Generalized Least-Powers Regressions I: Bivariate Regressions

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***Abstract*—** The bivariate theory of generalized least-squares is extended here to least-powers. The bivariate generalized least-powers problem of order p seeks a line which minimizes the average generalized mean of the absolute pth power deviations between the data and the line. Least-squares regressions utilize second order moments of the data to construct the regression line whereas least-powers regressions use moments of order p to construct the line. The focus is on even values of p, since this case admits analytic solution methods for the regression coefficients. A numerical example shows generalized least-powers methods performing comparably to generalized least-squares methods, but with a wider range of slope values.

***Keywords*—**Least-powers, generalized least-powers, least-squares, generalized least-squares, geometric mean regression, orthogonal regression, least-quartic regression.

I. Overview

FOR two variables x and y ordinary least-squares $y|x$ regression suffers from a fundamental lack of symmetry. It minimizes the distance between the data and the regression line in the dependent variable y alone. To predict the value of the independent variable x one cannot simply solve for this variable using the regression equation. Instead one must derive a new regression equation treating x as the dependent variable. This is called ordinary least-squares $x|y$ regression. The fact that there are two ordinary least-squares lines to model a single set of data is problematic. One wishes to have a single linear model for the data, for which it is valid to solve for either variable for prediction purposes. A theory of generalized least-squares was developed by this author to overcome this problem by minimizing the average generalized mean of the square deviations in both x and y variables [5]–[8]. For the resulting regression equation, one can solve for x in terms of y in order to predict the value of x. This theory was subsequently extended to multiple variables [9].

In this paper, the extension of the bivariate theory of generalized least-squares to least-powers is begun. The bivariate generalized least-powers problem of order p seeks a line which minimizes the average generalized mean of the absolute pth power deviations between the data and the line. Unlike least-squares regressions which utilize second order moments of the data to construct the regression line, least-pth powers regressions utilize moments of order p to construct the line.

In the interest of generality, the definitions here are formulated using arbitrary powers p. Nevertheless, the focus of this paper is on even values of p, since this case allows for the derivation of analytic formulas to find the regression coefficients, analogous to what was done for least-squares. The numerical example presented illustrates that bivariate generalized least-powers methods perform comparably to generalized least-squares methods but have a greater range of slope values.

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II. Bivariate Ordinary and Generalized Least-Powers Regression

A. Bivariate Ordinary Least-Powers Regression OLP_p

The generalization of ordinary least-squares regression (OLS) to arbitrary powers is called ordinary least-powers regression of order p and is denoted here by OLP_p. OLS is the same thing as OLP_2. The case of OLP_4, called least-quartic regression, is described in a paper by Arbia [1].

Definition 1: (The Ordinary Least-Powers Problem) Values of a and b are sought which minimize an error function defined by

$$E = \frac{1}{N}\sum_{i=1}^{N} |a + bx_i - y_i|^p . \quad (1)$$

The resulting line $y = a + bx$ is called the ordinary least-powers $y|x$ regression line.

The explicit bivariate formula for the ordinary least-squares error described by Ehrenberg [3] is generalized now to higher-order regressions using generalized product-moments.

Definition 2: Define the generalized bivariate product-moment of order $p = m + n$ as

$$\mu_{m,n} = \frac{1}{N}\sum_{i=1}^{N} (x_i - \mu_x)^m \left(y_i - \mu_y\right)^n \quad (2)$$

for whole numbers m and n.

Theorem 3: (Explicit Bivariate Error Formula) Let p be an even whole number and let $F = E|_{a=\mu_y - b\mu_x}$. Then

$$E = \sum_{r+s+t=p} (-1)^s \binom{p}{r,s,t} \mu_{r,s} b^r \left(a - \mu_y - b\mu_x\right)^t \quad (3)$$

or

$$E = \sum_{r+s+t=p, t\neq 0} (-1)^s \binom{p}{r,s,t} \mu_{r,s} b^r \left(a - \mu_y - b\mu_x\right)^t + F$$

where

$$F = \sum_{r=0}^{p} (-1)^{p-r} \binom{p}{r} \mu_{r,p-r} b^r . \quad (4)$$

Proof: Assume p is even and omit absolute values. Begin with the error expression and manipulate as follows:

$$\begin{aligned} E &= \frac{1}{N}\sum_{i=1}^{N}(a+bx_i-y_i)^p \\ &= \frac{1}{N}\sum_{i=1}^{N}\left(b(x_i-\mu_x)-(y_i-\mu_y)+(a-\mu_y-b\mu_x)\right)^p \\ &= \frac{1}{N}\sum_{i=1}^{N}\sum_{r+s+t=p}(-1)^s\binom{p}{r,s,t}b^r \\ &\quad \times (x_i-\mu_x)^r(y_i-\mu_y)^s(a-\mu_y-b\mu_x)^t \\ &= \sum_{r+s+t=p}(-1)^s\binom{p}{r,s,t}b^r \\ &\quad \times\left\{\frac{1}{N}\sum_{i=1}^{N}(x_i-\mu_x)^r(y_i-\mu_y)^s\right\}(a-\mu_y-b\mu_x)^t \\ &= \sum_{r+s+t=p}(-1)^s\binom{p}{r,s,t}\mu_{r,s}b^r(a-\mu_y-b\mu_x)^t. \end{aligned}$$

Now separate out the terms with $t=0$ and obtain

$$E=\sum_{r+s+t=p,t\neq 0}(-1)^s\binom{p}{r,s,t}\mu_{r,s}b^r(a-\mu_y-b\mu_x)^t+F$$

where

$$\begin{aligned} F &= \sum_{r+s=p}(-1)^s\binom{p}{r,s,0}\mu_{r,s}b^r \\ &= \sum_{r=0}^{p}(-1)^{p-r}\binom{p}{r}\mu_{r,p-r}b^r. \end{aligned}$$

Observe that applying the trinomial expansion theorem to the error expression E and then setting $a=\mu_y-b\mu_x$ produces the same result F as first setting $a=\mu_y-b\mu_x$ and then applying the binomial expansion:

$$\begin{aligned} F &= \frac{1}{N}\sum_{i=1}^{N}\left(b(x_i-\mu_x)-(y_i-\mu_y)\right)^p \\ &= \frac{1}{N}\sum_{i=1}^{N}\sum_{r=0}^{p}\binom{p}{r}b^r(x_i-\mu_x)^r(y_i-\mu_y)^{p-r}(-1)^{p-r} \\ &= \sum_{r=0}^{p}(-1)^{p-r}\binom{p}{r} \\ &\quad \times\left\{\frac{1}{N}\sum_{i=1}^{N}(x_i-\mu_x)^r(y_i-\mu_y)^{p-r}\right\}b^r \\ &= \sum_{r=0}^{p}(-1)^{p-r}\binom{p}{r}\mu_{r,p-r}b^r. \end{aligned}$$

■

Example 4: For $p=2$, which is least-squares,

$$F=\mu_{2,0}b^2-2\mu_{1,1}b+\mu_{0,2}. \quad (5)$$

In more familiar notation this is

$$F=\sigma_x^2b^2-2\sigma_{xy}b+\sigma_y^2. \quad (6)$$

For $p=4$,

$$F=\mu_{4,0}b^4-4\mu_{3,1}b^3+6\mu_{2,2}b^2-4\mu_{1,3}b+\mu_{0,4}. \quad (7)$$

For $p=6$,

$$\begin{aligned} F &= \mu_{6,0}b^6-6\mu_{5,1}b^5+15\mu_{4,2}b^4-20\mu_{3,3}b^3 \\ &\quad +15\mu_{2,4}b^2-6\mu_{1,5}b+\mu_{0,6}. \end{aligned} \quad (8)$$

Theorem 5: (Bivariate OLP Regression) The OLP$_p$ $y|x$ regression line $y=a+bx$ is obtained by solving

$$F'(b)=\sum_{r=1}^{p}(-1)^{p-r}r\binom{p}{r}\mu_{r,p-r}b^{r-1}=0 \quad (9)$$

for b and setting

$$a = \mu_y-b\mu_x. \quad (10)$$

Example 6: For $p=2$, which is least-squares, one solves

$$0=\mu_{2,0}b-\mu_{1,1}. \quad (11)$$

For $p=4$ one solves

$$0=\mu_{4,0}b^3-3\mu_{3,1}b^2+3\mu_{2,2}b-\mu_{1,3}. \quad (12)$$

For $p=6$ one solves

$$\begin{aligned} 0 &= \mu_{6,0}b^5-5\mu_{5,1}b^4+10\mu_{4,2}b^3 \\ &\quad -10\mu_{3,3}b^2+5\mu_{2,4}b-\mu_{1,5}. \end{aligned} \quad (13)$$

Now that the analog of OLS, called OLP, has been described, the corresponding bivariate theory of generalized least-powers can be described as well.

B. Bivariate Generalized Means and XMR$_p$ Notation

The axioms of a generalized mean were stated by us previously [8], [9] drawing from the work of Mays [11] and also from Chen [2]. They are stated here again for convenience.

Definition 7: A function $M(x,y)$ defines a generalized mean for $x,y>0$ if it satisfies Properties 1-5 below. If it satisfies Property 6 it is called a homogeneous generalized mean. The properties are:

1. (Continuity) $M(x,y)$ is continuous in each variable.
2. (Monotonicity) $M(x,y)$ is non-decreasing in each variable.
3. (Symmetry) $M(x,y)=M(y,x)$.
4. (Identity) $M(x,x)=x$.
5. (Intermediacy) $\min(x,y)\leq M(x,y)\leq\max(x,y)$.
6. (Homogeneity) $M(tx,ty)=tM(x,y)$ for all $t>0$.

All known means are included in this definition. All the means discussed in this paper are homogeneous. The generalized mean of any two generalized means is itself a generalized mean.

XMR$_p$ notation is used here to name generalized regressions: if ‘X’ is the letter used to denote a given generalized mean, then XMR$_p$ is the corresponding generalized least-pth power regression. XMR$_2$ is the same thing as XMR without a subscript. For example, 'G' is the letter usually used to denote the geometric mean and GMR_p is least-pth power geometric mean regression. AMR_p is least-pth power arithmetic mean regression. The generalization of orthogonal least-squares regression to least-powers is HMR_p since orthogonal regression is the same as harmonic mean regression.

C. The Two Generalized Least-Powers Problems and the Equivalence Theorem

The general symmetric least-powers problem is stated as follows.

Definition 8: (The General Symmetric Least-Powers Problem) Values of a and b are sought which minimize an error function defined by

$$E = \frac{1}{N}\sum_{i=1}^{N} M\left(|a+bx_i-y_i|^p, \left|\frac{a}{b}+x_i-\frac{1}{b}y_i\right|^p\right) \tag{14}$$

where $M(x,y)$ is any generalized mean.

Definition 9: (The General Weighted Ordinary Least-Powers Problem) Values of a and b are sought which minimize an error function defined by

$$E = g(b)\cdot\frac{1}{N}\sum_{i=1}^{N}|a+bx_i-y_i|^p. \tag{15}$$

where $g(b)$ is a positive even function that is non-decreasing for $b<0$ and non-increasing for $b>0$.

The next theorem states that every generalized least-powers regression problem is equivalent to a weighted ordinary least-powers problem with weight function $g(b)$.

Theorem 10: Every general symmetric least-powers error function can be written equivalently as

$$E = g(b)\cdot\frac{1}{N}\sum_{i=1}^{N}|a+bx_i-y_i|^p \tag{16}$$

where

$$g(b) = M\left(1,\frac{1}{|b|^p}\right). \tag{17}$$

Proof: Substitute $\frac{a}{b}+x_i-\frac{1}{b}y_i$ with $\frac{1}{b}(a+bx_i-y_i)$ and then use the homogeneity property:

$$\begin{aligned} E &= \frac{1}{N}\sum_{i=1}^{N} M\left(|a+bx_i-y_i|^p, \frac{|a+bx_i-y_i|^p}{|b|^p}\right) \\ &= \frac{1}{N}\sum_{i=1}^{N}|a+bx_i-y_i|^p\, M\left(1,\frac{1}{|b|^p}\right). \end{aligned}$$

Define

$$g(b) = M\left(1,\frac{1}{|b|^p}\right),$$

and factor $g(b)$ outside of the summation. ■

D. How to Find the Regression Coefficients

The fundamental practical question of bivariate generalized least-powers regression is how to find the coefficients a and b for the regression line $y=a+bx$. The case of least-even-power regressions can be solved analytically, analogous to how generalized least-squares regressions are solved. For p even, to find the regression coefficients a and b, take the first order partial derivatives of the error function E_a and E_b and set them equal to zero. Solving $E_a=0$ yields $a=\mu_y-b\mu_x$. Solving $E_b=\frac{\partial E}{\partial b}=0$ and then setting $a=\mu_y-b\mu_x$ is equivalent to setting $a=\mu_y-b\mu_x$ first and then solving $\frac{dF}{db}=\frac{\partial E}{\partial b}\big|_{a=\mu_y-b\mu_x}=0$. The latter procedure is employed here because it is simpler.

Theorem 11: (Solving for the Generalized Regression Coefficients) For p even, the generalized least-powers slope b is found by solving:

$$\frac{d}{db}\{g(b)F(b)\}=0 \tag{18}$$

where

$$F=\sum_{r=0}^{p}(-1)^{p-r}\binom{p}{r}\mu_{r,p-r}b^r \tag{19}$$

and the y-intercept a is given by

$$a=\mu_y-b\mu_x. \tag{20}$$

E. The Hessian Matrix

In order for the regression coefficients (a,b) to minimize the error function and be admissible, the Hessian matrix of second order partial derivatives must be positive definite when evaluated at (a,b). The general Hessian matrix is calculated next. As in the case of generalized least-squares, certain combinations of g and its first and second partial derivatives appear in the matrix. One combination is denoted here by J and another is denoted by G. They are called indicative functions.

Definition 12: Define the indicative functions

$$J(b)=\frac{g''(b)}{g(b)}-2\left(\frac{g'(b)}{g(b)}\right)^2 \tag{21}$$

and

$$G(b)=\frac{2g'(b)}{g(b)}-\frac{g''(b)}{g'(b)}. \tag{22}$$

The two indicative functions are related by the equation $F'/F=J/G$. The latter differential equation can be solved explicitly for $g(b)$. One obtains [6]

$$g(b)=\frac{1}{c+k\int\exp\left(-\int G(b)\,db\right)db}. \tag{23}$$

Theorem 13: (Hessian matrix) The Hessian matrix $\mathbf{H}$ of second order partial derivatives of the error function given by

$$\mathbf{H}=\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}. \tag{24}$$

is computed explicitly as follows:

$$H_{11} = E_{aa} = p(p-1)\,gF_{p-2} \tag{25}$$

$$H_{12} = H_{21} = E_{ab} = E_{ba} = pg\left(\frac{g'}{g}F_{p-1} + F'_{p-1}\right) \tag{26}$$

$$H_{22} = E_{bb} = g\left(JF + F''\right). \tag{27}$$

Alternatively,

$$H_{22} = g\left(GF' + F''\right). \tag{28}$$

Proof: Take second order partial derivatives of the error and evaluate at the solution (a, b). Let $E = E_{\mathrm{OLP}}$ for purposes of this proof and assume p is even.

$$\begin{aligned} H_{22} &= \frac{\partial^2}{\partial b^2}(gE) \\ &= \frac{\partial}{\partial b}\left(g'E + gE_b\right) \\ &= g''E + 2g'E_b + gE_{bb}. \end{aligned}$$

Since $g'E + gE_b = 0$ at the solution, substitute $E_b = -\frac{g'}{g}E$ into the middle term, simplify, and obtain

$$H_{22} = g\left(\left(\frac{g''}{g} - 2\left(\frac{g'}{g}\right)^2\right)E + E_{bb}\right)$$

which is the first form of the Hessian. Now substitute $E = -\frac{g}{g'}E_b$ and obtain the second form

$$H_{22} = g\left(\left(\frac{2g'}{g} - \frac{g''}{g'}\right)E_b + E_{bb}\right).$$

As before, upon substituting for a, $E = F$, $E_b = F'$ and $E_{bb} = F''$. For H_{11},

$$\begin{aligned} H_{11} &= \frac{\partial^2}{\partial a^2}(gE) \\ &= \frac{\partial^2}{\partial a^2}\left(g \cdot \frac{1}{N}\sum_{i=1}^{N}(a + bx_i - y_i)^p\right) \\ &= g \cdot p(p-1) \cdot \frac{1}{N}\sum_{i=1}^{N}(a + bx_i - y_i)^{p-2} \\ &= g \cdot p(p-1) \cdot \frac{1}{N}\sum_{i=1}^{N}\big(b(x_i - \mu_x) - (y_i - \mu_y) \\ &\quad + (a - \mu_y - b\mu_x)\big)^{p-2} \\ &= g \cdot p(p-1) \cdot \frac{1}{N}\sum_{i=1}^{N}\big(b(x_i - \mu_x) - (y_i - \mu_y)\big)^{p-2} \\ &= g \cdot p(p-1) \cdot F_{p-2}. \end{aligned}$$

For $H_{12} = H_{21}$,

$$\begin{aligned} H_{12} &= \frac{\partial^2}{\partial b \partial a}(gE) \\ &= p\frac{\partial}{\partial b}\left(g \cdot \frac{1}{N}\sum_{i=1}^{N}(a + bx_i - y_i)^{p-1}\right) \\ &= p\left(g'F_{p-1} + gF'_{p-1}\right). \end{aligned}$$

■

Theorem 14: The Hessian matrix $\mathbf{H}$ is positive definite at the solution point (a, b) provided that $\det \mathbf{H} > 0$.

Proof: $\mathbf{H}$ is positive definite provided that $H_{11} > 0$ and $\det \mathbf{H} > 0$. In our case $H_{11} = g(b) \cdot p(p-1) \cdot F_{p-2}(b) > 0$ for all b. Therefore it suffices to evaluate the determinant numerically at the slope b. ■

III. Regression Examples Based on Known Special Means

A. Arithmetic Mean Regression (AMR_p)

The arithmetic mean is given by

$$A(x, y) = \frac{1}{2}(x + y). \tag{29}$$

This mean generates arithmetic mean regression AMR_p. The weight function is given by

$$g(b) = \frac{1}{2}\left(1 + \frac{1}{|b|^p}\right). \tag{30}$$

The general AMR_p slope equation for p even is given by

$$\left(b^{p+1} + b\right)F'(b) - pF(b) = 0. \tag{31}$$

The AMR_2 slope equation is

$$0 = \mu_{2,0}b^4 - \mu_{1,1}b^3 + \mu_{1,1}b - \mu_{0,2}. \tag{32}$$

The AMR_4 slope equation is

$$\begin{aligned} 0 &= \mu_{4,0}b^8 - 3\mu_{3,1}b^7 + 3\mu_{2,2}b^6 - \mu_{1,3}b^5 \\ &\quad + \mu_{3,1}b^3 - 3\mu_{2,2}b^2 + 3\mu_{1,3}b - \mu_{0,4}. \end{aligned} \tag{33}$$

The AMR_6 slope equation is

$$\begin{aligned} 0 &= \mu_{6,0}b^{12} - 5\mu_{5,1}b^{11} + 10\mu_{4,2}b^{10} - 10\mu_{3,3}b^9 \\ &\quad + 5\mu_{2,4}b^8 - \mu_{1,5}b^7 + \mu_{5,1}b^5 - 5\mu_{4,2}b^4 \\ &\quad + 10\mu_{3,3}b^3 - 10\mu_{2,4}b^2 + 5\mu_{1,5}b - \mu_{0,6}. \end{aligned} \tag{34}$$

B. Geometric Mean Regression (GMR_p)

The geometric mean is given by

$$G(x, y) = (xy)^{1/2}. \tag{35}$$

This mean generates geometric mean regression GMR_p. The weight function is given by

$$g(b) = |b|^{-p/2}. \tag{36}$$

The general GMR_p slope equation for p even is

$$2bF'(b) - pF(b) = 0. \tag{37}$$

The GMR_2 slope equation is

$$0 = \mu_{2,0}b^2 - \mu_{0,2}. \tag{38}$$

The GMR_4 slope equation is

$$0 = \mu_{4,0}b^4 - 2\mu_{3,1}b^3 + 2\mu_{1,3}b - \mu_{0,4}. \tag{39}$$

The GMR_6 slope equation is

$$\begin{aligned} 0 &= \mu_{6,0}b^6 - 4\mu_{5,1}b^5 + 5\mu_{4,2}b^4 \\ &\quad - 5\mu_{2,4}b^2 + 4\mu_{1,5}b - \mu_{0,6}. \end{aligned} \tag{40}$$

C. Harmonic Mean (Orthogonal) Regression (HMR_p)

The harmonic mean is given by

$$H(x,y)=\frac{2xy}{x+y}. \tag{41}$$

This mean generates harmonic mean regression HMR_p. The weight function is given by

$$g(b)=\frac{2}{1+|b|^p}. \tag{42}$$

The general HMR_p slope equation for p even is

$$(b^p+1)\,F'(b)-pb^{p-1}F(b)=0. \tag{43}$$

The HMR_2 slope equation is

$$0=\mu_{1,1}b^2+\left(\mu_{2,0}-\mu_{0,2}\right)b-\mu_{1,1}. \tag{44}$$

The HMR_4 slope equation is

$$\begin{aligned}0 \;=\;& \mu_{3,1}b^6-3\mu_{2,2}b^5+3\mu_{1,3}b^4\\ &+\left(\mu_{4,0}-\mu_{0,4}\right)b^3-3\mu_{3,1}b^2\\ &+3\mu_{2,2}b-\mu_{1,3}.\end{aligned} \tag{45}$$

The HMR_6 slope equation is

$$\begin{aligned}0 \;=\;& \mu_{5,1}b^{10}-5\mu_{4,2}b^9+10\mu_{3,3}b^8-10\mu_{2,4}b^7\\ &+5\mu_{1,5}b^6+\left(\mu_{6,0}-\mu_{0,6}\right)b^5-5\mu_{5,1}b^4\\ &+10\mu_{4,2}b^3-10\mu_{3,3}b^2+5\mu_{2,4}b-\mu_{1,5}.\end{aligned} \tag{46}$$

Harmonic mean regression is the same thing as orthogonal regression. This is because of the Reciprocal Pythagorean Theorem [4], [12] which says that the diagonal deviation between a data point and the regression line is half the harmonic mean of the horizontal and vertical deviations.

D. Ordinary Least-Powers $x|y$ Regression (OLP_p $x|y$)

The selection mean given by

$$S_x(x,y)=x \tag{47}$$

generates OLP_p $y|x$ regression. The selection mean given by

$$S_y(x,y)=y \tag{48}$$

generates OLP_p $x|y$ regression. The weight function corresponding to OLP_p $y|x$ is given by

$$g(b)=1. \tag{49}$$

The weight function corresponding to OLP_p $x|y$ is given by

$$g(b)=\frac{1}{|b|^p}. \tag{50}$$

The general OLP_p $y|x$ slope equation for p even is

$$F'(b)=0 \tag{51}$$

The specific equations for $p=2,4,$ and 6 were already described.

The general OLP_p $x|y$ slope equation for p even is

$$bF'(b)-pF(b)=0 \tag{52}$$

The OLP_2 $x|y$ slope equation is

$$0=\mu_{1,1}b-\mu_{0,2}. \tag{53}$$

The OLP_4 $x|y$ slope equation is

$$0=\mu_{3,1}b^3-3\mu_{2,2}b^2+3\mu_{1,3}b-\mu_{0,4}. \tag{54}$$

The OLP_6 $x|y$ slope equation is

$$\begin{aligned}0 \;=\;& \mu_{5,1}b^5-5\mu_{4,2}b^4+10\mu_{3,3}b^3\\ &-10\mu_{2,4}b^2+5\mu_{1,5}b-\mu_{0,6}.\end{aligned} \tag{55}$$

IV. Regression Examples Based on Generalized Means

The generalized means in the next examples have free parameters which can be used to parameterize these and other known special cases.

A. Weighted Arithmetic Mean Regression

The weighted arithmetic mean with weight α in $[0,1]$ is given by

$$M_\alpha(x,y)=(1-\alpha)\,x+\alpha y \tag{56}$$

for $x\le y$.

The weight function corresponding to the weighted arithmetic mean is

$$g(b)=(1-\alpha)+\alpha\,|b|^{-p}. \tag{57}$$

For weighted $\mathrm{AMR}_{p,\alpha}$ the general slope equation for p even is

$$\left((1-\alpha)\,b^{p+1}+\alpha b\right)F'(b)-\alpha pF(b)=0. \tag{58}$$

The weighted $\mathrm{AMR}_{2,\alpha}$ slope equation is

$$0=(1-\alpha)\,\mu_{2,0}b^4-(1-\alpha)\,\mu_{1,1}b^3+\alpha\mu_{1,1}b-\alpha\mu_{0,2} \tag{59}$$

The weighted $\mathrm{AMR}_{4,\alpha}$ slope equation is

$$\begin{aligned}0 \;=\;& (1-\alpha)\,\mu_{4,0}b^8-3\,(1-\alpha)\,\mu_{3,1}b^7\\ &+3\,(1-\alpha)\,\mu_{2,2}b^6-(1-\alpha)\,\mu_{1,3}b^5\\ &+\alpha\mu_{3,1}b^3-3\alpha\mu_{2,2}b^2\\ &+3\alpha\mu_{1,3}b-\alpha\mu_{0,4}.\end{aligned} \tag{60}$$

The weighted $\mathrm{AMR}_{6,\alpha}$ slope equation is

$$\begin{aligned}0 \;=\;& (1-\alpha)\,\mu_{6,0}b^{12}-5\,(1-\alpha)\,\mu_{5,1}b^{11}\\ &+10\,(1-\alpha)\,\mu_{4,2}b^{10}-10\,(1-\alpha)\,\mu_{3,3}b^9\\ &+5\,(1-\alpha)\,\mu_{2,4}b^8-(1-\alpha)\,\mu_{1,5}b^7\\ &+\alpha\mu_{5,1}b^5-5\alpha\mu_{4,2}b^4+10\alpha\mu_{3,3}b^3\\ &-10\alpha\mu_{2,4}b^2+5\alpha\mu_{1,5}b-\alpha\mu_{0,6}\end{aligned} \tag{61}$$

B. Weighted Geometric Mean Regression

The weighted geometric mean with weight β in $[0,1]$ is given by

$$M_\beta(x,y) = x^{1-\beta}y^\beta \tag{62}$$

for $x \leq y$.

The weight function corresponding to the weighted geometric mean is given by

$$g(b) = |b|^{-p\beta}. \tag{63}$$

For weighted $\mathrm{GMR}_{p,\beta}$ the general slope equation for p even is

$$bF'(b) - p\beta F(b) = 0. \tag{64}$$

The weighted $\mathrm{GMR}_{2,\beta}$ slope equation is

$$0 = (1-\beta)\mu_{2,0}b^2 + (2\beta-1)\mu_{1,1}b - \beta\mu_{0,2} \tag{65}$$

The weighted $\mathrm{GMR}_{4,\beta}$ slope equation is

$$\begin{aligned} 0 \;=\;& (1-\beta)\mu_{4,0}b^4 - (3-4\beta)\mu_{3,1}b^3 \\ & +3(1-2\beta)\mu_{2,2}b^2 - (1-4\beta)\mu_{1,3}b - \beta\mu_{0,4}. \end{aligned} \tag{66}$$

The weighted $\mathrm{GMR}_{6,\beta}$ slope equation is

$$\begin{aligned} 0 \;=\;& (1-\beta)\mu_{6,0}b^6 - (5-6\beta)\mu_{5,1}b^5 \\ & +5(2-3\beta)\mu_{4,2}b^4 - 10(1-2\beta)\mu_{3,3}b^3 \\ & +5(1-3\beta)\mu_{2,4}b^2 - (1-6\beta)\mu_{1,5}b - \beta\mu_{0,6}. \end{aligned} \tag{67}$$

C. Power Mean Regression

The power mean of order q, for $q \neq 0$, is given by

$$M_q(x,y) = \left(\frac{1}{2}(x^q + y^q)\right)^{1/q} \tag{68}$$

with $M_0(x,y) = G(x,y)$, $M_{-\infty}(x,y) = \min(x,y)$ and $M_\infty(x,y) = \max(x,y)$.

Many other special means are specific cases as well: $q=-1$ is the harmonic mean, $q = -\frac{1}{2}$ was the basis for squared harmonic mean regression (SHR), $q = \frac{1}{2}$ was the basis for square perimeter regression (SPR), $q=1$ is the arithmetic mean and $q=2$ is called the root-mean-square [5], [8].

Many other special means are approximated well by power means: $M_{-1/3}(x,y)$ approximates the second logarithmic mean $L_2(x,y)$ and $(HG^2)^{1/3}$ well, $M_{1/3}(x,y)$, called the Lorentz mean, approximates the first logarithmic mean $L_1(x,y)$ well, and $M_{2/3}(x,y)$ approximates both the Heronian mean $N(x,y)$ and the identric mean $I(x,y)$ well. This is proven in our earlier paper [8].

The weight function is given by

$$g(b) = \left(\frac{1}{2}\left(1 + |b|^{-pq}\right)\right)^{1/q}. \tag{69}$$

The general $\mathrm{PMR}_{p,q}$ slope equation for p even is

$$(b^{pq}+1)\,bF'(b) - pF(b) = 0. \tag{70}$$

The $\mathrm{PMR}_{2,q}$ slope equation is

$$0 = \mu_{2,0}b^{2q+2} - \mu_{1,1}b^{2q+1} + \mu_{1,1}b - \mu_{0,2} \tag{71}$$

The $\mathrm{PMR}_{4,q}$ slope equation is

$$\begin{aligned} 0 \;=\;& \mu_{4,0}b^{4q+4} - 3\mu_{3,1}b^{4q+3} \\ & +3\mu_{2,2}b^{4q+2} - b^{4q+1}\mu_{1,3} \\ & +\mu_{3,1}b^3 - 3\mu_{2,2}b^2 + 3\mu_{1,3}b - \mu_{0,4}. \end{aligned} \tag{72}$$

The $\mathrm{PMR}_{6,q}$ slope equation is

$$\begin{aligned} 0 \;=\;& \mu_{6,0}b^{6q+6} - 5\mu_{5,1}b^{6q+5} + 10\mu_{4,2}b^{6q+4} \\ & -10\mu_{3,3}b^{6q+3} + 5\mu_{2,4}b^{6q+2} - \mu_{1,5}b^{6q+1} \\ & +\mu_{5,1}b^5 - 5\mu_{4,2}b^4 + 10\mu_{3,3}b^3 \\ & -10\mu_{2,4}b^2 + 5\mu_{1,5}b - \mu_{0,6}. \end{aligned} \tag{73}$$

D. Regressions Based on Other Generalized Means

As was done in our previous paper [8] for $p=2$, regression formulas for $p > 2$ based on other generalized means can be worked out. The Dietel-Gordon Mean of order r, the Stolarsky mean of order s, the two-parameter Stolarsky mean of order (r,s), the Gini mean of order t, the two-parameter Gini mean of order (r,s) are alternative generalized means which parameterize the known specific cases. Details on these means and the corresponding references can be found in that paper. We leave the detailed slope equations in these cases for a future work in this series.

V. Equivalence Theorems

A. Solving for the Generalized Mean Parameter as a Function of the Slope

For the case of weighted AMR, weighted GMR, and PMR the free parameter in these cases can solved for explicitly in terms of the slope. The result of doing this yields an equivalence theorem.

Theorem 15: (Weighted Arithmetic Mean Equivalence Theorem) Let b be the slope of a generalized least-powers regression line with p even. Then the line can be generated by an equivalent weighted arithmetic mean regression with weight α given by

$$\alpha = \frac{b^{p+1}F'(b)}{(b^{p+1}-b)F'(b) + pF(b)} \tag{74}$$

with

$$b = b_{\mathrm{OLP}\ y|x} + \omega\left(b_{\mathrm{OLP}\ x|y} - b_{\mathrm{OLP}\ y|x}\right) \tag{75}$$

and ω in $[0,1]$.

Proof: Solve the weighted $\mathrm{AMR}_{p,\alpha}$ slope equation for α. ■

Theorem 16: (Weighted Geometric Mean Equivalence Theorem) Let b be the slope of a generalized least-powers regression line with p even. Then the line can be generated by an equivalent weighted geometric mean regression with weight β given by

$$\beta = \frac{bF'(b)}{pF(b)} \tag{76}$$

with

$$b = b_{\text{OLP }y|x} + \omega\left(b_{\text{OLP }x|y} - b_{\text{OLP }y|x}\right) \tag{77}$$

and ω in $[0,1]$.

Proof: Solve the weighted $\text{GMR}_{p,\beta}$ slope equation for β. ■

Theorem 17: (Power Mean Equivalence Theorem) Let b be the slope of a generalized least-powers regression line with p even. Then the line can be generated by an equivalent power mean regression of order q with

$$q = \frac{1}{p}\ln\left(\frac{pF(b) - bF'(b)}{bF'(b)}\right)/\ln b. \tag{78}$$

Proof: Solve the weighted $\text{PMR}_{p,q}$ slope equation for q. ■

For $b > 0$, if the interval $\left[b_{\text{OLP }y|x}, b_{\text{OLP }x|y}\right]$ does not contain 1, or for $b < 0$, if the interval $\left[b_{\text{OLP }x|y}, b_{\text{OLP }y|x}\right]$ does not contain -1, then every regression line lying between the two ordinary least-powers lines is generated by a power mean of order q for some q in $(-\infty,\infty)$ with the ordinary least-powers lines corresponding to $q = \pm\infty$. This was shown in detail for the case of least-squares [8].

B. The Exponential Equivalence Theorem and the Fundamental Formula for Generalized Least-Powers Regression

In the bivariate case of generalized least-squares it was shown [7], [8] that every weighted ordinary least-squares regression line can be generated by an equivalent exponentially weighted regression with weight function $g_0(b) = \exp(-\gamma P_0|b|)$ for γ in $[0,1]$. This theorem generalizes to least-powers regressions as well.

Theorem 18: (The Extremal Line) For exponentially weighted ordinary least-powers regression with weight function $g(b) = \exp(-P|b|)$, the regression line generated by the maximum value of P is called the extremal line. The slope of the extremal line b_{EXT} is computed by solving

$$F''(b)F(b) - (F'(b))^2 = 0 \tag{79}$$

and the maximum value of P, called P_0, is computed by

$$P_0 = \operatorname{sgn}(b_{\text{EXT}})\frac{F'(b_{\text{EXT}})}{F(b_{\text{EXT}})}. \tag{80}$$

As the parameter P varies over the interval $[0, P_0]$ all exponentially weighted least-powers regressions are generated.

Proof: Consider the weight function $g(b) = \exp(-P|b|)$. To find the slope b, one must solve

$$\frac{d}{db}\{\exp(-P|b|)F(b)\} = 0$$

or

$$\exp(-P|b|)(-P)(\operatorname{sgn} b)F(b) + \exp(-P|b|)F'(b) = 0$$

which is equivalent to writing

$$P(b) = \operatorname{sgn} b\frac{F'(b)}{F(b)}.$$

To find the maximum value of P one must solve

$$P'(b) = \operatorname{sgn} b\frac{F''(b)F(b) - (F'(b))^2}{(F(b))^2} = 0$$

or simply

$$F''(b)F(b) - (F'(b))^2 = 0$$

for b. Call the resulting value b_{EXT}. To insure that the value is a maximum, one must also verify that

$$P''(b_{\text{EXT}}) < 0.$$

The corresponding maximum value of P is $P_0 = P(b_{\text{EXT}})$. ■

Example 19: For $p = 2$ one obtains

$$\mu_{2,0}^2 b^2 - 2\mu_{2,0}\mu_{1,1}b + \left(\mu_{2,0}\mu_{0,2} - 2\mu_{1,1}^2\right) = 0 \tag{81}$$

which in covariance notation is

$$\sigma_x^4 b^2 - 2\sigma_x^2\sigma_{xy}b + \left(\sigma_x^2\sigma_y^2 - 2\sigma_{xy}^2\right) = 0. \tag{82}$$

This can be solved explicitly to obtain the slope of the extremal line. It can also be expressed using covariances or standard deviations and correlation coefficients as in the previous papers [7], [8]:

$$\begin{aligned} b &= \frac{\mu_{1,1} \pm \sqrt{\mu_{2,0}\mu_{0,2} - \mu_{1,1}^2}}{\mu_{2,0}} && (83)\\ &= \frac{\sigma_{xy} \pm \sqrt{\sigma_x^2\sigma_y^2 - \sigma_{xy}^2}}{\sigma_x^2} && (84)\\ &= \rho\frac{\sigma_y}{\sigma_x} \pm \frac{\sigma_y}{\sigma_x}\sqrt{1-\rho^2}. && (85)\end{aligned}$$

For $p = 4$ one obtains

$$\begin{aligned} 0 = {} & \mu_{4,0}^2 b^6 - 6\mu_{4,0}\mu_{3,1}b^5 \\ & + \left(12\mu_{3,1}^2 + 3\mu_{4,0}\mu_{2,2}\right)b^4 \\ & - \left(24\mu_{3,1}\mu_{2,2} - 4\mu_{4,0}\mu_{1,3}\right)b^3 \\ & - \left(3\mu_{4,0}\mu_{0,4} - 18\mu_{2,2}^2\right)b^2 \\ & - \left(12\mu_{2,2}\mu_{1,3} - 8\mu_{3,1}\mu_{0,4}\right)b \\ & - \left(3\mu_{2,2}\mu_{0,4} - 4\mu_{1,3}^2\right). \end{aligned} \tag{86}$$

For $p = 6$ one obtains

$$\begin{aligned} 0 = {} & \mu_{6,0}^2 b^{10} - 10\mu_{6,0}\mu_{5,1}b^9 \\ & + \left(30\mu_{5,1}^2 + 15\mu_{6,0}\mu_{4,2}\right)b^8 \\ & -120\mu_{5,1}\mu_{4,2}b^7 \\ & - \left(20\mu_{6,0}\mu_{2,4} - 80\mu_{5,1}\mu_{3,3} - 150\mu_{4,2}^2\right)b^6 \\ & + \left(30\mu_{5,1}\mu_{2,4} - 300\mu_{4,2}\mu_{3,3} + 18\mu_{6,0}\mu_{1,5}\right)b^5 \\ & + \left(75\mu_{4,2}\mu_{2,4} - 5\mu_{6,0}\mu_{0,6} \right. \\ & \left. -60\mu_{5,1}\mu_{1,5} + 200\mu_{3,3}^2\right)b^4 \\ & + \left(20\mu_{5,1}\mu_{0,6} - 200\mu_{3,3}\mu_{2,4} + 60\mu_{4,2}\mu_{1,5}\right)b^3 \\ & - \left(30\mu_{4,2}\mu_{0,6} - 75\mu_{2,4}^2\right)b^2 \\ & + \left(20\mu_{3,3}\mu_{0,6} - 30\mu_{2,4}\mu_{1,5}\right)b \\ & - \left(5\mu_{2,4}\mu_{0,6} - 6\mu_{1,5}^2\right). \end{aligned} \tag{87}$$

Since one cannot solve these equations explicitly when $p > 2$ one solves the equations numerically for b instead. One selects the real root that shares the same sign as b_{OLP} and is greater than b_{OLP} in absolute value.

Theorem 20: (Exponential Equivalence Theorem) Define the normalized exponential parameter $\gamma = P/P_0$ so that $g_0(b) = \exp(-\gamma P_0 |b|)$ for γ in $[0, 1]$. Then every weighted generalized least-powers regression line with slope b is also generated by an equivalent exponentially weighted regression with normalized exponential parameter

$$\gamma = \frac{1}{P_0} \operatorname{sgn} b \frac{F'(b)}{F(b)}. \tag{88}$$

The case $\gamma = 0$ is OLP$_p$ $y|x$ regression. The case $\gamma = 1$ is the extremal line.

Proof: For an arbitrary weight function $g(b)$, the slope b is found by solving $d\{g(b)F(b)\}/db = 0$ which is equivalent to $-g'(b)/g(b) = F'(b)/F(b)$ at the solution. However, it is already known that for a fixed slope b there exists a constant P in $[0, P_0]$ such that $P \operatorname{sgn} b = F'(b)/F(b)$. Thus $P = -(\operatorname{sgn} b) g'(b)/g(b)$ and for every weight function g and slope b there is a corresponding value for the exponential parameters P and $\gamma = P/P_0$. ■

Since b always lies in the interval between b_{OLP} and b_{EXT}, the fundamental formula of generalized least-powers regression follows.

Theorem 21: (Fundamental Formula of Generalized Least-Powers Regression) Every weighted generalized least-powers regression line $y = a + bx$ has the form

$$b = b_{\text{OLP}} + \lambda(b_{\text{EXT}} - b_{\text{OLP}}) \tag{89}$$

$$a = \mu_y - b\mu_x \tag{90}$$

for some $\lambda = \lambda(\gamma)$ in $[0, 1]$.

Once the slope b of a regression line is known, the corresponding value of λ can be determined numerically using

$$\lambda = \frac{b - b_{\text{OLP}}}{b_{\text{EXT}} - b_{\text{OLP}}}. \tag{91}$$

The function $\gamma = \gamma(\lambda)$ can be determined explicitly by composing $\gamma = \gamma(b)$ with $b = b(\lambda)$. The result is

$$\gamma = \frac{1}{P_0} \operatorname{sgn} b \frac{F'(b_{\text{OLP}} + \lambda(b_{\text{EXT}} - b_{\text{OLP}}))}{F(b_{\text{OLP}} + \lambda(b_{\text{EXT}} - b_{\text{OLP}}))}. \tag{92}$$

For $p = 2$ only, the parameters γ and λ are related by the simple formulas $\gamma = \sin(2\tan^{-1}\lambda)$ and $\lambda = \tan\left(\frac{1}{2}\sin^{-1}\gamma\right)$ [8].

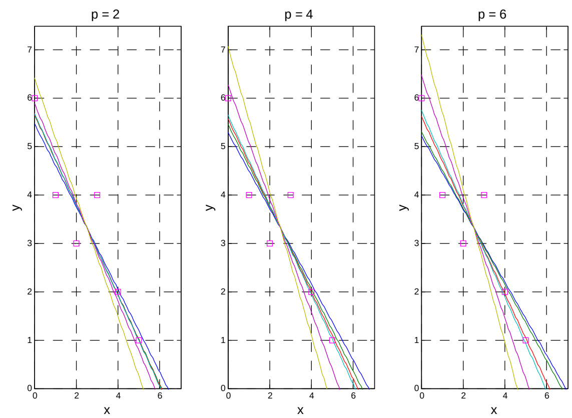
VI. Numerical Example

This section revisits an example explored in the previous work. Regressions corresponding to OLP$_p$, HMR$_p$, GMR$_p$, AMR$_p$ and the extremal line are computed for $p = 2, 4$, and 6. The corresponding effective weighted arithmetic and geometric mean parameters α and β are computed. The effective exponential parameters γ and λ are also computed.

Example 22: This example appears in our previous papers and is originally from Martin [10]. Six data values are given: $(0, 6)$, $(1, 4)$, $(2, 3)$, $(3, 4)$, $(4, 2)$, and $(5, 1)$. The reader can verify that $\rho = -0.9157$, $\mu_x = 2.5000$, $\mu_y = 3.3333$, $\sigma_x = 1.7078$, and $\sigma_y = 1.5986$.

The second order product-moments are: $\mu_{0,2} = 2.5556$, $\mu_{1,1} = -2.5000$, $\mu_{2,0} = 2.9167$. The fourth order product-moments are: $\mu_{0,4} = 13.9630$, $\mu_{1,3} = -13.8333$, $\mu_{2,2} = 13.9352$, $\mu_{3,1} = -14.1250$, $\mu_{4,0} = 14.7292$. The sixth order product-moments are: $\mu_{0,6} = 87.7956$, $\mu_{1,5} = -86.0802$, $\mu_{2,4} = 84.8200$, $\mu_{3,3} = -83.9583$, $\mu_{4,2} = 83.6227$, $\mu_{5,1} = -83.9063$, $\mu_{6,0} = 85.1823$.

The generalized regression lines are plotted together with the extremal line thereby displaying the region containing all admissible generalized regression lines.



The equation of each line is presented along with the exponential parameters γ and λ, the weighted arithmetic mean parameter α, and the weighted geometric mean parameter β. In all cases one uses $\gamma = \frac{1}{P_0}(\operatorname{sgn} b) F'(b)/F(b)$ and $\lambda = (b - b_{\text{OLP}})/(b_{\text{EXT}} - b_{\text{OLP}})$. According to the exponential equivalence theorem, all regression lines are generated for $p = 2$ by $g_0(b) = \exp(-2.6584\gamma|b|)$, for $p = 4$ by $g_0(b) = \exp(-4.3714\gamma|b|)$, and for $p = 6$ by $g_0(b) = \exp(-6.4294\gamma|b|)$.

$p=2$	$y=a+bx$	γ	λ	α	β
OLP$_2$ $y\|x$	$y=5.4762-0.8571x$	0.0000	0.0000	0.0000	0.0000
HMR$_2$	$y=5.6593-0.9304x$	0.3752	0.1947	0.4283	0.4640
GMR$_2$	$y=5.6735-0.9361x$	0.4019	0.2098	0.4670	0.5000
AMR$_2$	$y=5.6855-0.9409x$	0.4241	0.2226	0.5000	0.5304
OLP$_2$ $x\|y$	$y=5.8889-1.0222x$	0.7360	0.4389	1.0000	1.0000
EXT$_2$	$y=6.4166-1.2333x$	1.0000	1.0000		

$p=4$	$y=a+bx$	γ	λ	α	β
OLP$_4$ $y\|x$	$y=5.2993-0.7864x$	0.0000	0.0000	0.0000	0.0000
HMR$_4$	$y=5.4622-0.8515x$	0.3703	0.0920	0.2166	0.3446
GMR$_4$	$y=5.5750-0.8967x$	0.5102	0.1557	0.3926	0.5000
AMR$_4$	$y=5.6523-0.9276x$	0.5668	0.1993	0.5000	0.5746
OLP$_4$ $x\|y$	$y=6.2767-1.1774x$	0.7772	0.5519	1.0000	1.0000
EXT$_4$	$y=7.0703-1.4948x$	1.0000	1.0000		

$p=6$	$y=a+bx$	γ	λ	α	β
OLP$_6$ $y\|x$	$y=5.2239-0.7562x$	0.0000	0.0000	0.0000	0.0000
HMR$_6$	$y=5.3088-0.7902x$	0.2312	0.0407	0.0559	0.1958
GMR$_6$	$y=5.6291-0.9183x$	0.5081	0.1939	0.3749	0.5000
AMR$_6$	$y=5.7471-0.9655x$	0.5340	0.2504	0.5000	0.5524
OLP$_6$ $x\|y$	$y=6.4719-1.2554x$	0.7433	0.5973	1.0000	1.0000
EXT$_6$	$y=7.3135-1.5921x$	1.0000	1.0000		

It is readily observed that for $p = 2$, the slope values lie in the interval $[-1.2333, -0.8571]$ with a range of 0.3762. For $p = 4$, the slope values lie in the interval $[-1.4948, -0.7864]$

with a range of 0.7084. The range for $p = 4$ is approximately 1.9 times as long as the range for $p = 2$. For $p = 6$, the slope values lie in the interval $[-1.5921, -0.7562]$ with a range of 0.8359. The range for $p = 6$ is approximately 2.2 times as long as the range for $p = 2$.

For $p = 2$, the interval between the two OLP slopes is $[-1.0222, -0.8571]$ with a range of 0.1651. For $p = 4$, the interval between the two OLP slopes is $[-1.1774, -0.7864]$ with a range of 0.3910. The range for $p = 4$ is approximately 2.4 times as long as the range for $p = 2$. For $p = 6$, the interval between the two OLP slopes is $[-1.2554, -0.7562]$ with a range of 0.4992. The range for $p = 6$ is approximately 3.0 times as long as the range for $p = 2$.

VII. Summary

Least-powers regressions minimizing the average generalized mean of the absolute pth power deviations between the data and the regression line are described in this paper. Particular attention is paid to the case of p even, since this case admits analytic solution methods for the regression coefficients. Ordinary least-squares regression generalizes to ordinary least-powers regression. The case $p = 2$ corresponds to the generalized least-squares regressions of our previous works. The specific cases of arithmetic, geometric and harmonic mean (orthogonal) regression are worked out in detail for the case of $p = 2$, 4 and 6.

Regressions based on weighted arithmetic means of order α and weighted geometric means of order β are also worked out. The weights α and β continuously parameterize all generalized regression lines lying between the two ordinary least-powers lines. Power mean regression of order q has fixed values of q corresponding to many known special means and offers another way to parameterize the generalized mean regressions previously described.

Every generalized mean regression with error function given by

$$E = \frac{1}{N}\sum_{i=1}^{N} M\left(\left|a + bx_i - y_i\right|^p, \left|\frac{a}{b} + x_i - \frac{1}{b}y_i\right|^p\right) \quad (93)$$

is equivalent to a weighted ordinary least-powers regression with error function

$$E = g\left(b\right) \cdot \frac{1}{N}\sum_{i=1}^{N} \left|a + bx_i - y_i\right|^p \quad (94)$$

and weight function

$$g\left(b\right) = M\left(1, \frac{1}{\left|b\right|^p}\right) \quad (95)$$

where $M\left(x, y\right)$ is any generalized mean.

The exponential equivalence theorem states that every weighted ordinary least-powers regression line can be generated by an equivalent exponentially weighted regression with weight function $g_0\left(b\right) = \exp\left(-\gamma P_0\left|b\right|\right)$ for γ in $[0, 1]$. The case $\gamma = 0$ corresponds to $\text{OLP}_p\ y|x$ and the case $\gamma = 1$ corresponds to the extremal line. It follows that every generalized least-powers line has slope given by $b = b_{\text{OLP}} + \lambda\left(b_{\text{EXT}} - b_{\text{OLP}}\right)$ for $\lambda = \lambda\left(\gamma\right)$ in $[0, 1]$ and y-intercept given by $a = \mu_y - b\mu_x$. This is referred to here as the fundamental formula of generalized least-powers regression, since it characterizes all possible regression lines in a simple way.

A simple numerical example shows generalized least-powers regressions performing comparably to generalized least-squares but with a wider range of slope values. The application of bivariate generalized least-powers to non-normally distributed data and the potential advantage of these methods over generalized least-squares is a subject of the next paper in this series. The extension of this theory to multiple variables is also a subject of the next paper in this series.

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