The use of Navier-Stokes equations in modeling water quality in river-type systems

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***Abstract* —** This paper addresses the problem of water quality in river-type systems. The actuality of water quality modeling based on mathematical and numerical modeling is discussed. The water motion in river-type systems is presented on the basis of Navier-Stokes equations.

***Keywords* —** modeling, Navier-Stokes equations, river, turbulent motion, water quality.

I. Introduction

WATER quality is a primary problem for the sustainable development of the countries. Water is a complex physical, biochemical and ecological system, therefore the problem of water quality is a difficult problem. As a result of human activities, water quality has diminished dramatically. The evaluation of several water bodies, according to the requirements of Water Framework Directive, denotes a satisfactory or unsatisfactory ecological status [1, 2, 3]. Frequently for human necessities water from river is used [4]. Its quality is more often influenced by various chemical, physical and biological substances [5].

An effective solution for analysis and solving of various problems in aquatic systems is the use of informational systems consisting of two components: mathematical modeling of the studied sector and evaluation of water quality with numerical models obtained [6, 7, 8]. The use of mathematical modeling helps to predict aquatic systems behavior, as well as to determine results of actions of various processes on the aquatic systems [9, 10]. Choosing the appropriate mathematical model and simulation software will allow a proper evaluation of water quality [6, 7, 8].

In the modeling process are used algebraic equations and systems of algebraic equations, ordinary differential equations, partial differential equations et al. [9].

Navier-Stokes equations are at the core of modeling fluids. These equations describe the conservation of mass, momentum and energy. The solutions of the equations are the fields of variables in the modeled area [11, 12]. For solving the Navier-Stokes equations, numerical simulation models are used [13, 14]. The knowledge of characteristic measures of the aquatic systems makes it possible to study and determine the processes of transport and dispersion of pollutants.

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II. The modeling of turbulence in river-type systems using Navier-Stokes equations

The flow in river-type systems is a turbulent flow which can be described using the Navier-Stokes system of equations (1) and the continuity equation (2):

$$\frac{\partial \mathrm{v}}{\partial t} + \mathrm{v}\nabla\mathrm{v} = f - \frac{1}{\rho}\nabla p + \nu\Delta\mathrm{v}, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0, \tag{2}$$

where ∇ - Hamilton operator, Δ - Laplace operator, t - time, ν - the coefficient of viscosity, ρ - density, p - pressure, v - velocity, f - the field mass forces [13].

When applying these equations to river-type systems, the following simplifying assumptions should be considered:

- incompressible Newtonian fluids with free surface;
- neglecting vertical acceleration;
- hydrostatic equilibrium, under which pressure is balanced by the gravitational force:
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$$\nabla p + \rho g = 0 \tag{3}$$

where p - pressure; ρ – water density (kg/m^3);

- the action of gravitational acceleration:

$$G = -g\nabla h, \tag{4}$$

which components in the x and y directions, respectively, are:

$$G_x = -g\frac{\partial h}{\partial x}, \tag{5}$$

$$G_y = -g\frac{\partial h}{\partial y}, \tag{6}$$

where h is the normal to the surface of water from the earth surface, which, in this case, represents the water depth;

- the rotation of the Earth, which is taken into account by the Coriolis effect F_c. The components of Coriolis acceleration a_c in the x and y directions, respectively, are:

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$$a_{Cx} = 2\omega u sin\varphi, \quad (7)$$

$$a_{Cy} = -2\omega v sin\varphi, \quad (8)$$

where ω is the angular velocity of the Earth's rotation (rad/s); u – local velocity in the x direction (m/s); v – local velocity in the y direction (m/s); φ – latitude of the place;

- the transfer of energy by roughness. The change of the roughness ensures a certain control on the resultant velocity and fluid direction. The inferior shear stress or the tangential stress τ is defined as:

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$$\tau = \rho g R S, \quad (9)$$

where R is the hydraulic mean radius, and S is the bed slope (inclination).

The roughness is calculated using the Manning equation:

$$V = \frac{R^{2/3} \cdot S^{1/2}}{n}, \quad (10)$$

where V is the average velocity of the water particles in cross-section, and n – the value of Manning roughness.

Taking into account that the hydraulic mean radius $R \approx h$ and solving the Manning equation for S, we obtain:

$$\tau = \rho g n^2 \frac{V^2}{R^{1/3}} \quad (11)$$

or in the x and y directions:

$$\tau_x = \rho g n^2 \frac{u\sqrt{u^2+v^2}}{h^{1/3}}, \quad (12)$$

$$\tau_y = \rho g n^2 \frac{v\sqrt{u^2+v^2}}{h^{1/3}}; \quad (13)$$

- the effects of the wind, that are taken into account by the shear stress W of the wind. The shear stress of the wind at the surface is caused by the friction between air and water in motion. In the x and y directions:

$$W_x = \xi V_a^2 cos\psi, \quad (14)$$

$$W_y = \xi V_a^2 sin\omega, \quad (15)$$

where ξ is the empiric coefficient of the air friction; V_a – velocity of wind; ψ – direction of wind (is measured in degrees in the counterclockwise direction from the positive *x*-axis)

Based on the aspects mentioned, the equation (1) becomes:

$$\frac{\partial v}{\partial t} + v\nabla v + F_c = W - \frac{1}{\rho}\nabla p + \nu\Delta v + \tau \quad (16)$$

or, considering (3), the equation (16) becomes:

$$\frac{\partial v}{\partial t} + v\Delta v + F_c = W + G + \nu\Delta v + \tau. \quad (17)$$

Based on (4), the equation (17) takes the form:

$$\frac{\partial v}{\partial t} + v\Delta v + F_c = W - g\nabla h + \nu\Delta v + \tau. \quad (18)$$

In the *x* and *y* directions, taking into consideration the expressions (5) – (15), the equation (18) takes the form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + 2\omega u sin\varphi = \xi V^2 cos\psi - g\frac{\partial h}{\partial x} + E_{xx}\frac{\partial^2 u}{\partial x^2} + E_{xy}\frac{\partial^2 u}{\partial y^2} + \rho g n^2 \frac{u\sqrt{u^2+v^2}}{h^{1/3}}, \quad (19)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - 2\omega v sin\varphi = \xi V^2 sin\psi - g\frac{\partial h}{\partial y} + E_{yx}\frac{\partial^2 v}{\partial x^2} + E_{yy}\frac{\partial^2 v}{\partial y^2} + \rho g n^2 \frac{v\sqrt{u^2+v^2}}{h^{1/3}}, \quad (20)$$

where E are coefficients of turbulent viscosity (Pas or kg / m/s).

Coefficients of turbulent viscosity in the equations (19) and (20) represent the molecular viscosity and the effects of turbulence through turbulent stress or Reynolds stress:

$$E_{xx}\frac{\partial^2 u}{\partial x^2} = \nu\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\frac{\delta\overline{u'^2}}{\partial x}, \quad (21)$$

$$E_{xy}\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\frac{\delta\overline{u'v'}}{\partial y}, \quad (22)$$

$$E_{yx}\frac{\partial^2 v}{\partial x^2}=\nu\frac{\partial^2 v}{\partial x^2}+\frac{\partial}{\partial y}\frac{\delta\overline{v'u'}}{\partial x}, \tag{23}$$

$$E_{yy}\frac{\partial^2 v}{\partial y^2}=\nu\frac{\partial^2 v}{\partial y^2}+\frac{\partial}{\partial y}\frac{\delta\overline{v'^2}}{\partial y}, \tag{24}$$

where ν is the molecular viscosity; u' – fluctuations of the turbulent velocity in the x direction; v' – fluctuations of the turbulent velocity in the y direction; $\overline{a}$ – temporal mean value.

Integrating the equations (19) and (20) in depth *h*:

For the *x* direction:

$$\int_0^h\frac{\partial u}{\partial t}dz+u\int_0^h\frac{\partial u}{\partial x}dz+v\int_0^h\frac{\partial u}{\partial y}dz=2\omega u\sin\varphi\int_0^h dz+\xi V^2\cos\psi-$$
$$-g\int_h^H\frac{\partial h}{\partial x}dz+\int_0^h\left(E_{xx}\frac{\partial^2 u}{\partial x^2}+E_{xy}\frac{\partial^2 u}{\partial y^2}\right)dz+\rho g n^2\frac{u\sqrt{u^2+v^2}}{h^{1/3}}. \tag{25}$$

For the y direction:

$$\int_0^h\frac{\partial v}{\partial t}dz+u\int_0^h\frac{\partial v}{\partial x}dz+v\int_0^h\frac{\partial v}{\partial y}dz=-2\omega v\sin\varphi\int_0^h dz+\xi V^2\sin\psi-$$
$$-g\int_h^H\frac{\partial h}{\partial y}dz+\int_0^h\left(E_{yx}\frac{\partial^2 v}{\partial y^2}+E_{yy}\frac{\partial^2 v}{\partial y^2}\right)dz+\rho g n^2\frac{v\sqrt{u^2+v^2}}{h^{1/3}}. \tag{26}$$

As a result of the integration will be obtained the Navier-Stokes equations in the Reynolds form for the Cartesian coordinates *x* and *y:*

$$h\frac{\partial u}{\partial t}+hu\frac{\partial u}{\partial x}+hv\frac{\partial u}{\partial y}-2hu\omega\sin\varphi-\xi V^2\cos\psi+gh\left(\frac{\partial H}{\partial x}+\frac{\partial h}{\partial x}\right)-$$
$$-h\left(E_{xx}\frac{\partial^2 u}{\partial y^2}+E_{xy}\frac{\partial^2 u}{\partial y^2}\right)+\frac{\rho g u n^2}{\left(h^{1/6}\right)^2}(u^2+v^2)^{1/2}=0, \tag{27}$$

$$h\frac{\partial v}{\partial t}+hu\frac{\partial v}{\partial x}+hv\frac{\partial v}{\partial y}+2h\omega v\sin\varphi-\xi V^2\sin\psi+gh\left(\frac{\partial H}{\partial y}+\frac{\partial h}{\partial y}\right)-$$
$$-h\left(E_{yx}\frac{\partial^2 v}{\partial x^2}+E_{yy}\frac{\partial^2 v}{\partial y^2}\right)+\frac{\rho g v n^2}{\left(h^{1/6}\right)^2}(u^2+v^2)^{1/2}=0, \tag{28}$$

where *H* is the geodesic share bed (m).

The continuity equation results from the law of conservation of mass. For incompressible fluids, $\rho=const$, $\frac{\partial\rho}{\partial t}=0$, the equation (2) becomes:

$$\nabla u=0. \tag{29}$$

In Cartesian coordinates, the equation (29) has the form:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{30}$$

Integrating the equation (30) in depth:

$$\int_0^h\frac{\partial u}{\partial x}dz+\int_0^h\frac{\partial v}{\partial y}dz+\int_0^h\frac{\partial w}{\partial z}dz=0. \tag{31}$$

As a result of integration, the following equation is obtained:

$$h\frac{\partial u}{\partial x}+u\frac{\partial h}{\partial x}+h\frac{\partial v}{\partial y}+v\frac{\partial h}{\partial y}+\frac{\partial h}{\partial t}=0 \tag{32}$$

Or:

$$\frac{\partial h}{\partial t}+h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}=0. \tag{33}$$

The Navier-Stokes equations are a nonhomogeneous and nonlinear system of differential equations with second order partial derivative. The main source of turbulence is considered the inertial term $v\nabla v$, which represents the nonlinearity of the system. Solving these equations is possible only for simplified cases. Currently it is demonstrated only the existence of weak solutions [15]

The mathematical modeling of the turbulent flow in several sectors of the Prut river from the Republic of Moldova was conducted by the authors of this paper [16, 17, 18, 19]. To determine the hydrodynamics of the studied sectors were used the Navier-Stokes system of equations in the Reynolds form (1) and the continuity equation (2). The numerical solving of the mentioned equations was performed using the RMA2 (Resource Management Associates) software, which is a part of the SMS (Surface Water Modeling System).

Based on CFD numerical simulations of the water flow in the studied sectors, were determined:

- water depth;
- variation of the resultant velocity;
- velocity of the moving particles of water in the x direction;
- velocity of the moving particles of water in the y direction.

In Fig. 1 is presented the variation of the resultant velocity for the examined case studies.

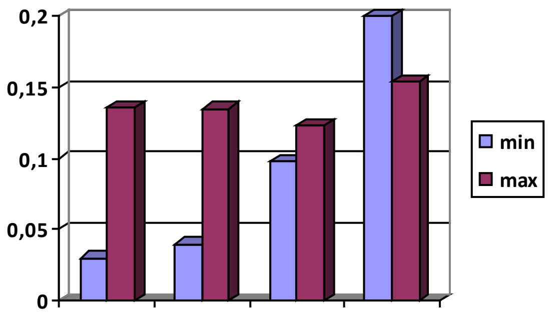

Fig. 1 Resulting velocity

The results obtained on hydrodynamics of the studied sectors were used to further determine the dispersion of pollutants in the studied sectors of the Prut River. The analysis of the results obtained showed a good correlation between the data measured from field and the data calculated using the numerical models [16, 17, 18, 19].

III. Conclusion

The study conducted proved that Navier-Stokes equations are a powerful tool to model turbulent water flow in river-type systems and to determine water quality.

Based on the mathematical and numerical modeling of water motion in the studied sectors were determined the most important parameters of water motion: the field of velocities and the field of depths. This allowed to determine the spatial-temporal evolution of the processes of transport and dispersion of pollutants.

The obtained results showed a good capacity of the mathematical model to faithfully reproduce real processes in river-type aquatic systems. This fact was confirmed by comparing the actual data, collected in situ, with the obtained data using the elaborated numerical models.

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