Long sine wave transformation in the framework of Gardner equation: spectral and statistical analysis

O. Kurkina, E. Rouvinskaya, A. Giniyatullin, A. Kurkin, T. Talipova, E. Pelinovsky

***Abstract*—**The main stages of the process of tidal wave evolution can be described within a weakly nonlinear model based on the Gardner equation (extended version of the Korteweg – de Vries equation with both quadratic and cubic nonlinear terms), which is actively applied in physical oceanography. These stages include nonlinear steepening, and then generation and development of undular bore. We carried out numerical modeling of long sine wave evolution in the framework of the Gardner model for different signs of its cubic nonlinear term and for different initial amplitudes to demonstrate the principal features of the process. The present study is focused on spectral and statistical characteristics of the generated wave field. If amplitude of sine wave is large enough, soliton-like impulses of different polarities depending on the sign of cubic nonlinearity are generated and their interactions may result in the formation of extreme amplitude waves. Statistical analysis of the wave field in time shows almost permanent substantial exceedance of the level of the significant wave height in some position in spatial coordinate. Spectrum behavior after a long time of initial wave evolution demonstrate the power asymptotic for small wave numbers and exponential asymptotics – for large wave numbers.

***Keywords*—**Long waves, Gardner equation, internal tide, undular bore, significant wave height, extreme amplitude waves, solitons, spectrum, exceedance probability distribution, skewness, kurtosis.

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I. Introduction

UNDULAR bores are very often observed in estuaries and river mouths during a tidal cycle, when long tidal wave entering shallow waters. Brilliant collection of undular bores' observations can be found in the book [1]. Besides, they were observed during the 1983 Japan Sea tsunami [2] and 2004 Indian Ocean tsunami [3]. In general, the criterion of undular bore formation is a relationship between bore height H, measured from the bottom, and unperturbed depth of reservoir h: $H < 1.5h$ [4].

Undular bores are also very often observed in the stratified ocean as the vertical displacements of the pycnocline lying at the depths of 50-200 m and manifesting on the ocean surface as a group of slicks of various intensity, see for instance, [5]. Similar phenomenon was found in lakes [6]. Sometimes internal undular bore is called as a solibore after [7].

The undular bore is generated in systems with weak dispersion and in the presence of nonlinearity, for example, when the initial disturbance is very long or due to "dam-break" process. Simplified model of such phenomenon is based on the famous Korteweg-de Vries equation with initial condition in the form of the "dam-break".

In the present paper we would like to study the evolution of the long sine wave in the framework of the non-dimensional Gardner equation with different signs of cubic nonlinearity. This problem is of practical interest because degeneration of the long tidal wave is often responsible for generation of intense undular bores, often observed in the river mouths and estuaries. These waves contain huge energy, so they are a major source of sediment transport, resuspension as well as turbulent mixing in the water column. Such waves have a significant influence on the propagation of sound in the water column and on the formation of the bottom sound channel. Another interesting aspect of the solibores' studying is the fact that extreme amplitude pulses may be generated in the process of their evolution under certain conditions. Main goal of this study is to carry out spectral and statistical analysis of long sine wave degeneration in the framework of Gardner equation..

II. Theoretical model

We will use the canonical form of the Gardner equation with positive or negative sign of cubic nonlinearity term:

$$\frac{\partial\eta}{\partial t}+6\eta(1\pm\eta)\frac{\partial\eta}{\partial x}+\frac{\partial^3\eta}{\partial x^3}=0\,, \qquad (1)$$

Initial value problem with periodic boundary conditions is solved for this equation:

$$\eta(x,0)=A\cos(2\pi x/L),\ \eta(x+L,t)=\eta(x,t)\,, \qquad (2)$$

where L is the length of the computational domain (taken to be equal to 100 or 200 nondimensional units in our computations), and amplitude of the wave A was ranged from 0.1 to 3 dimensionless units.

To solve the problem (1), (2) we use a numerical code based on the implicit pseudo-spectral method [8] with periodic boundary conditions and with the control of conservation of the values of the mass and momentum integrals:

$$M=\int\eta dx\,,\quad E=\int\eta^2 dx \qquad (3)$$

within the numerical domain.Numerical code, that we use, repeatedly was verified in simulation of wave processes of different nature (see, eg, [9], [10]).

The periodic solutions (“cnoidal” waves) of the Gardner equation can be found in [11]. The solitons of this equation are very good studied also [12]–[14] and their properties depends on values of nonlinear and dispersion coefficients.

In the present paper we eliminate the real values of the coefficients of the Gardner equation using appropriated scaling. But in context of the various physical applications they can be necessary to introduce them into practice.

The analytical one-soliton solution of Gardner equation is well known:

$$\eta_s(x,t)=\gamma^2/(1+B\cosh(\gamma(x-\gamma^2 t)))\,, \qquad (4)$$

where γ is inverse soliton width. The parameter B determine the soliton amplitude a as the extreme value of the function $\xi(x, t)$:

$$a=\gamma^2/(1+B)\,,\ B^2=1\pm\gamma^2\,, \qquad (5)$$

The parameters of the family of solutions can also be expressed through its amplitude a:

$$\gamma^2=a(2\pm a)\,,\quad B=1\pm a\,. \qquad (6)$$

There are three different branches of the soliton solutions depending on the sign of coefficient at the cubic nonlinear term in the Gardner equation (1), see Fig. 1. In case of negative cubic nonlinearity in (1) (Fig.1, left) parameter γ is changing from 0 (Korteweg – the Vries (KdV) limit) to 1 (table-top soliton), and parameter $B=\sqrt{1-\gamma^2}$. In case of positive cubic nonlinear term in (1) (Fig.1, right) coefficient γ is changing from 0 (KdV soliton) to infinity (soliton of modified KdV equation), and parameter $B=\pm\sqrt{1+\gamma^2}$;the sign “+” correspond to the solitons of positive polarity and “–” – to the solitons of negative polarity.

Let us briefly describe the properties of solitary waves for positive sign of the quadratic nonlinear term (In opposite case, the polarity of solitons should be inverted). If cubic nonlinear term is negative, the solitons have positive polarity. The soliton height has the limiting value $a_{\lim}$ = 1 for canonical form (1), and such a limit represents the infinitely wide table-top soliton. The soliton width varies non-monotonically with amplitude. For small amplitudes the KdV equation is a good model to describe soliton parameters. Solitons with amplitude exceeding 0.8 - 0.9 can be considered as wide (table-top) solitons.

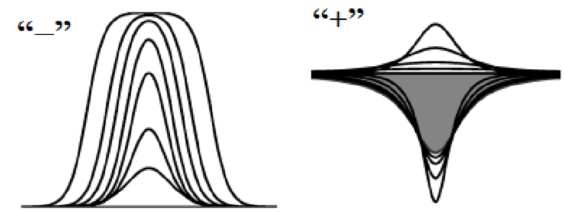

Fig. 1. Shapes of soliton solutions to Gardner equation: left panel – for positive, and right – for negative sign of the cubic nonlinear term in the Gardner equation (1)

In the case of positive cubic nonlinear term there are two branches of solitary waves. The first branch has the polarity determined by sign of quadratic nonlinearity, and its amplitude can be arbitrary with no limiting amplitude (within the applicability of Gardner equation). The second branch describes the solitons of alternative polarity. The soliton amplitude of this branch should exceed the minimal value corresponding to the so called algebraic soliton amplitude (which is equal to a_{alg} = – 2 for canonical Gardner equation (1))..

III. Soliton spectra

Spectrum of the Gardner eqution soliton (4) is

$$S(k)=\int_{-\infty}^{+\infty}\eta_s(x,t=0)\exp(-iKx)dx= \\ =2\int_0^{\infty}\eta_s(x,t=0)\cos(kx)dx. \qquad (7)$$

Gardner solitons for negative sign of cubic nonlinear term and their spectra are displayed in Fig. 2 for three values of γ. It is clearly seen how soliton transforms to table-top soliton with increase in amplitude, and the peaks appear in their spectrum similar to $\sin k/k$ as for step-like pulse.

In case of positive cubic nonlinearity soliton shapes and spectra for solution brahch of positive polarity are presented in Fig. 3 for three values of γ (1, 3 and 5). Qualitatively all pictures are similar to KdV solutions.

Solitons of family with negative polarity and their spactra are illustrated by Fig. 4. Qualitatively all pictures here similar to those for mKdV solutions.

IV. Soliton generation from sine wave

The detailed analysis of sine wave disintegration is given in [15]. Here we reproduce only briefly the description of key features of this process.

The scenario for evolution of small-amplitude long harmonic waves in the case of negative as well as positive values of cubic nonlinear term in the Gardner equation has many features in common with the process of disintegration such an impulse in the framework of the Korteweg-de Vries equation, see, for instance [16], [17]. The snapshots of evolution of wave (2) with amplitude A = 0.1 dimensionless units and negative cubic nonlinearity are shown in Fig. 5.

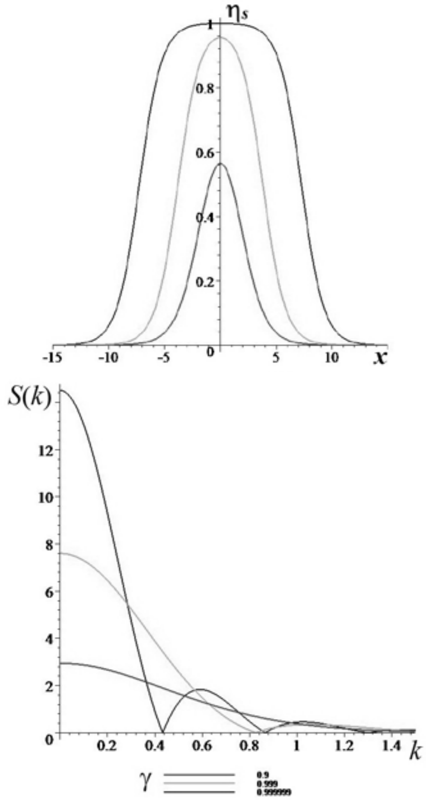

Fig. 2. Gardner solitons (upper panel) for negative sign of cubic nonlinear term and their spectra (lower panel) for three values of γ (shown in the legend).

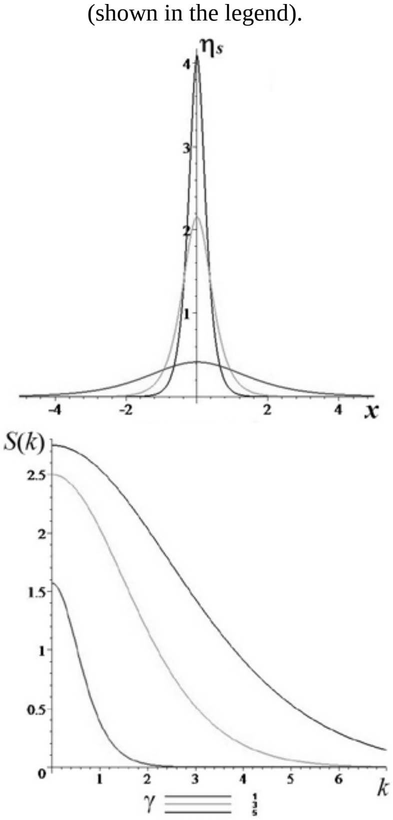

Fig. 3. Gardner solitons of positive polarity (upper panel) for positive sign of cubic nonlinear term and their spectra (lower panel) for three values of γ (shown in the legend)

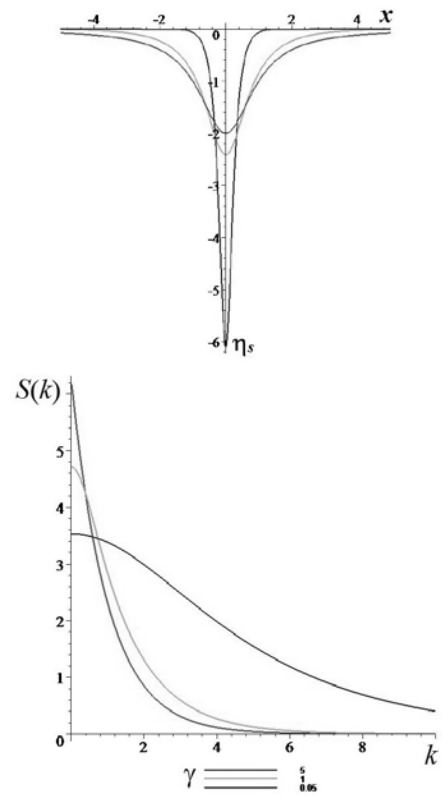

Fig. 4. Gardner solitons of negative polarity (upper panel) for positive sign of cubic nonlinear term and their spectra (lower panel) for three values of γ (shown in the legend).

After a time, one of the fronts becomes steeper due to nonlinearity, and cnoidal waves of variable, decreasing linearly amplitudes are generated on it. These waves interact with each other because of the periodicity of the boundary condition. These interactions lead to a negative phase shift and decreasing of waves' velocity as in the case of two-soliton interaction such as overtaking. But amplitudes of the resulting impulse in the moment of the interaction is less than the amplitude of cnoidal wave with greater amplitude.

The Fourier spectra of the evolving wave in terms of coefficients C_j:

$$C_j = (2/N)\sum_{k=1}^{N} \eta(k) w_N^{(k-1)(j-1)}, \qquad w_N = \exp(-2\pi i / N), \tag{8}$$

(we use discrete set of N harmonics $\eta(k)$) are presented in Fig. 6. Due to nonlinear steepness of initial sine wave, the spectrum on small time has the breaking asymptotic $j^{-4/3}$ for approximately 20 harmonics which is a common feature of nonlinear hyperbolic systems with weak dispersion [18], [19]. Then, forming of undular bore leads to generation of several spectral peaks in range 10-100 harmonics downshifting with time. The energy of the basic harmonics is decreased transferring the energy in high harmonics. The variable amplitude cnoidal-like structure of undular bore is not strongly periodic that leads to the wide overlapping peaks.

In more detail the evolution of small-amplitude sine waves is analyzed in [15].

Cubic nonlinear effects become noticeable with increasing amplitude. When the amplitude of initial wave amounts to 0.5 dimensionless units and the cubic nonlinear term is negative, the “breaking” point shifts to the trough. If cubic nonlinear term is positive and A = 0.5 the “breaking” point shifts to the wave crest (Fig. 7). In both cases envelope of wave crests becomes parabolic. It is worth noting that there are many nonlinear interactions of waves, that are similar to “overtaking”, if cubic nonlinearity is negative. But if cubic nonlinearity is positive, scenario of “exchange” takes place.

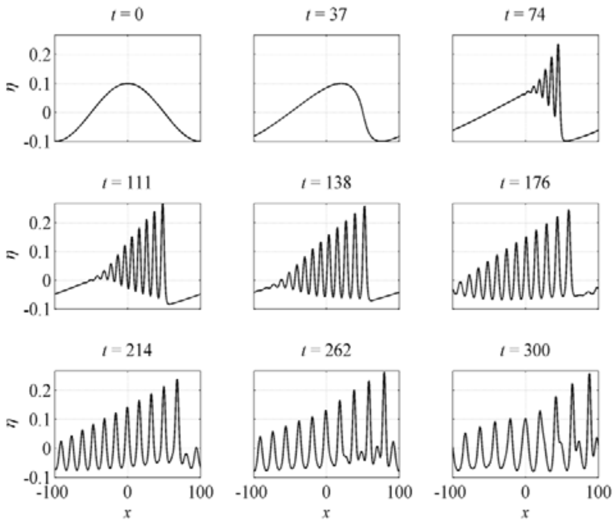

Fig. 5. Snapshots of wave dynamics with A = 0.1 and negative cubic nonlinear term for the Gardner equation

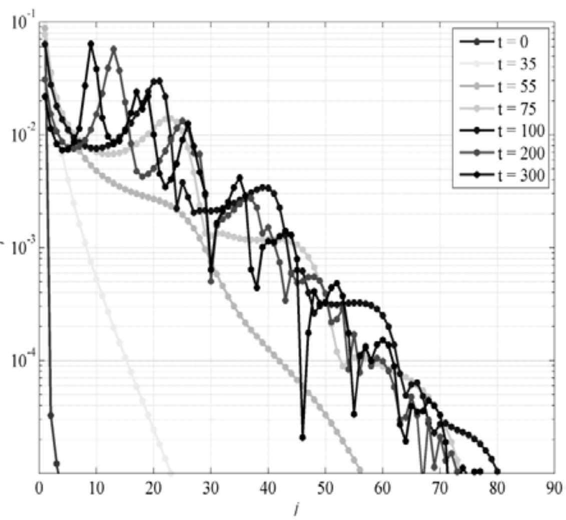

Fig. 6. Spectrum of wave records for A = 0.1 and negative cubic nonlinearity at different times.

Spectra of sine wave evolution have much in common for such amplitude and negative or positive cubic nonlinearity, therefore only graph for the latter case is given (Fig. 8). The spectra for this run are wider due to increased nonlinearity. Positive cubic nonlinearity accelerates the generation of higher harmonics in comparison with negative cubic nonlinearity. But again qualitatively, the shape of spectra are the same as in previous case with spectral peaks downshifting with time. The spectra after t = 10 are equidistant (with peaks on harmonics with multiple numbers).

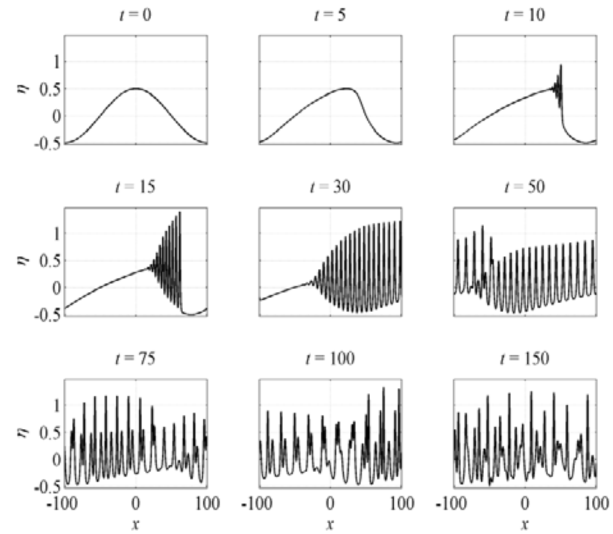

Fig. 7. Snapshots of wave dynamics with A = 0.5 and positive cubic nonlinear term for the Gardner equation

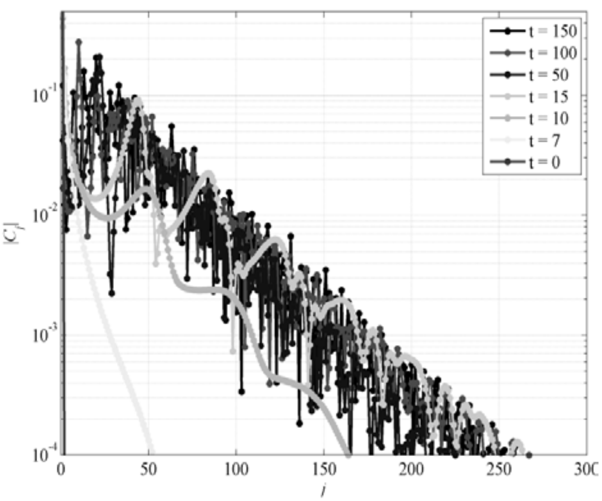

Fig. 8. Spectrum of wave records for A = 0.5 and positive cubic nonlinearity at different times.

Exceedance probability distribution of wave heights over time is shown in Fig. 9 separately for positive and negative parts of the wavefield. Significant wave height, which is defined as:

$$H_s = 4\,\sigma, \tag{9}$$

where σ is the standard deviation of ordinates η, H_s is indicated by the black line. Substantial asymmetry of negative and positive values of η is demonstrated by this plot.

Further amplitude increasing in the case of negative cubic nonlinearity causes appearance of second breaking point and generation of one table-top soliton with a group of solitary-like waves at the leading edge of the bore. Small solitons run on the crest of the table-top soliton and change their polarity. This process is described in detail in paper [15].

If the coefficient of cubic nonlinearity is positive and amplitude of initial wave amounts to A = 1.5 dimensionless units, a second "breaking" point appears and the pulses of both positive and negative polarity are generated. This process is demonstrated in Fig 10. Interactions of waves of opposite polarities result in an increase of the maximum amplitude of the wave field.

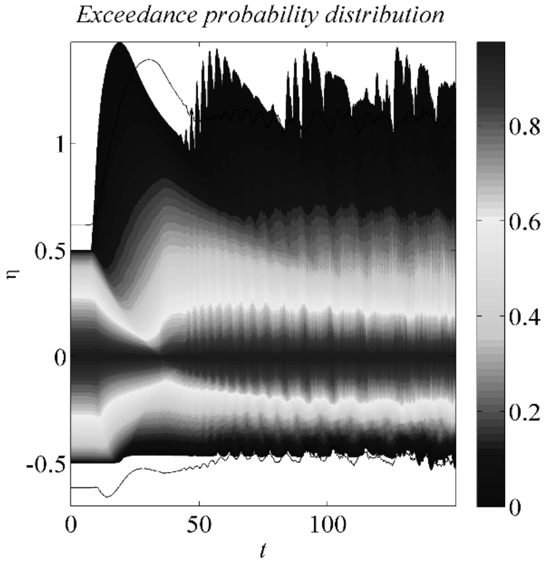

Fig. 9. Exceedance probability distribution of ordinates for $A = 0.5$ and positive cubic nonlinearity at various moments in time. Black line – significant height.

The spectra are qualitatively similar to those shown above but are significantly wider (Fig. 11). After $t = 2$ they contain almost equidistant peaks corresponding to multiple numbers of harmonics.

Statistical analysis of the wave field is shown in Fig. 12. Even in the case of $A = 1.5$ the wave interactions cause appearance of impulses with amplitude greater than significant wave height.

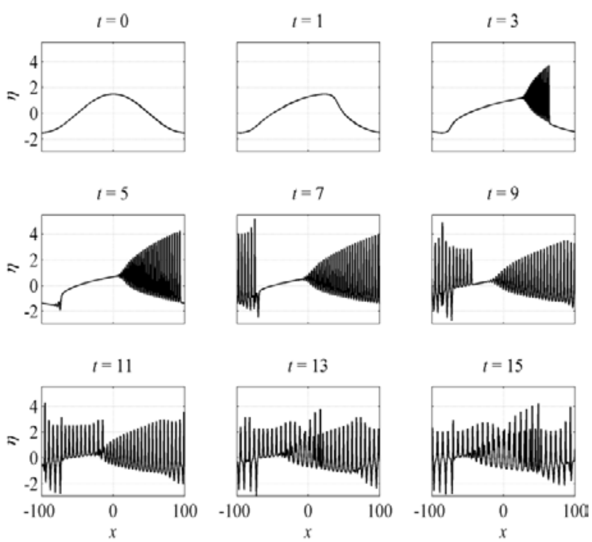

Fig. 10. Snapshots of wave dynamics with $A = 1.5$ and positive cubic nonlinear term for the Gardner equation

Nonlinear interactions of pulses of opposite polarities are more intense with increasing of initial sine wave amplitude. So to study the possible mechanisms of generating of extreme waves in the canonical Gardner equation with a positive cubic nonlinearity we increase the amplitude of sine impulse to a value $A = 3$ dimensionless units. The process of undular bore development and generation of wave field, which is represented interactions of ensembles of positive and negative polarity solitons, is shown in Fig. 13 for this initial amplitude.

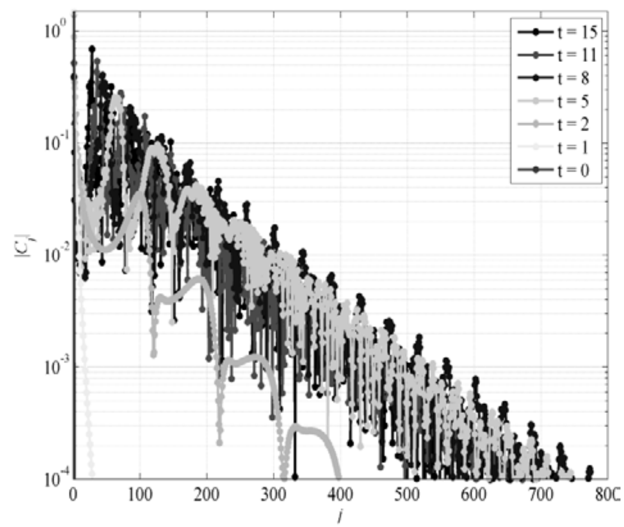

Fig. 11. Spectrum of wave records for $A = 1.5$ and positive cubic nonlinearity at different times.

One can see in Fig. 14 for exceedance probability distribution of wave heights, that large values of η are observed when paired collisions of different polarities solitons began after $t = 1.6$. Amplitudes of such impulses are four times greater than initial sine wave amplitude and often more than H_s.

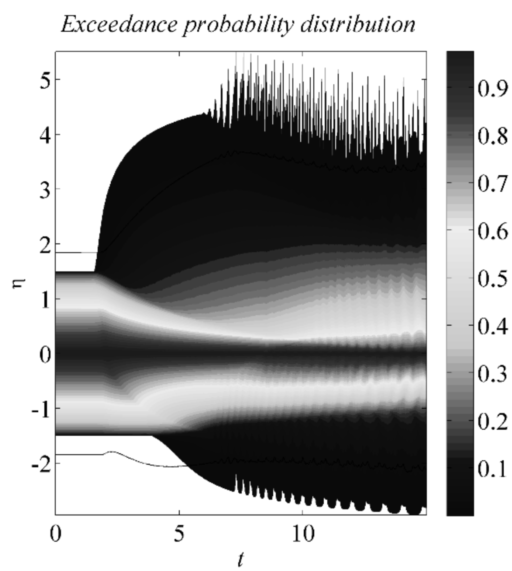

Fig. 12. Exceedance probability distribution of ordinates for $A = 1.5$ and positive cubic nonlinearity at various moments in time. Black line – significant wave height.

Graphs of kurtosis and skewness (Fig. 15) are also characterized by presence of peaks at times of higher probability of large-amplitude waves.

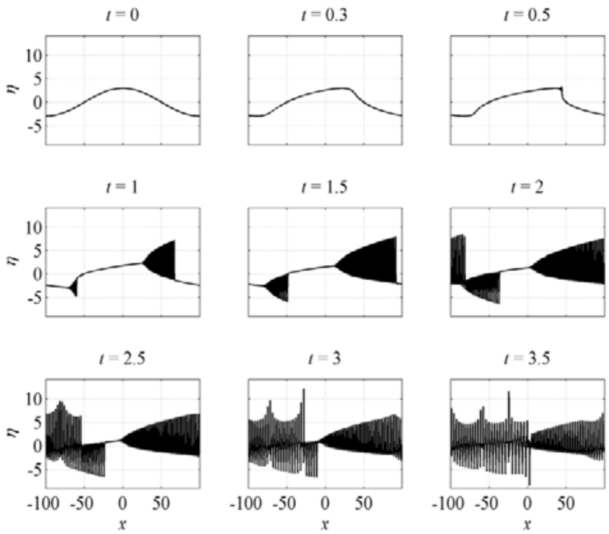

Fig. 13. Snapshots of wave dynamics with $A = 3$ and positive cubic nonlinear term for the Gardner equation

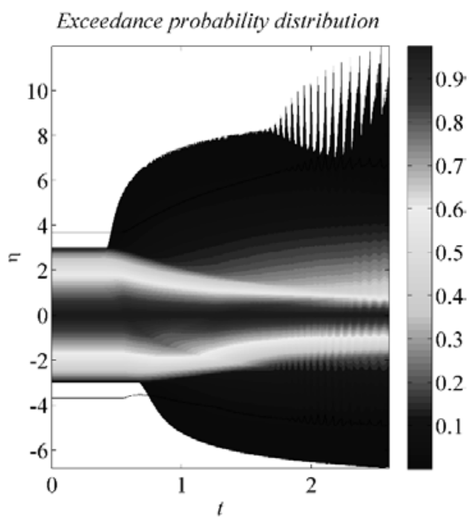

Fig. 14. Exceedance probability distribution of ordinates for $A = 3$ and positive cubic nonlinearity at various moments in time. Black line – significant height.

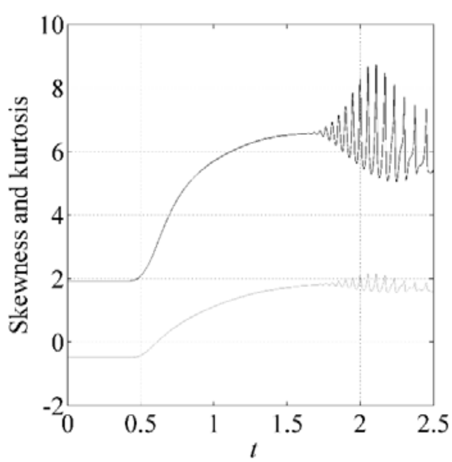

Fig. 15. Skewness (lower line) and kurtosis (upper line) of η for different points in time for $A = 3$ and positive cubic nonlinearity

The analysis of the shapes of spectra of the calculated wave fields vividly demonstrated the presence of exponential asymptotics for large wave numbers after some time of evolution (see Figs. 6, 8, 11, which illustrate linear dependence of C_j in semi logarithmic in ordinate axes):

$$S(k \to \infty) \propto \exp\left[-k / K_0\right]. \qquad (10)$$

We calculated scaling parameter K_0 for all our numerical experiments, and found out that it can be approximated in terms of amplitude A and length L of initial sine wave (2). The results are shown in Fig. 16.

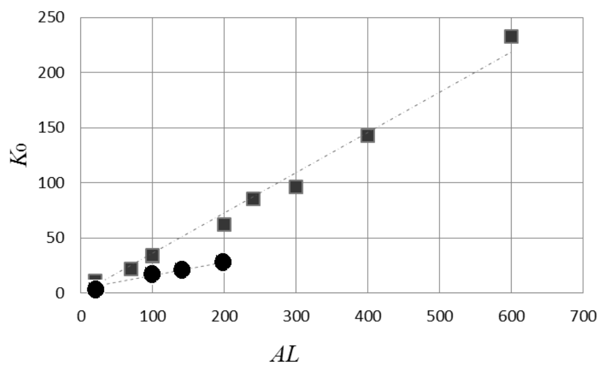

Fig. 16. Parameter K_0 of spectral exponential asymptotics versus the product AL of amplitude and length of initial wave (2). Symbols "■" denote numerical experiments for positive term of cubic nonlinearity in the Gardner equation (1), symbols "•" are for the runs with negative coefficient of cubic nonlinear term. Dashed lines – approximating linear dependencies.

V. Conclusion

The present paper studies the process of long sine wave disintegration in weakly nonlinear and weakly dispersive media within the Gardner equation with positive and negative cubic nonlinearity. We made a series of numerical computations to demonstrate the features of undular bores development for different signs of the cubic nonlinear term. If the cubic nonlinear term is positive and the wave amplitude is large, the solitons of both polarities appear. These waves interact and extreme amplitude waves can generate as a result of such collisions. This process is demonstrated well in exceedance probability distribution graphs for wave height as ordinates substantially exceeding significant amplitude H_s. Nonlinear interactions lead to the generation of higher harmonics forming the breaking asymptotic $j^{-4/3}$ for small times (as it was predicted within the dispersionless Gardner equation). For larger times the spectral peaks appear due to generation of solitary waves, they downshift with time. Exponential asymptotic of wavefield spectra for large values of wave numbers for longer evolution times are shown to be valid for both signs of cubic nonlinearity. The parameter of exponent is shown to be well described as a linear function of the product of initial sine wave amplitude and length.

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