On Approximations by polynomial and trigonometrical integro-differential splines

I. G. Burova, S. V. Poluyanov

***Abstract*—**Here we construct continuously differentiable approximation using middle and left basis integro-differential splines of fifth order. The goal of this work is the presentation of some new formulas which are useful for the approximation of the functions with one and two variables. Here we construct the basic one-dimensional polynomial and trigonometrical integro-differential splines of the fifth order approximation. For each interval we construct the approximation separately. In order to construct the approximation in each interval we need the values of the function, its first derivative in the points of interpolation, and the value of the integral of the function over the interval. If we don't know the values of the first derivative of the function in the points of interpolation and/or the value of the integral of the function over the interval then we use the expressions which were obtained for this instance and the error of the approximation will be of the fifth order. The one-dimensional case can be extended to multiple dimensions through the use of the tensor product spline constructs. The examples of the approximations functions of two variables are included. The spline approximation schemes discussed in this paper allow us to control the effect of knot placement on the accuracy of spline approximation. Numerical examples are presented.

***Keywords*—**Integro-Differential Splines, Interpolation, Polynomial splines, Trigonometrical splines.

I. INTRODUCTION

NOWADAYS splines are widely used for solving various mathematical problems. A great deal of research has been devoted to the application of various splines with different properties for approximation and estimation of data. Special attention is given to methods of constructing images [1]–[12]. When using splines to construct approximations for each interval we are not constrained by the basic limitation of single polynomial approximation — namely the Runge phenomenon as it has previously been established [14]. This method of approximation using polynomial splines is widely used for the interpolation and approximation of discrete data. As is well known, the one-dimensional case can be extended to multiple dimensions through the use of the tensor product spline constructs. [13]–[15].

Kireev V.I. (in 1993) became the first to use values of one-variable integrals of a function over subintervals for the construction of approximations. Recently, many researchers have devoted much attention to construct integro splines [16]–[24].

This paper takes as its topic the investigation of cases in which both the values of the function and its first derivative are known at the ends of each subinterval. In addition, values of the integrals over the subintervals are known. Polynomial and nonpolynomial splines of one variable were constructed by the authors of the paper in [26]–[28].

Suppose that n, m are natural numbers, while a, b, c, d are real numbers, $h = (b-a)/n$, $h_y = (d-c)/m$. We can build the grid of interpolation nodes $x_j = a + jh$, $j = 0, 1, \ldots, n$, $y_k = c + kh_y$, $k = 0, 1, \ldots, m$.

Let us consider a rectangular domain Ω, where

$$\Omega = \{(x, y) \,|\, a \le x \le b,\ c \le y \le d\}.$$

We introduce a mesh of lines on Ω which divides the domain Ω into the rectangles $\Omega_{j,k}$:

$$\Omega_{j,k} = \{(x, y)|x \in [x_j, x_{j+1}],\ y \in [y_k, y_{k+1}]\}.$$

II. MIDDLE POLYNOMIAL SPLINES OF ONE VARIABLE

Let the function $u(x)$ be such that $u \in C^5([a, b])$.

Suppose that we know $u(x_j)$, $u'(x_j)$, $j = 0, 1, \ldots, n$, and $\int_{x_j}^{x_{j+1}} u(t)dt$, $j = 0, \ldots, n-1$.

We denote by $\widetilde{u}(x)$ an approximation of the function $u(x)$ on the interval $[x_j, x_{j+1}] \subset [a, b]$:

$$\begin{aligned}\widetilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x) + \\ u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x) + \\ + \int_{x_j}^{x_{j+1}} u(t)dt\, \omega_j^{<0>}(x). \end{aligned} \quad (1)$$

The basic splines $\omega_{j,0}(x)$, $\omega_{j+1,0}(x)$, $\omega_{j,1}(x)$, $\omega_{j+1,1}(x)$, $\omega_j^{<0>}(x)$, we obtain from the system:

$$\widetilde{u}(x) \equiv u(x), \quad u(x) = x^{i-1}, i = 1, 2, 3, 4, 5. \quad (2)$$

Suppose that $supp\, \omega_{k,\alpha} = [x_{k-1}, x_{k+1}]$, $\alpha = 0, 1$, $supp\, \omega_k^{<0>} = [x_k, x_{k+1}]$. We have for $x = x_j + th$, $t \in [0, 1]$, the next formulas:

$$\omega_{j,0}(x_j + th) = -18\,t^2 + 32\,t^3 - 15\,t^4 + 1, \quad (3)$$

$$\omega_{j+1,0}(x_j + th) = -12\,t^2 + 28\,t^3 - 15\,t^4, \quad (4)$$

$$\omega_{j,1}(x_j + th) = -(9/2)\,h\,t^2 + 6\,h\,t^3 - (5/2)\,h\,t^4 + t\,h, \quad (5)$$

$$\omega_{j+1,1}(x_j + th) = (3/2)\,h\,t^2 - 4\,h\,t^3 + (5/2)\,h\,t^4, \quad (6)$$

$$\omega_j^{<0>}(x_j + th) = (30\,t^2 - 60\,t^3 + 30\,t^4)/h. \quad (7)$$

Fig. 1, 2, 3 show the graphics of the basic functions $\omega_{j,0}(x)$, $\omega_{j+1,0}(x)$, $\omega_{j,1}(x)$, $\omega'_{j+1,1}(x)$, $\omega_j^{<0>}(x)$, when $h = 1$. Fig. 3 (b) shows the error of approximation of the Runge function

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(a) (b)

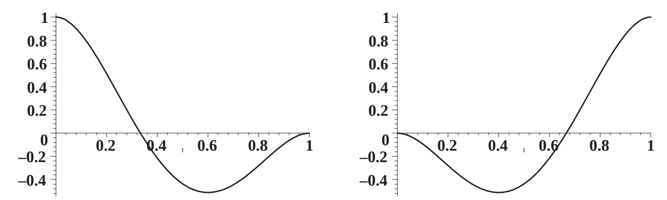


Fig. 1. Plots of the basic functions: (a) $\omega_{j,0}(x)$; (b) $\omega_{j+1,0}(x)$

(a) (b)

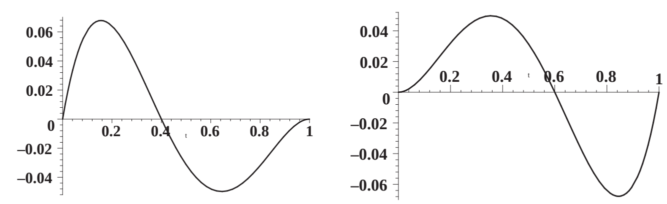


Fig. 2. Plots of the basic functions: (a) $\omega_{j,1}(x)$, when $h = 1$; (b) $\omega_{j+1,1}(x)$, when $h = 1$

$u(x) = 1/(1 + 25x^2)$ with the polynomial splines, $h = 0.1$, $x \in [-1, 1]$.

From (1), (2), (3)–(7) we have

$$\omega_{j,0}(x_j+th)=\begin{cases}(5t+1)(1-3t)(t-1)^2, & t\in[0,1],\\ (3t+1)(1-5t)(1+t)^2, & t\in[-1,0],\\ 0, & t\notin[-1,1],\end{cases}$$

$$\omega_{j,1}(x_j+th)=\begin{cases}-\frac{1}{2}th(5t-2)(t-1)^2, & t\in[0,1],\\ \frac{1}{2}th(2+5t)(1+t)^2, & t\in[-1,0],\\ 0, & t\notin[-1,1],\end{cases}$$

$$\omega_j^{<0>}(x_j+th)=\begin{cases}\frac{30t^2}{h}(t-1)^2, & t\in[0,1],\\ 0, & t\notin[0,1].\end{cases}$$

It is easy to see that $\omega_{k,0}, \omega_{k,1}, \omega_k^{<0>} \in C^1(R^1)$.

Let us take $\tilde{U}(x)$, $x \in [a, b]$, such that $\tilde{U}(x) = \tilde{u}(x)$, $x \in [x_j, x_{j+1}]$. Let $\|u\|_{[a,b]} = \max\limits_{[a,b]} |u(x)|$.

Theorem 1 Let function $u(x)$ be such that $u \in C^5([a, b])$. For approximation $u(x)$, $x \in [x_j, x_{j+1}]$ by (1), (2) – (7) we have $\tilde{u} \in C^1(R^1)$,

$$|\tilde{u}(x) - u(x)|_{[x_j,x_{j+1}]} \le K_1 h^5 \|u^{(5)}\|_{[x_j,x_{j+1}]}, \qquad (8)$$

$K_1 = 0.0138$,

$$|\tilde{u}'(x) - u'(x)|_{[x_j,x_{j+1}]} \le K_2 h^4 \|u^{(5)}\|_{[x_j,x_{j+1}]}, \qquad (9)$$

$K_2 = 0.125$,

$$|\tilde{U}(x) - u(x)|_{[a,b]} \le K_1 h^5 \|u^{(5)}\|_{[a,b]},\ K_1 = 0.0138, \qquad (10)$$

Proof: Inequality (8) follows from Taylor's theorem and the inequalities:

$|\omega_{j,0}(x)| \le 1$, $|\omega_{j+1,0}(x)| \le 1$,
$|\omega_{j,1}(x)| \le 0.06779h$, $|\omega_{j+1,1}(x)| \le 0.06779h$,
$|\omega_j^{<0>}(x)| \le 1.875/h$.

Inequality (10) follows from (8).

(a) (b)

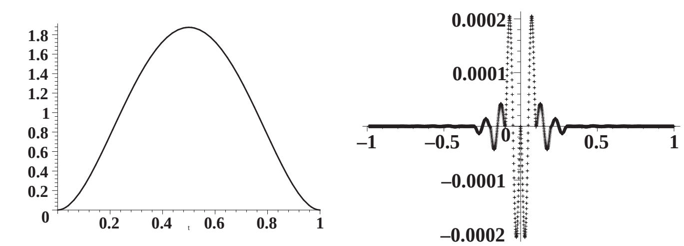


Fig. 3. Plots of the basic functions: (a) $\omega_j^{<0>}(x)$, when $h = 1$, and (b) the error of approximation of the Runge function with the polynomial splines, $h = 0.1$, $x \in [-1, 1]$

Inequality (9) follows from Taylor's theorem and the inequalities:

$\omega'_{j,0}(x_j + th) = -12t(5t - 3)(t - 1)/h$,
$\omega'_{j+1,0}(x_j + th) = -12t(5t^2 - 7t + 2)/h$,
$\omega'^{<0>}_j(x_j + th) = 60t(2t - 1)(t - 1)/h^2$,
$\omega'_{j,1}(x_j + th) = -(t - 1)(10t^2 - 8t + 1)$,
$\omega'_{j+1,1}(x_j + th) = t(3 - 12t + 10t^2)$.

Let us take an irregular mesh of nodes. If $x \in [x_j, x_{j+1}]$, then $x_{j+1} = x_j + th_j$, where $h_j = x_{j+1} - x_j$, $t \in [0, 1]$. Now we can use the basic splines in the form:

$\omega_{j,0}(x_j + th) = -(1 + 5t)(-1 + 3t)(t - 1)^2$,
$\omega_{j+1,0}(x_j + th) = -t^2(-2 + 3t)(-6 + 5t)$,
$\omega_{j,1}(x_j + th) = -(1/2)th_j(5t - 2)(t - 1)^2$,
$\omega_{j+1,1}(x_j + th) = (1/2)t^2 h_j(t - 1)(5t - 3)$,
$\omega_j^{<0>}(x_j + th) = (30t^2)(t - 1)^2/h_j$.

Fig. 3 (b) shows the error of the approximation of the Runge function when $x \in [-1, 1]$, $x_{j+1} = x_j + h_j$, $h_j = 0.1$, $j = 0, 1, \ldots, n - 1$, $n = 20$, here $\|u - \tilde{U}\|_{[-1,1]} = \varepsilon$, $\varepsilon = 0.207 \cdot 10^{-3}$. Our aim is to reduce n and receive the same or less error of approximation.

We construct the approximation on every $[x_j, x_{j+1}]$ separately. The spline approximation scheme allows us to control the effect of knot placement on the accuracy of spline approximation. So we can change the stepsize $h_j = x_{j+1} - x_j$. To improve the quality of the approximation we can choose the nodes $x_j \in [a, b]$ as the follows. Beginning with an initial stepsize of $h_0 = x_1 - x_0$, we obtain $x_{j+1} = x_j + h_j$, where h_j we get from the relation:

$$I_j = \int_{x_j}^{x_j+h_j} \sqrt{1 + (u'(x))^2}\, dx = I_0, \qquad (11)$$

$$\text{where } I_0 = \int_{x_0}^{x_1} \sqrt{1 + (u'(x))^2}\, dx.$$

Example 1a. Fig. 4 (a) shows the error of approximation of the Runge function, $x \in [-1, 1]$, with the polynomial splines, when h_j we obtain from (11), $n = 15$, $h_0 = 0.2$, here $\|u - \tilde{U}\|_{[-1,1]} = \varepsilon$, $\varepsilon = 0.00018$. Fig. 4 (b) shows the error of approximation of the Runge function, when h_j we obtain from (11), $n = 15$, $h_0 = 0.205$, here $\varepsilon = 0.000077$.

(a) (b)

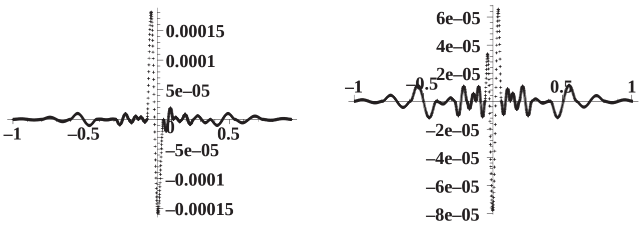


Fig. 4. Plots of the errors of approximation of the Runge function, h_j we obtain from (11), $n = 15$, when (a) $h_0 = 0.2$, (b) $h_0 = 0.205$

Example 1b. Fig. 5 show the error of the approximation of the function $\sin(exp(3x))$, $x \in [-1, 1]$, when $x_{j+1} = x_j + h$, $h = 1/50$ (fig. 5 (a)), and when h_j we obtain from (11), $j = 0, 1, \ldots, 47$, $h_0 = 0.28$ (fig. 5 (b)).

(a) (b)

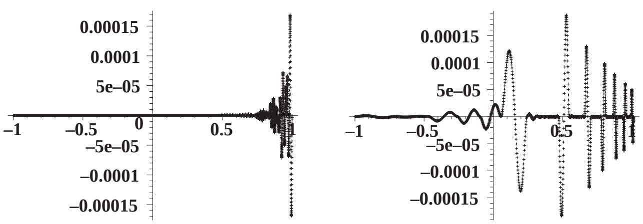


Fig. 5. Plots of the errors approximation of the function $\sin(exp(3x))$, when (a) $x_{j+1} = x_j + h$, $h = 1/50$, $n = 100$, and (b) when $h_0 = 0.28$, h_j we obtain from (11), $n = 48$

It is important to approximate the function well using a spline with as few knots as possible. Fig. 3 (b) shows the error of approximation of the Runge function, $x \in [-1, 1]$, when $h_i = 0.1$, $n = 20$.

Example 2. Fig. 6 (a) shows the error of approximation of the Runge function with the polynomial splines, when $h_0 = 0.37$, $n = 10$. Here h_j we obtain using (11), $j = 0, 1, 2, 7, 8, 9$, and $x_{j+1} = x_j + h_j$, $h_j = h_2/2$, $j = 3, 4, 5, 6$, $\|u - \tilde{U}\|_{[-1,1]} = \varepsilon$, $\varepsilon = 0.000206$. Fig. 6 (b) shows the error of approximation of the Runge function with the polynomial splines, when $x_{j+1} = x_j + h_j$, $h_j = 0.2$, $n = 10$. Here $\varepsilon = 0.00248$.

(a) (b)

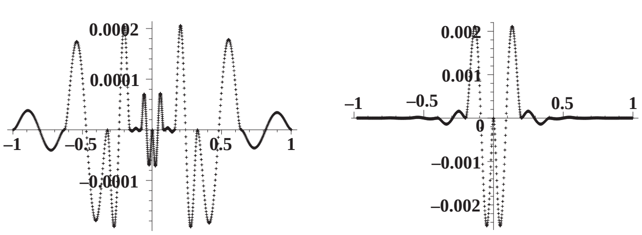


Fig. 6. Plots of the errors of approximation of the Runge function, when (a) $n = 10$, h_i we obtain using (11), $j = 0, 1, 2, 7, 8, 9$, $x_{j+1} = x_j + h_j$, $h_j = h_2/2$, $j = 3, 4, 5, 6$, and (b) $n = 10$, $x_{j+1} = x_j + h_j$, $h_j = 0.2$,

Fig. 7 (a) shows the error of approximation of the Runge function with the polynomial splines, when $h_0 = 0.344$, $n = 9$. Here h_j we obtain using (11), $j = 0, 1, 2, 6, 7, 8$, and $x_{j+1} = x_j + h_j$, $h_j = 0.73h_2$, $j = 3, 4, 5$, $\|u - \tilde{U}\|_{[-1,1]} = \varepsilon$, $\varepsilon = 0.000139$. Fig. 7 (b) shows the error of approximation of the Runge function with the polynomial splines, when $x_{j+1} = x_j + h_j$, $h_j = 2/9$, $n = 9$. Here $\varepsilon = 0.00222$.

(a) (b)

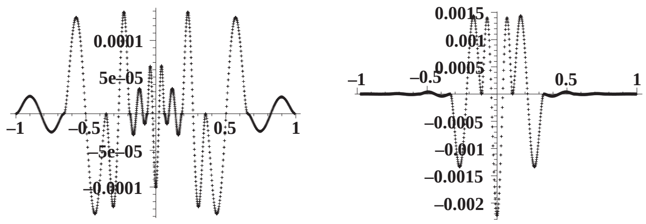


Fig. 7. Plots of the errors of approximation of the Runge function, $n = 9$, when (a) h_j we obtain using (11), $j = 0, 1, 2, 5, 6, 7, 8, 9$; $x_{j+1} = x_j + h_j$, $h_j = 0.7h_2$, $j = 3, 4$, and (b) $x_{j+1} = x_j + h_j$, $h_j = 2/9$, $j = 0, \ldots, 8$

Example 3. Fig. 8 (a) shows the error of approximation of the Runge function with the polynomial splines, when $x_0 = -1$, $x_1 = -0.4107$, $x_2 = -0.123$, $x_3 = 0.123$, $x_4 = 0.4107$, $x_5 = 1$, $I_k \approx 0.617$, $k = 0, 1, 2, 3, 4$, $\|u - \tilde{U}\|_{[-1,1]} = \varepsilon$, $\varepsilon = 0.00361$.

Fig. 8 (b) shows the error of approximation of the Runge function with the polynomial splines, when $x_0 = -1$, $x_1 = -0.361$, $x_2 = -0.1105$, $x_3 = 0.1105$, $x_4 = 0.361$, $x_5 = 1$, $I_0 = I_4 \approx 0.683$, $I_1 = I_3 \approx 0.593$, $I_2 \approx 0.532$. Here $\varepsilon = 0.00219$.

(a) (b)

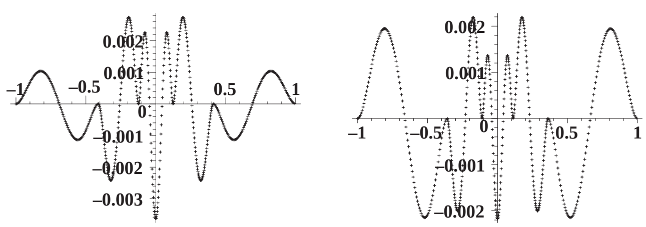


Fig. 8. Plots of the errors of approximation of the Runge function with the polynomial splines, (a) when $I_k \approx 0.617$, and (b) when $I_k \approx 0.683$, $k = 0, 4$; $I_k \approx 0.593$, $k = 1, 3$; $I_2 \approx 0.532$

Example 4. Fig. 9 (a) shows the error of approximation of the Runge function with the polynomial splines, when $x_0 = -1$, $x_1 = -0.310428$, $x_2 = 0$, $x_3 = 0.310428$, $x_4 = 1$, $I_0 = I_3 =\approx 0.7602$, $I_1 = I_2 \approx 0.7817$, $\|u - \tilde{U}\|_{[-1,1]} = \varepsilon$, $\varepsilon = 0.00388$.

Fig. 9 (b) shows the error of approximation of the Runge function with the polynomial splines, when $x_0 = -1$, $x_1 = -0.30404$, $x_2 = 0$, $x_3 = 0.30404$, $x_4 = 1$, $I_k \approx 0.771$, $k = 0, 1.2.3$. Here $\varepsilon = 0.00416$.

On every line parallel to axis y, we can construct the approximation in the form:

$$\widetilde{u}(y) = u(y_k)\omega_{k,0}(y) + u(y_{k+1})\omega_{k+1,0}(y) +$$
$$u'(y_j)\omega_{k,1}(y) + u'(y_{k+1})\omega_{k+1,1}(y) +$$
$$+ \int\limits_{y_k}^{y_{k+1}} u(t)dt\, \omega_k^{<0>}(y), \quad y \in [y_k, y_{k+1}]. \qquad (12)$$

(a) (b)

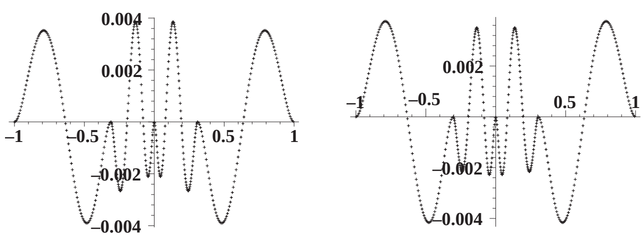


Fig. 9. Plots of the errors of approximation of the Runge function with the polynomial splines, when (a) $I_k \approx 0.7602$, $k = 0, 3$; $I_k \approx 0.7817$, $k = 1, 2$, and (b) when $I_k \approx 0.771$, $k = 0, 1, 2, 3$

Now we have the next formulas for $y = y_k + t_1 h$, $t_1 \in [0, 1]$:

$$\omega_{k,0}(y_k + t_1 h) = -18\, t_1^2 + 32\, t_1^3 - 15\, t_1^4 + 1, \qquad (13)$$

$$\omega_{k+1,0}(y_k + t_1 h) = -12\, t_1^2 + 28\, t_1^3 - 15\, t_1^4, \qquad (14)$$

$$\omega_{k,1}(y_k+t_1 h)=-(9/2)\, h\, t_1^2+6\, h\, t_1^3-(5/2)\, h\, t_1^4+t_1\, h, \qquad (15)$$

$$\omega_{k+1,1}(y_k + t_1 h) = (3/2)\, h\, t_1^2 - 4\, h\, t_1^3 + (5/2)\, h\, t_1^4, \qquad (16)$$

$$\omega_k^{<0>}(y_k + t_1 h) = (30\, t_1^2 - 60\, t_1^3 + 30\, t_1^4)/h. \qquad (17)$$

If $(x, y) \in \Omega_{j,k}$ then we get the next expression using the tensor product:

$$\tilde{u}(x, y) = \sum_{i=0}^{1}\sum_{p=0}^{1} u(x_{j+i}, y_{k+p})\omega_{j+i,0}(x)\omega_{k+p,0}(y)+$$
$$+\sum_{i=0}^{1}\sum_{p=0}^{1} u'_y(x_{j+i}, y_{k+p})\omega_{j+i,0}(x)\omega_{k+p,1}(y)+$$
$$\sum_{i=0}^{1}\int_{y_k}^{y_{k+1}} u(x_{j+i}, t)dtdy\omega_{j+i,0}(x)\omega_k^{<0>}(y)$$
$$+\sum_{i=0}^{1}\int_{x_j}^{x_{j+1}} u(t, y_{k+i})dt\omega_j^{<0>}(x)\omega_{k+i,0}(y)+$$
$$+\sum_{i=0}^{1}\int_{x_j}^{x_{j+1}} u'_y(t, y_{k+i})dt\omega_j^{<0>}(x)\omega_{k+i,1}(y)+$$
$$\int_{y_k}^{y_{k+1}}\int_{x_j}^{x_{j+1}} u(x, y)dxdy\omega_k^{<0>}(y)\omega_j^{<0>}(x)$$
$$+\sum_{i=0}^{1} u'_x(x_j, y_{k+i})dt\omega_{j,0}(x)\omega_{k+i,0}(y)+$$
$$+\sum_{i=0}^{1} u''_{xy}(x_j, y_{k+i})dt\omega_{j,0}(x)\omega_{k+i,1}(y)+$$
$$+\int_{y_k}^{y_{k+1}} u'_x(x_j, t)dt\omega_{j,1}(x)\omega_k^{<0>}(y). \qquad (18)$$

A. Trigonometrical splines of one variable

We denote by $\widetilde{\widetilde{u}}(x)$ an approximation of the function $u(x)$ on the interval $[x_j, x_{j+1}]$:

$$\widetilde{\widetilde{u}}(x) = u(x_j)\widetilde{\omega}_{j,0}(x) + u(x_{j+1})\widetilde{\omega}_{j+1,0}(x)+$$
$$u'(x_j)\widetilde{\omega}_{j,1}(x) + u'(x_{j+1})\widetilde{\omega}_{j+1,1}(x)+$$
$$+\int_{x_j}^{x_{j+1}} u(t)dt\, \widetilde{\omega}_j^{<0>}(x), \quad x \in [x_j, x_{j+1}]. \qquad (19)$$

The basic splines $\widetilde{\omega}_{j,0}(x)$, $\widetilde{\omega}_{j+1,0}(x)$, $\widetilde{\omega}_{j,1}(x)$, $\widetilde{\omega}_{j+1,1}(x)$, $\widetilde{\omega}_j^{<0>}(x)$, we obtain from the system:

$$\widetilde{\widetilde{u}}(x) \equiv u(x),\ u(x) = 1, \sin(kx), \cos(kx),\ k = 1, 2. \qquad (20)$$

We have for $x = x_j + th$, $t \in [0, 1]$, the next formulas:

$\widetilde{\omega}_{j,0}(x_j+th) = (6\, \sin(t\, h+h)-8\, \cos(t\, h)\, h+12\, h\, \cos(t\, h+h) - 4\, h\, \cos(-3\, h + t\, h) + 15\, \sin(h) - 6\, \sin(2\, h + t\, h) + 6\, \sin(-2\, h + t\, h) + 3\, \sin(3\, h) - 12\, \sin(2\, h) + 3\, \sin(-3\, h + 2\, t\, h)+2\, h\, \cos(-3\, h+2\, t\, h)+6\, h\, \cos(-h+2\, t\, h)+3\, \sin(h+2\, t\, h) - 6\, \sin(-h + 2\, t\, h) - 8\, h\, \cos(2\, t\, h) - 6\, \sin(-3\, h + t\, h))/(30\, \sin(h) - 24\, \sin(2\, h) - 16\, h + 18\, h\, \cos(h) - 2\, h\, \cos(3\, h) + 6\, \sin(3\, h))$,

$\widetilde{\omega}_{j+1,0}(x_j + th) = (3\, \sin(h + 2\, t\, h) - 6\, \sin(-h + 2\, t\, h) + 8\, h\, \cos(t\, h - h) - 6\, h\, \cos(-h + 2\, t\, h) - 2\, h\, \cos(h + 2\, t\, h) + 6\, \sin(t\, h + h) + 12\, \sin(2\, h) - 3\, \sin(3\, h) - 15\, \sin(h) + 8\, h\, \cos(2\, t\, h - 2\, h) - 6\, \sin(-3\, h + t\, h) + 3\, \sin(-3\, h + 2\, t\, h)-12\, h\, \cos(-2\, h+t\, h)+4\, h\, \cos(2\, h+t\, h)-6\, \sin(2\, h+t\, h) + 6\, \sin(-2\, h+t\, h))/(-30\, \sin(h)+24\, \sin(2\, h)+16\, h-18\, h\, \cos(h) + 2\, h\, \cos(3\, h) - 6\, \sin(3\, h))$,

$\widetilde{\omega}_{j,1}(x_j + th) = (10 + 6\, \cos(2\, h) - 2\, \cos(2\, t\, h) - 4\, h\, \sin(2\, t\, h)-15\, \cos(h)-8\, \cos(t\, h)+5\, \cos(-3\, h+2\, t\, h)+ 6\, h\, \sin(-h + 2\, t\, h) - 2\, h\, \sin(-3\, h + 2\, t\, h) - 8\, h\, \sin(t\, h) + 6\, \cos(-2\, h + t\, h) + 2\, \cos(2\, h + t\, h) - 4\, \cos(-3\, h + t\, h) + 4\, \cos(t\, h - h) + 12\, \cos(-h + 2\, t\, h) - \cos(h + 2\, t\, h) - 14\, \cos(2\, t\, h-2\, h)+6\, h\, \sin(t\, h+h)-\cos(3\, h)+2\, h\, \sin(-3\, h+ t\, h))/(30\, \sin(h) - 24\, \sin(2\, h) - 16\, h + 18\, h\, \cos(h) - 2\, h\, \cos(3\, h) + 6\, \sin(3\, h))$,

$\widetilde{\omega}_{j+1,1}(x_j + th) = (10 + 6\, \cos(2\, h) + 2\, \cos(-3\, h + t\, h) - 15\, \cos(h) - \cos(3\, h) - 2\, h\, \sin(2\, h + t\, h) - 6\, h\, \sin(-2\, h + t\, h) + 12\, \cos(-h + 2\, t\, h) + 5\, \cos(h + 2\, t\, h) - \cos(-3\, h + 2\, t\, h)+2\, h\, \sin(h+2\, t\, h)-6\, h\, \sin(-h+2\, t\, h)-2\, \cos(2\, t\, h- 2\, h)-14\, \cos(2\, t\, h)-4\, \cos(2\, h+t\, h)+4\, h\, \sin(2\, t\, h-2\, h)+ 8\, h\, \sin(t\, h - h) + 4\, \cos(t\, h) - 8\, \cos(t\, h - h) + 6\, \cos(t\, h + h))/(-30\, \sin(h) + 24\, \sin(2\, h) + 16\, h - 18\, h\, \cos(h) + 2\, h\, \cos(3\, h) - 6\, \sin(3\, h))$,

$\widetilde{\omega}_j^{<0>}(x_j + th) = (8 + 6\, \cos(-h + 2\, t\, h) + \cos(h + 2\, t\, h) - 9\, \cos(h) + \cos(3\, h) - 4\, \cos(2\, t\, h) - 2\, \cos(-3\, h + t\, h) - 4\, \cos(2\, t\, h - 2\, h) + \cos(-3\, h + 2\, t\, h) + 6\, \cos(-2\, h + t\, h) - 2\, \cos(2\, h + t\, h) - 4\, \cos(t\, h - h) + 6\, \cos(t\, h + h) - 4\, \cos(t\, h))/(-15\, \sin(h) + 12\, \sin(2\, h) + 8\, h - 9\, h\, \cos(h) + h\, \cos(3\, h) - 3\, \sin(3\, h))$.

It can be shown that the next formulas are valid:
$\tilde{\omega}_{j,0}(x_j + th) = (1 + 32\, t^3 - 18\, t^2 - 15t^4) + O(h^2)$,
$\tilde{\omega}_{j+1,0}(x_j + th) = (28\, t^3 - 12\, t^2 - 15t^4) + O(h^2)$,
$\tilde{\omega}_{j,1}(x_j+th)=(6t^3-(9/2)t^2-(5/2)t^4 + t)h+O(h^3)$,
$\tilde{\omega}_{j+1,1}(x_j+th)=(-4t^3+(3/2)t^2+(5/2)t^4)h+O(h^3)$,
$\tilde{\omega}_j^{<0>}(x_j + th) = (-60t^3 + 30t^2 + 30t^4)h^{-1} + O(h)$.

Theorem 2 The error of the approximation by the splines (19) is as follows:

$$|\widetilde{\widetilde{u}}(x) - u(x)| \leq Kh^5\|4u' + 5u''' + u^V\|_{[x_j,x_{j+1}]}, \qquad (21)$$

where $x \in [x_j, x_{j+1}]$, $K > 0$.

Proof: The function $u(x)$ on $[x_j, x_{j+1}]$ can be written in the form (see [26]): $u(x) = \frac{2}{3}\int_{x_j}^{x}(4u'(\tau) + 5u'''(\tau) + u^V(\tau))\sin^4(x/2 - \tau/2)d\tau + c_1 + c_2\sin(x) + c_3\cos(x) + c_4\sin(2x) + c_5\cos(2x)$, where c_i, $i = 1, 2, 3, 4, 5$ are arbitrary constants. Using the method from [26] we obtain (21).

Fig. 10 (a) shows the error of approximation of the function $7\cos(2x) + 5\sin(x)$ when $x_{j+1} = x_j + h_j$, $h_j = 2/9$, $n = 9$, with the polynomial splines, here $\varepsilon = 0.00000896$. Fig. 10 (b) shows the error of approximation of the function $7\cos(2x) + 5\sin(x)$ when $x_{j+1} = x_j + h_j$, $h_j = 2/9$, $n = 9$, with the trigonometrical splines, $Digits = 25$.

(a) (b)

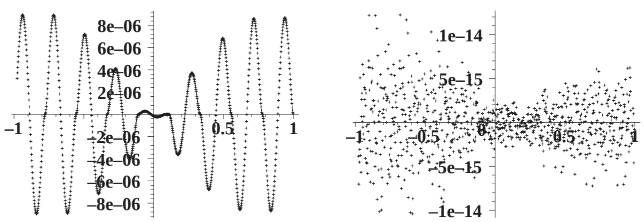


Fig. 10. Plots of the errors of approximation of the function $7\cos(2x) + 5\sin(x)$ with the polynomial splines (a), and with the trigonometrical splines (b)

Fig. 11 shows the error of approximation of the function $15\cos(2x)\sin(exp(x))$ with the polynomial splines, here $\varepsilon = 0.000677$ (a) and with the trigonometrical splines, here $\varepsilon = 0.000579$ (b), when $x_{j+1} = x_j + h$, $h = 2/9$, $n = 9$, Digits=25.

(a) (b)

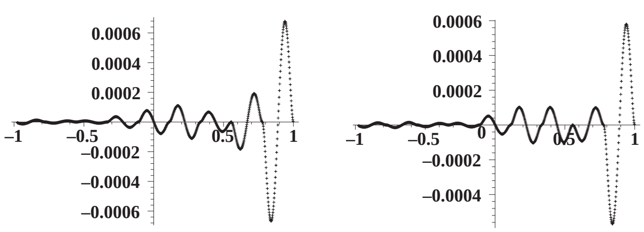


Fig. 11. Plots of the errors of approximation of the function $15\cos(2x)\sin(exp(x))$ with the polynomial splines (a), and with the trigonometrical splines (b)

We denote by $\widetilde{\widetilde{u}}(y)$ an approximation of the function $u(y)$ on the interval $[y_k, y_{k+1}]$. The basic splines $\widetilde{\omega}_{k,0}(y)$, $\widetilde{\omega}_{k+1,0}(y)$, $\widetilde{\omega}_{k,1}(y)$, $\widetilde{\omega}_{k+1,1}(y)$, $\widetilde{\omega}_k^{<0>}(y)$, we obtain from the system:

$$\widetilde{\widetilde{u}}(y) \equiv u(y), \; u(y) = 1, \sin(ky), \cos(ky), \; k = 1, 2. \qquad (22)$$

If $(x, y) \in \Omega_{j,k}$ then we use the expression (18), where we take trigonometrical $\widetilde{\omega}_{s,i}(x)$, $\widetilde{\omega}_{s,i}(y)$, $\widetilde{\omega}_j^{<0>}(x)$, $\widetilde{\omega}_k^{<0>}(y)$ instead of polynomial functions $\omega_{s,i}(x)$, $\omega_{s,i}(y)$, $\omega_j^{<0>}(x)$, $\omega_j^{<0>}(y)$.

Let us take $\tilde{\tilde{U}}(x)$, $x \in [a, b]$, such that $\tilde{\tilde{U}}(x) = \tilde{\tilde{u}}(x)$, $x \in [x_j, x_{j+1}]$.

Table 1 shows the error of the approximations $\tilde{\tilde{u}}(x, y) - u(x, y)$ with the tensor product of the trigonometric splines obtained from (20), (22) and the error of the approximations $\tilde{u}(x, y) - u(x, y)$ with the tensor product of the polynomial splines of the functions

$$u_1(x, y) = \frac{\cos(x)\cos(y)}{((1 + 25\sin^2(x))(1 + 25\sin^2(y)))},$$

$$u_2(x, y) = \frac{xy}{((1 + 25x^2)(1 + 25y^2))},$$

where $[a, b] = [-1, 1]$, $[c, d] = [-1, 1]$, $h = 0.1$, $y = 0.05$.

Here $R^I = \max_{x\in[-1,1]} |\tilde{U} - u|$, $R^{II} = \max_{x\in[-1,1]} |\tilde{\tilde{U}} - u|$.

Calculations were done in Maple, Digits=25.

Table 1 The error of the approximations with the tensor product

$u(x,y)$	R^I	R^{II}
$u_1(x,y)$	$0.70998e-5$	$0.70395e-5$
$u_2(x,y)$	$0.48035e-6$	$0.47316e-6$

III. Method 2

A. Left polynomial splines of one variable

Here we denote by $\widetilde{u}(x)$ an approximation of the function $u(x)$ in the interval $[x_j, x_{j+1}]$:

$$\widetilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x) + u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x) + \int_{x_{j-1}}^{x_j} u(t)dt\,\omega_j^{<-1>}(x). \qquad (23)$$

The basic splines $\omega_{j,0}(x)$, $\omega_{j+1,0}(x)$, $\omega_{j,1}(x)$, $\omega_{j+1,1}(x)$, $\omega_j^{<-1>}(x)$, we obtain from the system:

$$\widetilde{u}(x) \equiv u(x), \quad u(x) = x^{i-1}, i = 1, 2, 3, 4, 5. \qquad (24)$$

We have for $x = x_j + th$, $t \in [0, 1]$, the following formulas:

$$\omega_{j,0}(x_j+th) = (32/31)t^3 - (78/31)t^2 + (15/31)t^4 + 1, \qquad (25)$$

$$\omega_{j+1,0}(x_j+th) = (48/31)t^2 + (28/31)t^3 - (45/31)t^4, \qquad (26)$$

$$\omega_{j,1}(x_j+th) = (h/62)(85t^4 - 39t^2 - 108t^3 + 62t), \qquad (27)$$

$$\omega_{j+1,1}(x_j+th) = (h/62)(35t^4 - 27t^2 - 8t^3), \qquad (28)$$

$$\omega_j^{<-1>}(x_j+th) = (1/31)\,(30\,t^2 - 60\,t^3 + 30\,t^4)/h, \qquad (29)$$

Fig. 12, 13, 14 (a) show the plots of the basic functions $\omega_{j,0}(x)$, $\omega_{j+1,0}(x)$, $\omega_{j,1}(x)$, $\omega_{j+1,1}(x)$, $\omega_j^{<-1>}(x)$, when $h = 1$. Fig. 14 (b) shows the error of approximation of the Runge function $u(x) = 1/(1 + 25x^2)$ with the polynomial splines, $h = 0.1$, $x \in [-1, 1]$, $\|u - \tilde{U}\| = \varepsilon$, $\varepsilon = 0.00141$.

To improve the quality of the approximation we can chose the nodes $x_j \in [a, b]$ as the follows:

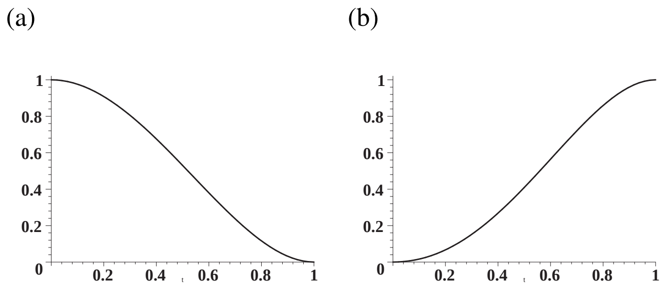


Fig. 12. Plots of the basic functions: $\omega_{j,0}(x)$ (a), $\omega_{j+1,0}(x)$ (b)

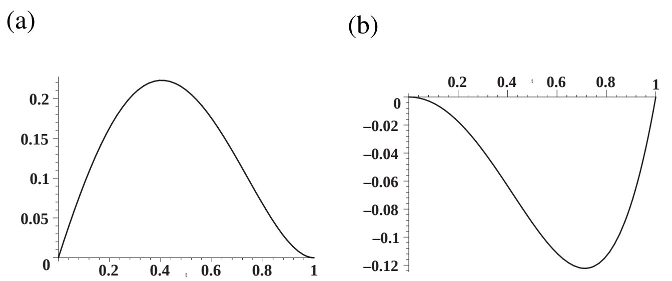


Fig. 13. Plots of the basic functions: $\omega_{j,1}(x)$, when $h = 1$(a), $\omega_{j+1,1}(x)$, when $h = 1$ (b)

$h_{-1} = x_0 - x_{-1}$, $x_{j+1} = x_j + h_j$, where h_j we obtain from the relation:

$$\int_{x_j}^{x_j+h_j} \sqrt{1+(u'(x))^2}\, dx = \int_{x_0-h_{-1}}^{x_0} \sqrt{1+(u'(x))^2}\, dx. \quad (30)$$

Example 4. Fig. 15 (a) shows the error of approximation of the Runge function with the polynomial splines, when $h_{-1} = 0.25$, h_j we obtain from (30), $j = 0, 1, 3, 4, 12, 13, 14$, and $h_j = 0.8h_4$, $j = 5, 6, 7, 8, 9, 10, 11$, here $\|u - \tilde{U}\| = \varepsilon$, $\varepsilon = 0.00019$. Fig. 15 (b) shows the error of approximation of the Runge function with the polynomial splines, when $h_{-1} = 0.151$, h_j we obtain from (30), $j = 0, 1, \ldots, 19$, here $\varepsilon = 0.000203$.

Theorem 3 Let function $u(x)$ be such that $u \in C^5([a, b])$. For the approximation $u(x)$, $x \in [x_j, x_{j+1}]$ by (23) – (29) we have

$$|\tilde{u}(x) - u(x)|_{[x_j,x_{j+1}]} \le K_3 h^5 \|u^{(5)}\|_{[x_{j-1},x_{j+1}]}, \quad (31)$$

$K_3 = 0.0135$,

$$|\tilde{u}'(x) - u'(x)|_{[x_j,x_{j+1}]} \le K_4 h^4 \|u^{(5)}\|_{[x_{j-1},x_{j+1}]}, \quad (32)$$

$K_4 = 0.064$,

$$|\tilde{U}(x) - u(x)|_{[a,b]} \le K_3 h^5 \|u^{(5)}\|_{[a,b]},\ K_3 = 0.0135, \quad (33)$$

Proof: Inequality (31) follows from Taylor's theorem and
$|\omega_{j,0}(x)| \le 1$, $|\omega_{j+1,0}(x)| \le 1$,
$|\omega_{j,1}(x)| \le 0.223h$, $|\omega_{j+1,1}(x)| \le 0.1223h$,
$\omega_j^{<-1>}(x)| \le 0.0605/h$.
Inequality (33) follows from (31).
Inequality (32) follows from Taylor's theorem and
$|\omega'_{j,0}(x)| \le 1.51/h$, $|\omega'_{j+1,0}(x)| \le 1.58/h$,
$|\omega'_{j,1}(x)| \le 1$, $|\omega'_{j+1,1}(x)| \le 1$,

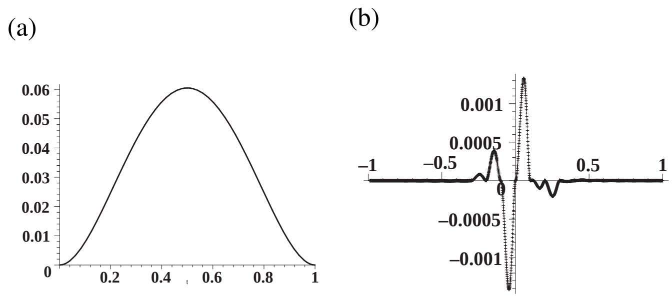


Fig. 14. Plots of the basic functions: $\omega_j^{<-1>}(x)$, when $h = 1$ (a), and the error of approximation of the Runge function with the polynomial splines, $h = 0.1$, $x \in [-1, 1]$ (b)

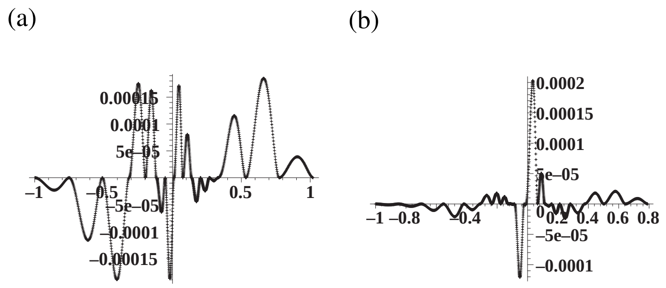


Fig. 15. Plots of the error of approximation of the Runge function with the polynomial splines: (a) $h_{-1} = 0.25$, h_j we obtain from (30), $j = 0, 1, 3, 4, 12, 13, 14$; $h_j = 0.8h_4$, $j = 5, 6, 7, 8, 9, 10, 11$, and (b) $h_{-1} = 0.151$, h_j we obtain from (30), $j = 0, \ldots, 19$

$$|\omega'^{<-1>}_j(x)| \le 1.187/h^2.$$

If $(x, y) \in \Omega_{j,k}$ then we put the approximation in the form of the tensor product:

$$\tilde{u}(x, y) = \sum_{i=0}^{1}\sum_{p=0}^{1} u(x_{j+i}, y_{k+p})\omega_{j+i,0}(x)\omega_{k+p,0}(y) +$$

$$+\sum_{i=0}^{1}\sum_{p=0}^{1} u'_y(x_{j+i}, y_{k+p})\omega_{j+i,0}(x)\omega_{k+p,1}(y) +$$

$$\sum_{i=0}^{1}\int_{y_{k-1}}^{y_k} u(x_{j+i}, t)dtdy\omega_{j+i,0}(x)\omega_k^{<-1>}(y)$$

$$+\sum_{i=0}^{1}\int_{x_{j-1}}^{x_j} u(t, y_{k+i})dt\omega_j^{<-1>}(x)\omega_{k+i,0}(y) +$$

$$+\sum_{i=0}^{1}\int_{x_{j-1}}^{x_j} u'_y(t, y_{k+i})dt\omega_j^{<-1>}(x)\omega_{k+i,1}(y) +$$

$$\int_{y_{k-1}}^{y_k}\int_{x_{j-1}}^{x_j} u(x, y)dxdy\omega_k^{<-1>}(y)\omega_j^{<-1>}(x)$$

$$+\sum_{i=0}^{1} u'_x(x_j, y_{k+i})dt\omega_{j,0}(x)\omega_{k+i,0}(y) +$$

$$+\sum_{i=0}^{1} u''_{xy}(x_j, y_{k+i})dt\omega_{j,0}(x)\omega_{k+i,1}(y) +$$

$$+\int_{y_{k-1}}^{y_k} u'_x(x_j,t)dt\omega_{j,1}(x)\omega_k^{<-1>}(y), \tag{34}$$

where

$$\omega_{k,0}(y_k+t_1h)=-\frac{78}{31}t_1^2+\frac{32}{31}t_1^3+\frac{15}{31}t_1^4+1, \tag{35}$$

$$\omega_{k+1,0}(y_k+t_1h)=\frac{48}{31}t_1^2+\frac{28}{31}t_1^3-\frac{45}{31}t_1^4, \tag{36}$$

$$\omega_{k,1}(y_k+t_1h)=-\frac{39}{62}ht_1^2-\frac{54}{31}ht_1^3+\frac{85}{62}ht_1^4+t_1h, \tag{37}$$

$$\omega_{k+1,1}(y_k+t_1h)=-\frac{27}{62}\,h\,t_1^2-\frac{4}{31}\,h\,t_1^3+\frac{35}{62}\,h\,t_1^4, \tag{38}$$

$$\omega_k^{<-1>}(y_k+t_1h)=\frac{1}{31}\,(30\,t_1^2-60\,t_1^3+30\,t_1^4)/h. \tag{39}$$

B. Trigonometrical splines of one variable

We denote by $\widetilde{\widetilde{u}}(x)$ an approximation of the function $u(x)$ on the interval $[x_j,x_{j+1}]$:

$$\widetilde{\widetilde{u}}(x)=u(x_j)\widetilde{\omega}_{j,0}(x)+u(x_{j+1})\widetilde{\omega}_{j+1,0}(x)+$$
$$u'(x_j)\widetilde{\omega}_{j,1}(x)+u'(x_{j+1})\widetilde{\omega}_{j+1,1}(x)+$$
$$+\int_{x_{j-1}}^{x_j} u(t)dt\,\widetilde{\omega}_j^{<-1>}(x),\quad x\in[x_j,x_{j+1}]. \tag{40}$$

The basic splines $\widetilde{\omega}_{j,0}(x)$, $\widetilde{\omega}_{j+1,0}(x)$, $\widetilde{\omega}_{j,1}(x)$, $\widetilde{\omega}_{j+1,1}(x)$, $\widetilde{\omega}_j^{<1>}(x)$, we obtain from the system:

$$\widetilde{\widetilde{u}}(x)\equiv u(x),\ u(x)=1,\sin(kx),\cos(kx),\ k=1,2. \tag{41}$$

We get from (41) and $x=jh+th$, $t\in[0,1]$, the following formulas:

$\widetilde{\omega}_{j,0}(x_j+th) = (4\sin(2h)-\sin(h)-6\sin(3h)+4\sin(4h)-\sin(5h)-8\sin(3h+th)+2\sin(-3h+th)-2\sin(2th+h)-\sin(2th-h)-\sin(3h+2th)+4\sin(-3h+2th)-2\sin(th+h)+8\sin(th-h)+2\sin(4h+th)-6\sin(-2h+th)+6\sin(2h+th)-2\sin(-4h+th)+4\sin(2th+2h)-4\sin(2th-2h)+2h\cos(-3h+2th)+6h\cos(2th-h)+12h\cos(th+h)-4h\cos(-3h+th)-8h\cos(th)-8h\cos(2th))/(-16h+24\sin(2h)-12\sin(h)-21\sin(3h)+8\sin(4h)+18h\cos(h)-2h\cos(3h)-\sin(5h))$,

$\widetilde{\omega}_{j+1,0}(x_j+th) = (4\sin(4h)-8h\cos(th-h)+20\sin(2h)+2\sin(2th+h)+\sin(2th-h)-4\sin(2th+2h)+4\sin(2th-2h)+12h\cos(-2h+th)-4h\cos(2h+th)-8h\cos(2th-2h)+\sin(3h+2th)-4\sin(-3h+2th)-15\sin(3h)-11\sin(h)+8\sin(3h+th)-2\sin(-3h+th)+6h\cos(2th-h)+2h\cos(2th+h)-2\sin(4h+th)+2\sin(-4h+th)+2\sin(th+h)-8\sin(th-h)-6\sin(2h+th)+6\sin(-2h+th))/(-16h+24\sin(2h)-12\sin(h)-21\sin(3h)+8\sin(4h)+18h\cos(h)-2h\cos(3h)-\sin(5h))$,

$\widetilde{\omega}_{j,1}(x_j+th) = (16-2\cos(2h)-17\cos(h)-\cos(5h)+2\cos(3h)+6\cos(-3h+2th)+13\cos(2th-h)+12\cos(th+h)-8\cos(th)-8\cos(2th)-6\cos(th-h)+6\cos(-2h+th)+2\cos(2h+th)-10\cos(2th-2h)-2\cos(2th+h)+2\cos(4h+th)-\cos(3h+2th)-7\cos(3h+th)+4\cos(2th+2h)-2\cos(-4h+2th)+\cos(-5h+th)-2\cos(-4h+th)+2\cos(4h)-6h\sin(th+h)-2h\sin(-3h+th)+8h\sin(th)-6h\sin(2th-h)+2h\sin(-3h+2th)+4h\sin(2th))/(16h-24\sin(2h)+12\sin(h)+21\sin(3h)-8\sin(4h)-18h\cos(h)+2h\cos(3h)+\sin(5h))$,

$\widetilde{\omega}_{j+1,1}(x_j+th) = (14\cos(2h)-7\cos(h)-9\cos(3h)+2\cos(-3h+2th)+3\cos(2th-h)-10\cos(th+h)+2\cos(-3h+th)+6\cos(th)-6\cos(2th)+4\cos(th-h)-4\cos(-2h+th)-4\cos(2th-2h)+10\cos(2th+h)-\cos(4h+th)+\cos(3h+2th)+4\cos(3h+th)-6\cos(2th+2h)-\cos(-4h+th)+2\cos(4h)+6h\sin(2th-h)+2h\sin(2h+th)+6h\sin(-2h+th)-4h\sin(2th-2h)-8h\sin(th-h)-2h\sin(2th+h))/(-16h+24\sin(2h)-12\sin(h)-21\sin(3h)+8\sin(4h)+18h\cos(h)-2h\cos(3h)-\sin(5h))$,

$\widetilde{\omega}_k^{<-1>}(x_j+th) = (-16-2\cos(-3h+2th)-2\cos(2th+h)-12\cos(2th-h)-2\cos(3h)+8\cos(th)+18\cos(h)+4\cos(-3h+th)-12\cos(th+h)+8\cos(th-h)+8\cos(2th-2h)+8\cos(2th)-12\cos(-2h+th)+4\cos(2h+th))/(-16h+24\sin(2h)-12\sin(h)-21\sin(3h)+8\sin(4h)+18h\cos(h)-2h\cos(3h)-\sin(5h))$.

It can be shown that the next relations are valid:

$\widetilde{\omega}_{j,0}(x_j+th) = (1+(32/31)t^3-(78/31)t^2+(15/31)t^4)+O(h^2)$,

$\widetilde{\omega}_{j+1,0}(x_j+th) = ((28/31)t^3+(48/31)t^2-(45/31)t^4)+O(h^2)$,

$\widetilde{\omega}_{j,1}(x_j+th) = (t-(54/31)t^3-(39/62)t^2+(85/62)t^4)h+O(h^3)$,

$\widetilde{\omega}_{j+1,1}(x_j+th) = ((-4/31)t^3-(27/62)t^2+(35/62)t^4)h+O(h^3)$,

$\widetilde{\omega}_k^{<-1>}(x_j+th) = (-(60/31)t^3+(30/31)t^2+(30/31)t^4)h^{-1}+O(h)$.

Fig. 16 shows the error of approximation of the function $7\cos(2x)+5\sin(x)$: (a) with the polynomial splines, here $\varepsilon=0.00176$, and (b) with the trigonometrical splines, when $x_{j+1}=x_j+h$, $h=2/9$, $n=9$, $Digits=25$.

(a) (b)

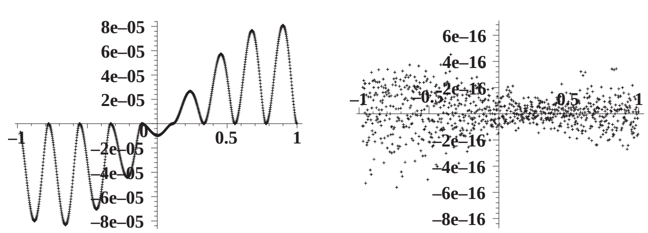
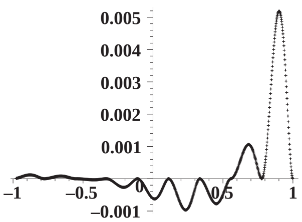


Fig. 16. Plots of the errors of approximation of the function $7\cos(2x)+5\sin(x)$: (a) with the polynomial splines, and (b) with the trigonometrical splines

Fig. 17 shows the error of approximation of the function $15\cos(2x)\sin(exp(x))$: (a) with the polynomial splines, here $\varepsilon=0.00519$, and (b) with the trigonometrical splines, here $\varepsilon=0.00425$, when $x_{j+1}=x_j+h_j$, $h_j=2/9$, $n=9$.

Fig. 18 shows the approximation of the function $((\sin(x-y)\cos(x-y))$ and the error of its approximation, where we use the middle polynomial splines, $h=0.2$, $\Omega=[-1,1]\times[-1,1]$.

(a)



(b)

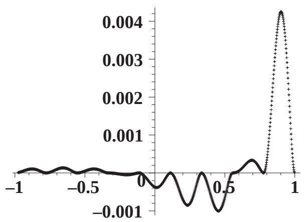
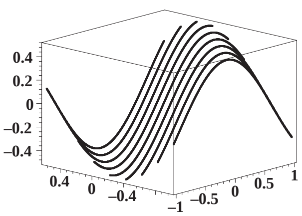


Fig. 17. Plots of the errors of approximation of the function $15\cos(2x)\sin(exp(x))$: (a) with the polynomial splines, and (b) with the trigonometrical splines

(a)



(b)

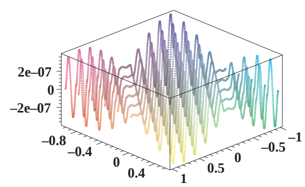
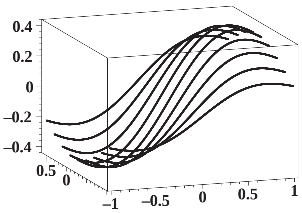


Fig. 18. Plots of the approximation of the function $((\sin(x-y)\cos(x-y))$ with the middle polynomial splines (*left*), the error of its approximation with the polynomial splines (*right*), here $h = 0.2$, $\Omega = [-1,1]\times[-1,1]$

Fig. 19 shows the approximation of the function $x\,exp(-x^2-y^2)$ with the middle polynomial splines and the error of the approximation, where $h = 0.2$, $\Omega = [-1,1]\times[-1,1]$.

(a)



(b)

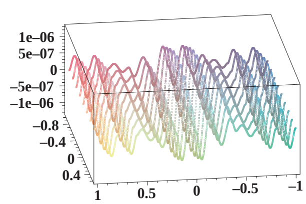


Fig. 19. Plots of the approximation of the function $x\ exp(-x^2-y^2)$ by the polynomial splines (a), the error of its approximation with the middle polynomial splines (b), here $h = 0.2$, $\Omega = [-1,1]\times[-1,1]$

Fig. 20 shows the approximation with the middle polynomial splines of the function $u_3(x,y) = 1/((1+25x^2)(1+25y^2))$ and the error of the approximation, where $h = 0.2$, $\Omega = [-1,1]\times[-1,1]$.

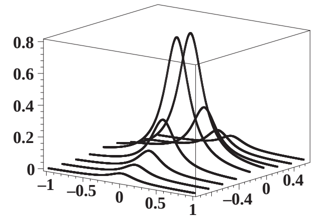
IV. Discrete version of polynomial integro-differential splines

Consider the solution to the problem of finding some approximating function that describes an experimental law of a distribution using the proposed polynomial splines.

We use the quadrature formula for the approximate calculation of the integrals in (23) and in (1):

$$\int_{x_k}^{x_{k+1}} u(x)dx = \frac{3}{5}hu(x_k) + \frac{2}{5}hu(x_{k+1}) + \frac{3}{20}h^2u'(x_k)$$

(a)



(b)

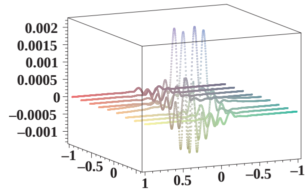


Fig. 20. Plots of the approximation of the function $u_3 = 1/((1+25x^2)(1+25y^2))$ (a), and the error of approximation of the function u_3 (b), here $h = 0.2$, $\Omega = [-1,1]\times[-1,1]$

$$-\frac{h^2}{20}u'(x_{k+1}) + \frac{h^3}{60}u''(x_k) + O(h^5),$$

$$\int_{x_{k-1}}^{x_k} u(x)dx = \frac{3h}{5}u(x_{k-1}) + \frac{2h}{5}u(x_k) +$$

$$\frac{3h^2}{20}u'(x_{k-1}) - \frac{h^2}{20}u'(x_k) + \frac{h^3}{60}u''(x_{k-1}) + O(h^5),$$

and use formulas for numerical differentiation if $k = 2,3,\ldots,n-2$,

$$u''(x_k) = \frac{1}{24h^2}(-2u(x_{k-2}) +$$

$$32u(x_{k-1}) - 60u(x_k) + 32u(x_{k+1}) - 2u(x_{k+2})) + O(h^6);$$

$$u'(x_k) = \frac{1}{12h}(u(x_{k-2}) - 8u(x_{k-1}) +$$

$$8u(x_{k+1}) - u(x_{k+2})) + O(h^5).$$

For $k = 0, 1$

$$u''(x_k) = \frac{1}{24h^2}(70u(x_k) - 208u(x_{k+1})$$

$$+228u(x_{k+2}) - 112u(x_{k+3}) + 22u(x_{k+4})) + O(h^5),$$

$$u'(x_k) = \frac{1}{12h}(-25u(x_k) + 48u(x_{k+1})$$

$$-36u(x_{k+2}) + 16u(x_{k+3}) - 3u(x_{k+4})) + O(h^5);$$

for $k = n-1, n$

$$u''(x_k) = \frac{1}{24h^2}(70u(x_k) - 208u(x_{k-1}) +$$

$$228u(x_{k-2}) - 112u(x_{k-3}) + 22u(x_{k-4})) + O(h^5),$$

$$u'(x_k) = \frac{1}{12h}(25u(x_k) - 48u(x_{k-1}) +$$

$$36u(x_{k-2}) - 16u(x_{k-3}) + 3u(x_{k-4})) + O(h^5).$$

Table 2 shows the actual errors of the approximation function and their first derivatives with the left integro-differential splines. Here $\widetilde{R}^L$ is the actual error of the approximation function and $\widetilde{R}_1^L$ is the actual error of the approximation's first derivative in a case in which we only have the values of the function $u(x_k)$, $k = 0,\ldots,n$, $x \in [-1,1]$, $h = 0.1$.

Table 3 shows the theoretical errors $\widetilde{R}^{TL}$ of the approximation function and its first derivative $\widetilde{R}_1^{TL}$ with the left integro-differential splines.

Table 2 The actual errors of the approximation function and its first derivative, $h = 0.1$

No	$u(x)$	$\widetilde{R}^L$	$\widetilde{R}_1^L$
1	$\sin(3x)\cos(5x)$	$0.22 \cdot 10^{-3}$	$0.73 \cdot 10^{-2}$
2	$tg(x)$	$0.32 \cdot 10^{-4}$	$0.10 \cdot 10^{-2}$
3	$\cos(2x)$	$0.49 \cdot 10^{-6}$	$0.16 \cdot 10^{-4}$
4	$\dfrac{1}{(1+25x^2)}$	$0.12 \cdot 10^{-2}$	$0.45 \cdot 10^{-1}$

Table 3 The theoretical errors of the approximation functions, $h = 0.1$

No	$u(x)$	$\widetilde{R}^{TL}$	$\widetilde{R}_1^{TL}$
1	$\sin(3x)\cos(5x)$	$0.36 \cdot 10^{-2}$	0.10
2	$tg(x)$	$0.76 \cdot 10^{-3}$	$0.22 \cdot 10^{-1}$
3	$\cos(2x)$	$0.70 \cdot 10^{-5}$	$0.20 \cdot 10^{-3}$
4	$\dfrac{1}{(1+25x^2)}$	$0.69 \cdot 10^{-1}$	1.98

In [30] noted that "in practice, the use of a probabilistic approach to the evaluation of errors of measurement results primarily assumes the knowledge of the analytical model of the distribution law of the considered error. Occurring in metrology distributions diverse enough". A large part of these distributions are bimodal.

Fig. 21 (a) shows the histogram, the density distribution and the approximation with the left polynomial splines of the density of bimodal distribution $f = (f_1 + f_2)/2$, where

$$f_i = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-(x-\alpha_i)^2/(2\sigma_i^2)},$$

$i = 1, 2,\ \sigma_1 = 0.5, \sigma_2 = 0.8, \alpha_1 = -0.8, \alpha_2 = 1$, by the left polynomial splines of the interval $[-2, 3]$. Fig. 21 (b) shows the error of the approximation of the density of distributions

(a) (b)

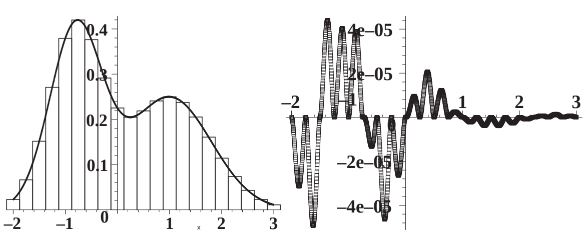


Fig. 21. The histogram, the density distribution and the approximation of the density distribution (a); the errors of the approximation of the density distributions (b)

Fig. 22 (a) shows the histogram, the density distribution and the approximation of the density distribution

$$f_0 = \frac{1}{\sqrt{2\pi}\sigma} e^{\left(\frac{-(x-\alpha)^2}{2\sigma^2}\right)}, \text{ where } \sigma = 0.5, \alpha = 0$$

of the interval $[-2, 2]$.

Fig. 22 (b) shows the error of the approximation of the density of distributions

(a) (b)

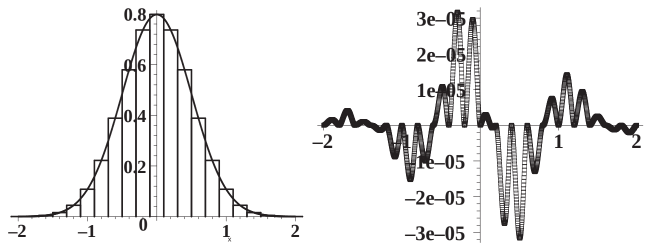


Fig. 22. The histogram, the density distribution and the approximation of the density distribution f_0 (a), and the error of the approximation of the density of distributions (b)

V. Conclusion

We can take as an approximation of the functions of two variables in the form of the tensor product of polynomial splines in one direction and trigonometric splines in other direction if it is necessary to improve the properties of the approximation.

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