}$ longer cycles, for $j = 2,3,\ldots,\frac{t-1}{2},\frac{t+1}{2}$,

$$(s-1,j) \to (0,t-j) \to \cdots \to (s-1,t-j+2) \to$$
$$\to (0,t-(t-j+2) = (0,j-2) \to \cdots \to$$
$$\to (s-1,j).$$

The edges from B form one (short) cycle

$$(s-1,t-2) \to (0,0) \to (1,t-1) \to \cdots \to (s-1,t-2)$$

and $\frac{t-1}{2}$ longer cycles, for $j = \frac{t-1}{2}-1,\frac{t-1}{2},\ldots,t-3$,

$$(s-1,j) \to (0,t-j-2) \to \cdots \to (s-1,t-j-4) \to$$
$$(0,t-(t-j-4)-2) = (0,j+2) \to \cdots \to (s-1,j).$$

Observe that each long cycle determined by the set A (for $j = 2,3,\ldots,\frac{t-1}{2},\frac{t+1}{2}$) in fiber 0 contains exactly two vertices $(0,t-j)$ and $(0,j-2)$. Clearly, in the second coordinate one of these vertices is odd and the other even. Similarly, each long cycle determined by the set B (for $j = \frac{t-1}{2}-1,\frac{t-1}{2},\ldots,t-3$) contains a vertex with even and a vertex with odd second coordinate: $(0,t-j-2)$ and $(0,j+2)$. Vertices $(0,t-1)$ and $(0,0)$ are on the short cycle determined by A and B, respectively, and have even second coordinate. Thus, vertices $(0,2k)$, $k = 0,1,2,\ldots,\frac{t-1}{2}$ lie on different cycles determined by A and in on different cycles determined by B. Equivalently, vertices $(0,2k+1)$, $k = 0,1,2,\ldots,\frac{t-1}{2}-1$ lie in different cycles determined by A and in different cycles determined by B.

Further, edges from set A between fibres 0,1 and 2 go “up" (these edges are directed “up", reading from left to right) and edges from set B between fibres 0,1 and 2 go “down" (these edges are directed “down", reading from left to right).

Now we will show how one can always combine the cycles determined by A into a hamiltonian cycle, $\mathscr{H}_1$, and the cycles determined by B into a hamiltonian cycle, $\mathscr{H}_2$.

For $i = 0,2,4,\ldots,t-3$ delete edges $(1,i)(2,i+1),(0,i+1)(1,i+2)$ from A and put them into B and similarly (for $i = 0,2,4,\ldots,t-3$) delete edges $(0,i+1)(1,i),(1,i+2)(2,i+1)$ from B and put them into A. We claim that this replacement gives two hamiltonian cycles.

An edge $(1,i)(2,i+1)$ is on the cycle determined by A with vertex $(0,i-1)$ in fiber 0 and edge $(0,i+1)(1,i+2)$ on the cycle determined by A with vertex $(0,i+1)$ in fibre 0. These two cycles are different, because $i-1$ and $i+1$ are consecutive odd numbers. By adding edges $(0,i+1)(1,i),(1,i+2)(2,i+1)$ from the set B combines these two cycles into a larger one (see Figure 10a)). More precisely, if $i > 0$ then the cycle with vertex $(0,i-1)$ in fiber 0 is actually a union of cycles that contains vertex $(0,i-1)$. Replacement of these edges gives hamiltonian cycle $\mathscr{H}_1$.

Due to obvious symmetry, the cycles determined by the set B give rise, after the replacement above, to a second hamiltonian cycle $\mathscr{H}_2$.

On the left picture in Figure 8 we can see which edges are deleted (dotted lines) in the case of the set A. The second and the third picture shows which vertices lie on the same cycle (left) and how the replacement of edges these cycles combine into one cycle (right). In the middle picture cycles are determined by the set A, in the right picture by the set B. ■

An example is given in Figure 9.

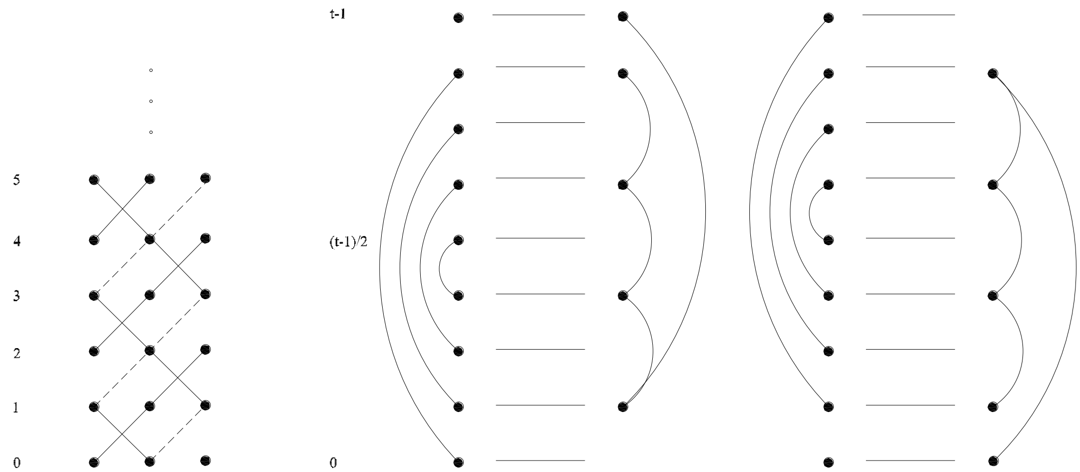


Figure 8: The replacement of edges $\frac{t+1}{2}$ cycles combine into one

VII. Hamiltonian decomposition of direct graph bundles - cyclic shifts

We first recall the constructions providing hamiltonian cycles [29].

Construction 1. Let $\bar{X}$ be the subgraph of connected direct graph bundle $X = C_s \times^{\sigma_\ell} C_t$ in which only edges $(i, j)(i+1, (j+1) \bmod t)$, $i = 0, 1, \ldots, s-2$, $j = 0, 1, \ldots, t-1$ and $(s-1, j)\ (0, (j+1+\ell) \bmod t)$, $j = 0, 1, \ldots, t-1$ are present. □

Informally, one can also say that in $\bar{X}$, reading from left to right, only edges directed "up" are taken from X.

Lemma 18 *Let $C_s \times^{\sigma_\ell} C_t$ be connected direct graph bundle. Let $\bar{X}$ be obtained by Construction 1. Then $\bar{X}$ is isomorphic to a union of p cycles of length $\frac{st}{p}$. Moreover, p is an odd number and the i-th cycle meets the first fibre in vertices $(0, (i+p) \bmod t)$.*

If $p = 1$ then $\bar{X}$ gives a hamiltonian cycle of X, but this is of course not always the case. (Examples with $p = 1$ and $p = 3$ are given on Figure 11.a) and b).)

The next construction makes it possible to combine the cycles resulting from Construction 1 into one cycle by a small modification, i.e. by replacing only a few edges.

Construction 2. Let $\bar{X}$ be the subgraph of X that is a union of cycles. Delete edges $(1, i)(2, i+1)$ and $(0, i+1)(1, (i+2) \bmod t)$ and replace them with edges $(0, i+1)(1, i)$ and $(1, (i+2) \bmod t)(2, i+1)$ for $i = 0, 1, 2, \ldots, p-2$ to obtain $\widetilde{X}$. □

Assuming that the edges of $\bar{X}$ between fibres 0,1, and 2 are as given by Construction 1, (i.e. all edges go "up") we have the situation on Figure 10.a) and 10.b). The result of Construction 2 on the graph from Figure 11.b) is given on Figure 11.c).

By Lemma 18, the edges $(1, i)(2, i+1)$ and $(0, i+1)(1, (i+2) \bmod t)$ are on the $i-1$-th and $i+1$-th cycle. The replacement thus combines the two cycles into a larger one. Note that the edges involved in Construction 2 for different i are disjoint. Therefore

Lemma 19 *Let $\bar{X}$ be obtained by Construction 1 and assume it has $p > 1$ cycles. Then $\widetilde{X}$, the result of Construction 2 (replacing $p-1$ pairs of edges) gives a hamiltonian cycle.*

Observe that Construction 1 decomposes the graph X into $\bar{X}$ and $\tilde{X} = X - \bar{X}$. Clearly, $\tilde{X}$ is 2- regular graph, and let us say it is a union of q cycles. If $q = p$ then Construction 2 would result in hamiltonian decomposition of X. However, in most cases $q \neq p$.

Let us consider variations of Construction 1 in which we replace some of the "up" directed edges with "down" directed edges.

This results in the next possible values of p and q:

- $$p = \gcd((s+\ell) \bmod t, t),$$
  when all edges are "up" directed and
  $$q = \gcd((-s+\ell) \bmod t, t),$$
  when all edges are "down" directed

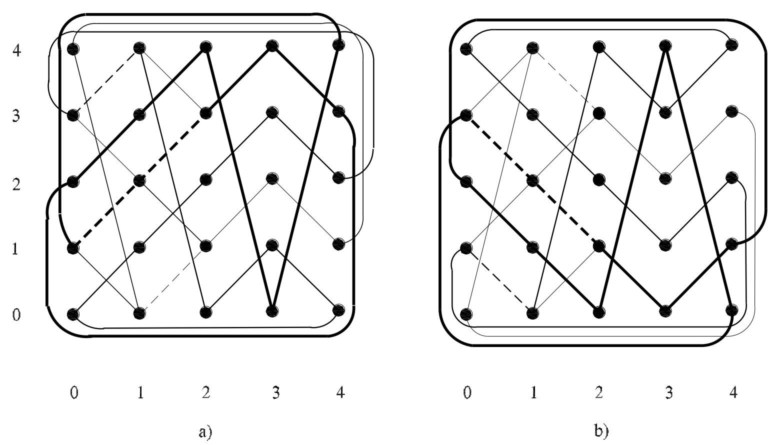


Figure 9: Hamiltonian cycles in the direct graph bundle $C_5 \times^{\rho_1} C_5$

- $$p = \gcd((s-2+\ell) \bmod t, t),$$
  when exactly one edge is "down" directed and all other edges are "up" directed and
  $$q = \gcd((-s+2+\ell) \bmod t, t),$$
  when exactly one edge is "up" directed and all other edges are "down" directed
- $$p = \gcd((s-4+\ell) \bmod t, t),$$
  when exactly two edges are "down" directed and all other edges are "up" directed and
  $$q = \gcd((-s+4+\ell) \bmod t, t),$$
  when exactly two edges are "up" directed and all other edges are "down" directed
- and so on
- until we get, for odd s:
  $$p = \gcd((1+\ell) \bmod t, t),$$
  when $\frac{s-1}{2}$ edges are "down" directed and $\frac{s+1}{2}$ edges are "up" directed and
  $$q = \gcd((-1+\ell) \bmod t, t)$$
  when $\frac{s-1}{2}$ edges are "up" directed and $\frac{s+1}{2}$ edges are "down" directed
- and, for s even:
  $$p = \gcd(\ell, t) = q$$
  when $\frac{s}{2}$ edges are "down' directed and $\frac{s}{2}$ edges are "up" directed.

Therefore, if the base of X is cycle C_s on even number of vertices, a variation of Construction 1 gives a decomposition of X into $\bar{X}$ and $\tilde{X}$ that have $p = q$ cycles. Hence, Construction 2 in this case gives a hamiltonian decomposition of X, and we have

Proposition 20 *Let $X = C_s \times^{\sigma_\ell} C_t$ be connected direct graph bundle with cycle on even number of vertices as base. Then X admits a hamiltonian decomposition.*

When the base cycle is odd, the variation of Construction 1 may never give $p = q$. In this paper we will look at one partical situation where $p \neq q$ but we can construct a hamiltonian decomposition of X. We need results about Hamiltonian cycles in circulant digraphs with two stripes [44].

Let $n, a_1, a_2, \ldots, a_m$ be arbitrary integers with $0 < a_1 < \ldots < a_m < n$ and let $V = \{0, 1, \ldots, n-1\}$. A digraph $G = (V, E)$ with vertex set V and arc set E is called a circulant digraph generated by $a_1, a_2, \ldots, a_m$ and denote by $G(n, a_1, a_2, \ldots, a_m)$ if its arc set E consists only of the arcs (i, j) with $j - i \equiv a_t (\bmod n)$ for any $t \in \{1, 2, \ldots, m\}$. An arc (i, j) of G with $j - i \equiv a_t (\bmod n)$ is called an a_t-arc. The pattern of the path $\mathscr{P} = (v_1, v_2, \ldots, v_k\}$ in $G(n, a_1, a_2)$ is the sequence of numbers $(a^1, a^2, \ldots, a^{k-1})$ where $a^i = a_1$ if (v_i, v_{i+1}) is an a_1-arc and $a^i = a_2$ otherwise.

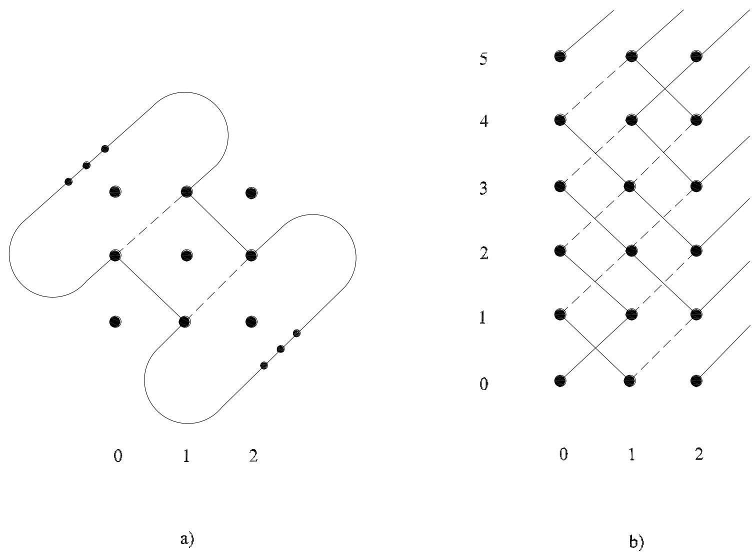


Figure 10: a) A switch that joins two parallel cycles into one cycle, b) $p-1$ switches that connect p parallel cycles into one (hamiltonian) cycle.

Theorem 21 ([44]) *Let a circulant 2-digraph $G(n,a_1,a_2)$ be given and let $t = \gcd(n,a_2-a_1), n' = \frac{n}{t}, a' = \frac{(a_2-a_1)}{t}$. The digraph $G(n,a_1,a_2)$ is Hamiltonian if and only if $gcd(n,a_1,a_2) = 1$ and there exists a number $0 \le h \le t$ such that $\gcd(n', a_1 + a'h) = 1$.*

Theorem 22 ([44]) *Let a hamiltonian circulant 2-digraph $G(n,a_1,a_2)$ be given. Each Hamiltonian cycle of $G(n,a_1,a_2)$ is periodic with period n' and its pattern consists of $t-h$ a_1-arcs and h a_2- arcs, for some h with $\gcd(n', a_1 + a'h) = 1$ and t, n', a' defined as in Theorem 21.*

Proposition 23 *Let $X = C_s \times^{\sigma_\ell} C_t$ be connected direct graph bundle with cycle on odd number of vertices as base and cycle on even number of vertices as fiber. Let $\gcd(\frac{t}{2}, \ell) = 1$. Then X admits a hamiltonian decomposition.*

Proof: Recall that direct graph bundle with cycle on odd number of vertices as base and cycle on even number of vertices as fiber is connected exactly when ℓ is even.

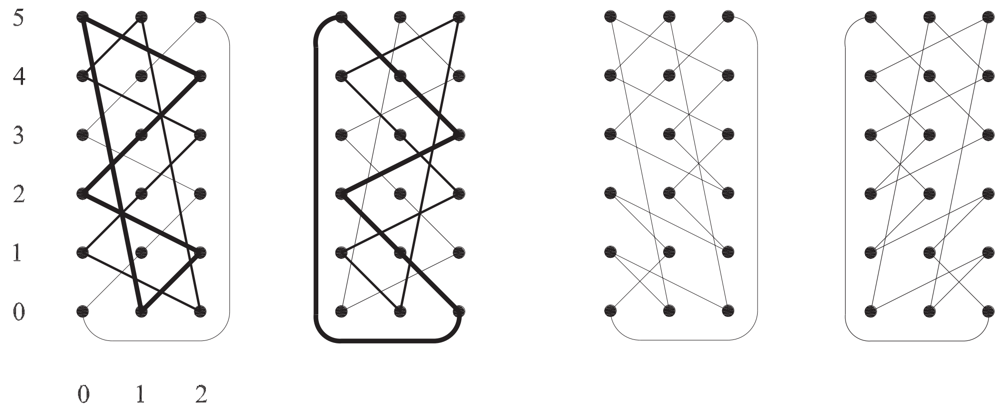
Form t disjoint paths of length $s-1$ from $(0,j)$ to $(s-1,j)$, $j = 0,1,\dots,t-1$, by taking edges $(i,j)(i+1,(j+1) \bmod t))$ for even i and edges $(i,j)(i+1,(j-1) \bmod t)$ for odd i (and $j = 0,1,\dots,t-1$). Obviously, if all edges directed up replace with edges directed down and vice versa, we get a second set of t disjoint paths of length $s-1$ from $(0,j)$ to $(s-1,j)$, $j = 0,1,\dots,t-1$. Note that each vertex $(0,j)$ is connected with $(s-1,j)$ by exactly two different paths.

Constructed paths will now be connected into two disjoint hamiltonian cycles by suitable choice of edges between fibres s-1 and 0. This wll be done by application of above theorems for circulant digraphs.

The cycle C_t is associatd with the circulant 2-digraph $G(t,a_1,a_2)$ where $a_1 = (-1+\ell) \bmod t$ and $a_2 = (1+\ell) \bmod t$. Since t is even, while $(-1+\ell) \bmod t$ and $(1+\ell) \bmod t$ are odd, it can be shown that $\gcd(t,a_1,a_2) = 1$. It is easily seen that $h = 1$ is in accordance with the conditions of Theorem 21 and that therefore $G(t,a_1,a_2)$ is hamiltonian. Namely $\gcd(t,a_2-a_1) = \gcd(t,2) = 2 = k$ and $\gcd(\frac{t}{k}, a_1 + \frac{a_2-a_1}{k}h) = \gcd(\frac{t}{2}, -1+\ell+1) = \gcd(\frac{t}{2}, \ell) = 1$. Further, by Theorem 22 each hamiltonian cycle of $G(t,a_1,a_2)$ is periodic with period $\frac{t}{k} = \frac{t}{2}$ and its pattern consists of $k-h = 2-1 = 1$ a_1-arcs and $h = 1$ a_2- arcs. Clearly, we have two different hamiltonian cycles whose patterns have exactly one a_1- arc and one a_2- arc. First cycle denoted by $\mathscr{H}'_1$ then has pattern (a_1,a_2) and the second cycle denoted by $\mathscr{H}'_2$ has pattern (a_2,a_1).

The first hamiltonian cycle $\mathscr{H}_1$ on X is constructed as follows. Take paths from $(0,j)$ to $(s-1,j)$, $j = 0,1,\dots,t-1$ from the first set of paths. The cycle $\mathscr{H}'_1$ provides edges between fibres $s-1$ and 0: an edge $v_i v_j$ of cycle $\mathscr{H}'_1$ provides the edge $(s-1,v_i)(0,v_j)$ of bundle X. It is easy to check that these gives a hamiltonian cycle $\mathscr{H}_1$. Similarly, paths from the second set of paths together with edges inherited from the cycle $\mathscr{H}'_2$ give the second hamiltonian cycle $\mathscr{H}_2$. ■

Example: Let us show $C_3 \times^{\sigma_8} C_{30}$ admits a hamiltonian decomposition. Notice $\gcd(t,(-1+\ell) \bmod t,(1+\ell) \bmod t) = \gcd(30,7,9) = 1$ and $\gcd(\frac{t}{2},\ell) = \gcd(15,8) = 1$. Hamiltonian cycles $\mathscr{H}'_1$ and $\mathscr{H}'_2$ on $G(30,7,9)$ are cycles with pattern $(7,9)$ and $(9,7)$, recpectively. By taking edges $(i,j)(i+$



a) b)

Figure 11: a) There are three cycles in $C_3 \times C_6$ (left and right), b) Hamiltonian cycles in $C_3 \times C_6$

$1,(j+1)\,\mathrm{mod}\,t)$), $i=0$ and $(i,j)(i+1,(j-1)\,\mathrm{mod}\,t))$, $i=1$ (and $j=0,1,\ldots,14$) and edges between fibers 2 and 0 detemnined by cycle $\mathscr{H}'_1$ construct first hamiltonian cycle. The second construct similar.

Example: Does $C_3 \times^{\sigma_4} C_{20}$ admit a hamiltonian decomposition? Because $\gcd(\frac{t}{2},\ell)=\gcd(10,4)=2$, above construction does not give a hamiltonian decomposition. In fact, we obtain two pairs of cycles so that each pair covers the graph. It is not clear at present how to combine the two pairs of cycles simultaneously.

VIII. Summary

We have stated two conjectures:

Conjecture 12 Let $X=C_s \times^{\alpha} C_t$ be a direct graph bundle with fibre C_t and base C_s. Then X has a hamiltonian decomposition if and only if X is connected.

Conjecture 13 Let B and F be hamiltonian graphs that have a hamiltonian decomposition, and $t=|V(F)|$ odd. Then any direct graph bundle X with fiber F and base graph B admits a hamiltonian decoposition if and only if X is connected.

We know that Conjecture 12 implies Conjecture 13.

In this paper we have provided a proof of conjecture for some special cases, in particular, Conjecture 12 is true for graph bundles with α any reflection (Proposition 14, Proposition 15,Proposition 16, and Proposition 17).

The validity of Conjecture 12 is open when the authomorphism is a cyclic shift. Besides the trivial case, $\ell=0$ (shift zero, i.e. $\alpha=id$) when X is the product of cycles which is known to admit a hamiltonian decomposition whenever it is connected, we know that Conjecture 12 is true for graph bundles with α a cyclic shift when the base cycle is even (Proposition 20). In Proposition 23 is considered some special cases of direct graph bundles with α a cyclic shift and odd base cycle. Only in those particular cases, we can explicitly find construction of hamiltonian cycles that give us hamiltonian decomposition. On the other hand, we are not aware of any counterexample, and therefore believe Conjecture 12 and Conjecture 13 are correct.

Acknowledgment

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