Investigation and Rothe's Type Scheme for Nonlinear Integro-Differential Multi-Dimensional Equations Associated with the Penetration of a Magnetic Field in a Substance

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***Abstract*—**The paper is devoted to the construction and study of the additive average semi-discrete scheme for two nonlinear multi-dimensional integro-differential equations of parabolic type. The studied equation is based on well-known Maxwell's system arising in mathematical simulation of electromagnetic field penetration into a substance. Existence, uniqueness and long-time behavior of solutions of initial-boundary value problems for nonlinear systems of parabolic integro-differential equations are fixed too.

***Keywords*—**Nonlinear parabolic multi-dimensional integro-differential equations, existence and uniqueness of solutions, long-time behavior, additive averaged semi-discrete schemes.

I. Introduction

INTEGRO-differential models arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, and so forth (see, for example, [7], [8], [13], [20] and references wherein). Such systems arise for instance for mathematical modeling of the process of penetrating of electromagnetic field in the substance. In a quasistationary case the corresponding system of Maxwell's equations has the form [9]:

$$\frac{\partial H}{\partial t} = -rot(\nu_m rotH), \tag{1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m \left(rotH\right)^2, \tag{2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, c_ν and ν_m characterize the thermal heat capacity and electroconductivity of the substance. Equations (1) defines the process of diffusion of the magnetic field and equation (2) - change of the temperature at the expense of Joule's heating. If c_ν and ν_m depend on temperature θ, i.e., $c_\nu = c_\nu(\theta)$, $\nu_m = \nu_m(\theta)$, then the system (1), (2) can be rewritten in the following form [6]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t \left| rotH \right|^2 d\tau \right) rotH \right], \tag{3}$$

where function $a = a(S)$ is defined for $s \in [0,\infty)$.

In [10] some generalization of the system of type (3) is proposed. Here the same process of penetration of the magnetic field into the material is simulated by the following averaged integro-differential model:

$$\frac{\partial H}{\partial t} = a\left(\int_0^t \int_\Omega \left| rotH \right|^2 dx d\tau \right) \Delta H, \tag{4}$$

where Ω is an area occupied by the conductor.

Note that integro-differential parabolic models of (3) and (4) type are complex and still yields to the investigation only for special cases (see, for example, [1], [3]-[6], [10]-[12], [14]-[21], [23]-[28]).

Let us consider the following magnetic field H, with the form $H = (0,0,U)$, where $U = U(x,y,t)$ is a scalar function of time and of two spatial variables. Then $rotH = \left(\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial x}, 0 \right)$ and systems (3) and (4) will take the forms:

$$\frac{\partial U}{\partial t} = \nabla\left[a\left(\int_0^t \left| \nabla U \right|^2 d\tau \right) \nabla U \right] \tag{5}$$

and

$$\frac{\partial U}{\partial t} = a\left(\int_0^t \int_\Omega \left| \nabla U \right|^2 d\tau \right) \Delta U. \tag{6}$$

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Study of the models of type (3) and (5) have begun in the work [6]. In this work, in particular, are proved the theorems of existence of solution of the first boundary value problem for scalar and one-dimensional space case while $a(S)=1+S$ and uniqueness for more general cases. One-dimensional scalar variant for the case $a(S)=(1+S)^p$, $0<p\le 1$ is studied in [5]. Investigations for multidimensional space cases at first are carried out in the work [4] and then have continued in the following works [1], [3], [10]-[12], [14]-[21], [23]-[28] and in a number over works as well.

Study of the models of type (4) and (6) have started in the work [16], where the existence, uniqueness and asymptotic behavior of the solutions of the first initial-boundary value problem for the one-dimensional scalar variant with $a(S)=(1+S)^p$, $0<p\le 1$ is studied.

The solvability of the initial-boundary value problems for (3), (4) type models in scalar cases is studied using a modified version of the Galerkin's method and compactness arguments that are used in [13], [29] for investigation elliptic and parabolic equations.

One must note that for the cylindrical conductors to the study of modeling of physical process of penetrating of the electromagnetic field some amounts of works were again devoted. To the investigation of periodic problem for one-dimensional (3) type model in cylindrical coordinates was devoted work [14].

Particular attention should be paid to construction of numerical solutions and to their importance for integro-differential models (see, for example, [2], [13], [15], [17]-[23], [25], [26], [28]).

The paper is devoted to the existence and uniqueness of solution of the initial-boundary problem for two (5) and (6) type nonlinear multi-dimensional integro-differential equations. Construction and study of the additive averaged Rothe's type scheme is also given.

Principal characteristic peculiarity of the equations (5) and (6) is connected with the appearance in the coefficient with derivative of higher order nonlinear term depended on the integral of time and space variables. These circumstances requires different discussions, than it is usually necessary for the solution of local differential problems.

Many authors study the Rothe's scheme for a integro-differential models (see, for example, [13], [17], [23]).

It is very important to study decomposition analogs for above-mentioned multi-dimensional differential and integro-differential models as well. At present there are some effective algorithms for solving the multi-dimensional problems (see, for example, [13], [30] and references therein).

Our paper is dedicated to the global existence and uniqueness of solutions of initial-boundary value problem. Investigations are given in usual Sobolev spaces. Attention is paid to investigation of semi-discrete additive average schemes. In this paper we shall focus our attention to the particular case of (5), (6) type multi-dimensional integro-differential equations.

This article is organized as follows. In the Section 2 the formulation of the problem and some of its properties are given for so-called not-averaged (5) type equation. Especially existence and uniqueness of the solution of the stated problem are fixed there. Main attention is paid to construction and investigation of semi-discrete additive average scheme. This question is discussed in Section 3. In Section 4 analogical results for averaged (6) type equation are fixed. Some conclusions are given in Section 5.

II. Existence and Uniqueness for Not-Averaged Equation

Let Ω is bounded domain in the n-dimensional Euclidean space R^n, with sufficiently smooth boundary $\partial\Omega$. In the domain $Q=\Omega\times(0,T)$ of the variables $(x,t)=(x_1,x_2,...,x_n,t)$ let us consider the following first type initial-boundary value problem:

$$\frac{\partial U}{\partial t}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\left(1+\int_0^t\left|\frac{\partial U}{\partial x_i}\right|^2 d\tau\right)\frac{\partial U}{\partial x_i}\right] = f(x,t),\quad (x,t)\in Q, \tag{7}$$

$$U(x,t)=0,\quad (x,t)\in\partial\Omega\times[0,T], \tag{8}$$

$$U(x,0)=0,\quad x\in\overline{\Omega}, \tag{9}$$

where T is a fixed positive constant, f is a given function of its arguments.

Using modified version of the Galerkin's method and compactness arguments [13], [29] as in [16] the following statement can be proven for problem (7) - (9).

Theorem 1 *If*

$$f,\ \frac{\partial f}{\partial t},\ \sqrt{\psi}\,\frac{\partial f}{\partial x_i}\in L_2(Q),\quad f(x,0)=0,$$

then there exists the unique solution U of problem (7) - (9) satisfying properties:

$$U\in L_4\left(0,T,\overset{\circ}{W}{}_4^1(\Omega)\right),\ \frac{\partial U}{\partial t}\in L_2(Q),$$

$$\sqrt{\psi}\,\frac{\partial^2 U}{\partial x_i\partial x_j}\in L_2(Q),\quad \sqrt{T-t}\,\frac{\partial^2 U}{\partial t\partial x_i}\in L_2(Q),$$

$$i,j=1,...,n,$$

where

$$\psi\in C^\infty(\Omega),\quad \psi(x)>0, x\in\Omega;$$

$$\psi=\frac{\partial\psi}{\partial\nu}=0, x\in\partial\Omega,$$

is outer normal of $\partial\Omega$.

Using the scheme of investigation as in, e.g., [16], [18], [20], [23], [24], [26], [27] it is not difficult to get the result of exponentially asymptotic behavior of solution as $t \to \infty$ for the (7) equation with $f(x,t) \equiv 0$ and homogeneous boundary (8) and nonhomogeneous initial (9) conditions.

III. Rothe's Type Additive Scheme for Not-Averaged Equation

On $[0,T]$ let us introduce a net with mesh points denoted by $t_j = j\tau$, $j = 0,1\ldots J$, with $\tau = 1/J$.

Coming back to problem (7) - (9) and let us construct additive average Rothe's type semi-discrete scheme:

$$\eta_i \frac{u_i^{j+1} - u^j}{\tau} =$$

$$\frac{\partial}{\partial x_i}\left[\left(1 + \tau \sum_{k=1}^{j+1}\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right)\frac{\partial u_i^{j+1}}{\partial x_i}\right] + f_i^{j+1}, \tag{10}$$

$$u_i^0 = u^0 = 0,$$

$$i = 1,\ldots,n, \quad j = 0,1\ldots J-1,$$

with homogeneous boundary conditions, where $u_i^j(x)$, $j = 0,1\ldots J$ is solution of the problem (10) and following notations are introduced:

$$u^j(x) = \sum_{i=1}^{n} \eta_i u_i^j(x), \quad \sum_{i=1}^{n} \eta_i = 1, \quad \eta_i > 0,$$

$$\sum_{i=1}^{n} f_i^{j+1}(x) = f^{j+1}(x) = f(x, t_{j+1}),$$

where u^j denotes approximation of exact solution U of problem (7) - (9) at t_j.

The object of this section is to prove one main statement of this paper. Here we use usual scalar product $(,)$ and norm $\|\cdot\|$ of the space $L_2(\Omega)$.

Theorem 2 *If problem (7) - (9) has sufficiently smooth solution then functions u^m defined by the solutions of problems (10) converge to the solution of problem (7) - (9) and the following estimate is true*

$$\left\| U^m - u^m \right\| = O(\tau^{1/2}), \quad m = 1\ldots J.$$

Proof. Let us introduce following notations:

$$z^k = U^k - u^k, \quad z_i^k = U^k - u_i^k.$$

For the exact solution of problem (7) - (9) we have

$$\eta_i \frac{U^{j+1} - U^j}{\tau} =$$

$$\eta_i \sum_{\ell=1}^{n} \frac{\partial}{\partial x_\ell}\left[\left(1 + \tau \sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_\ell}\right)^2\right)\frac{\partial U^{j+1}}{\partial x_\ell}\right]$$

$$+ \eta_i f^{j+1} + O(\tau).$$

After subtracting (10) from relation above we get

$$\eta_i\left(\frac{U^{j+1} - U^j}{\tau} - \frac{u_i^{j+1} - u^j}{\tau}\right) =$$

$$\eta_i \sum_{\ell=1}^{n} \frac{\partial}{\partial x_\ell}\left[\left(1 + \tau \sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_\ell}\right)^2\right)\frac{\partial U^{j+1}}{\partial x_\ell}\right]$$

$$- \frac{\partial}{\partial x_i}\left[\left(1 + \tau \sum_{k=1}^{j+1}\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right)\frac{\partial u_i^{j+1}}{\partial x_i}\right]$$

$$+ \eta_i f^{j+1} - f_i^{j+1} + O(\tau).$$

Thus, introducing the notation

$$\frac{z_i^{j+1} - z^j}{\tau} = z_{i\bar{t}}^{j+1},$$

we have

$$\eta_i z_{i\bar{t}}^{j+1} = \frac{\partial}{\partial x_i}\left[\left(1 + \tau \sum_{k=0}^{j+1}\left(\frac{\partial U^k}{\partial x_i}\right)^2\right)\frac{\partial U^{j+1}}{\partial x_i}\right]$$

$$- \frac{\partial}{\partial x_i}\left[\left(1 + \tau \sum_{k=0}^{j+1}\left(\frac{\partial U^k}{\partial x_i}\right)^2\right)\frac{\partial U^{j+1}}{\partial x_i}\right]$$

$$+ \eta_i \sum_{\ell=1}^{n} \frac{\partial}{\partial x_\ell}\left[\left(1 + \tau \sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_\ell}\right)^2\right)\frac{\partial U^{j+1}}{\partial x_\ell}\right]$$

$$- \frac{\partial}{\partial x_i}\left[\left(1 + \tau \sum_{k=1}^{j+1}\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right)\frac{\partial u_i^{j+1}}{\partial x_i}\right]$$

$$+ \eta_i f^{j+1} - f_i^{j+1} + O(\tau).$$

Here we add and subtract the first and second terms in the right side.

Using (7) and (10) we have the following problem:

$$\eta_i z_{i\bar{t}}^{j+1} =$$

$$\frac{\partial}{\partial x_i}\left[\left(1 + \tau \sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_i}\right)^2\right)\frac{\partial U^{j+1}}{\partial x_i} - \right.$$

$$\left.\left(1 + \tau \sum_{k=1}^{j+1}\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right)\frac{\partial u_i^{j+1}}{\partial x_i}\right] + \psi_i^{j+1}(x), \tag{11}$$

$$z_i^0 = 0,$$

with homogeneous boundary conditions and where:

$$\psi_i^{j+1}(x) = -\frac{\partial}{\partial x_i}\left[\left(1 + \tau \sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_i}\right)^2\right)\frac{\partial U^{j+1}}{\partial x_i}\right]$$

$$+ \eta_i \sum_{\ell=1}^{n} \frac{\partial}{\partial x_\ell}\left[\left(1 + \tau \sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_\ell}\right)^2\right)\frac{\partial U^{j+1}}{\partial x_\ell}\right]$$

$$+\eta_i f^{\,j+1}(x) - f_i^{\,j+1}(x) + O(\tau) = \overline{\psi}_i^{\,j+1}(x) + O(\tau).$$

Using assumptions on $f_i^{\,j+1}$ and η_i we have

$$\sum_{i=1}^{n} \overline{\psi}_i^{\,j+1}(x) =$$
$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\left(1+\tau\sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_i}\right)^2\right)\frac{\partial U^{\,j+1}}{\partial x_i}\right]$$
$$+\sum_{i=1}^{n}\eta_i \sum_{\ell=1}^{n}\frac{\partial}{\partial x_\ell}\left[\left(1+\tau\sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_\ell}\right)^2\right)\frac{\partial U^{\,j+1}}{\partial x_\ell}\right] \tag{12}$$
$$+\sum_{i=1}^{n}\eta_i f^{\,j+1}(x) - \sum_{i=1}^{n} f_i^{\,j+1}(x) = 0.$$

So,

$$\sum_{i=1}^{n}\psi_i^{\,j+1}(x) = O(\tau).$$

Multiplying (11) scalarly on $2\tau z_i^{\,j+1}$ we obtain

$$+2\tau\left(\left(1+\tau\sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_i}\right)^2\right)\frac{\partial U^{\,j+1}}{\partial x_i}\right.$$
$$\left.-\left(1+\tau\sum_{k=1}^{j+1}\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right)\frac{\partial u_i^{\,j+1}}{\partial x_i}, \frac{\partial z_i^{\,j+1}}{\partial x_i}\right) \tag{13}$$
$$-2\tau\left(\psi_i^{\,j+1}, z_i^{\,j+1}\right) = 0.$$

It can be easily checked that

$$\left(\left(1+\tau\sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_i}\right)^2\right)\frac{\partial U^{\,j+1}}{\partial x_i}\right.$$
$$\left.-\left(1+\tau\sum_{k=1}^{j+1}\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right)\frac{\partial u_i^{\,j+1}}{\partial x_i}, \frac{\partial z_i^{\,j+1}}{\partial x_i}\right)$$
$$=\frac{1}{2}\left[\left(2+\tau\sum_{k=1}^{j+1}\left(\frac{\partial U^k}{\partial x_i}\right)^2\right.\right.$$
$$\left.+\tau\sum_{k=1}^{j+1}\left(\frac{\partial u_i^k}{\partial x_i}\right)^2, \left(\frac{\partial z_i^{\,j+1}}{\partial x_i}\right)^2\right)$$
$$+\left(\tau\sum_{k=1}^{j+1}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right],\right.$$
$$\left.\left.\left(\frac{\partial U^{\,j+1}}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^{\,j+1}}{\partial x_i}\right)^2\right)\right]$$
$$\geq \frac{1}{2}\left(\tau\sum_{k=1}^{j+1}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right],\right.$$
$$\left.\left(\frac{\partial U^{\,j+1}}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^{\,j+1}}{\partial x_i}\right)^2\right).$$

From (13) for the error we get

$$2\tau\eta_i\left(z_{i\bar{t}}^{\,j+1}, z_i^{\,j+1}\right)$$
$$+\tau\left(\tau\sum_{k=1}^{j+1}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right],\right.$$
$$\left.\left(\frac{\partial U^{\,j+1}}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^{\,j+1}}{\partial x_i}\right)^2\right)$$
$$\leq 2\tau(\psi_i^{\,j+1}, z_i^{\,j+1}).$$

Using identities:

$$z_i^{\,j+1} = z^j + \tau z_{i\bar{t}}^{\,j+1},$$
$$2\tau\left(z_{i\bar{t}}^{\,j+1}, z_i^{\,j+1}\right) = \left\|z_i^{\,j+1}\right\|^2 + \tau^2\left\|z_{i\bar{t}}^{\,j+1}\right\|^2 - \left\|z^j\right\|^2$$

after simple transformations we have from the last inequality

$$\eta_i\left(\left\|z_i^{\,j+1}\right\|^2 + \tau^2\left\|z_{i\bar{t}}^{\,j+1}\right\|^2\right)$$
$$+\frac{1}{2}\left\|\tau\sum_{k=1}^{j+1}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right]\right\|^2$$
$$+\frac{\tau^2}{2}\left\|\left(\frac{\partial U^{\,j+1}}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^{\,j+1}}{\partial x_i}\right)^2\right\|^2$$
$$\leq \eta_i\left\|z^j\right\|^2 + \frac{1}{2}\left\|\tau\sum_{k=1}^{j}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right]\right\|^2$$
$$+2\tau\left(\psi_i^{\,j+1}, z^j + \tau z_{i\bar{t}}^{\,j+1}\right).$$

Summing this equality from 1 to n we arrive at

$$\sum_{i=1}^{n}\eta_i\left(\left\|z_i^{\,j+1}\right\|^2 + \tau^2\left\|z_{i\bar{t}}^{\,j+1}\right\|^2\right) +$$
$$\frac{1}{2}\sum_{i=1}^{n}\left\|\tau\sum_{k=0}^{j+1}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right]\right\|^2$$
$$+\frac{\tau^2}{2}\sum_{i=1}^{n}\left\|\left(\frac{\partial U^{\,j+1}}{\partial x_i}\right)^2 - \left(\frac{\partial u_i^{\,j+1}}{\partial x_i}\right)^2\right\|^2 \tag{14}$$
$$\leq \sum_{i=1}^{n}\eta_i\left\|z^j\right\|^2 +$$

$$\frac{1}{2}\sum_{i=1}^{n}\left\|\tau\sum_{k=1}^{j}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2-\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right]\right\|^2$$

$$+2\tau\sum_{i=1}^{n}(\psi_i^{j+1},z^j)+2\tau\sum_{i=1}^{n}(\psi_i^{j+1},\tau z_{i\bar{t}}^{j+1}).$$

Note that,

$$\sum_{i=1}^{n}\eta_i z_i^{j+1}=\sum_{i=1}^{n}\eta_i(U^{j+1}-u_i^{j+1})=z^{j+1},$$

$$\sum_{i=1}^{n}\eta_i\left\|z^j\right\|^2=\left\|z^j\right\|^2,$$

$$\sum_{i=1}^{n}\eta_i\left\|z_i^{j+1}\right\|^2\geq\left\|\sum_{i=1}^{n}\eta_i z_i^{j+1}\right\|^2=\left\|z^{j+1}\right\|^2.$$

Using these relations, identity of sum approximation (12) and Schwarz's inequality we get from (14)

$$\left\|z^{j+1}\right\|^2+\sum_{i=1}^{n}\eta_i\tau^2\left\|z_{i\bar{t}}^{j+1}\right\|^2+$$

$$\frac{1}{2}\sum_{i=1}^{n}\left\|\tau\sum_{k=1}^{j+1}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2-\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right]\right\|^2$$

$$\leq\left\|z^j\right\|^2+\frac{1}{2}\sum_{i=1}^{n}\left\|\tau\sum_{k=1}^{j}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2-\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right]\right\|^2$$

$$+2\tau(O(\tau),z^j)+\tau^2\left\|\psi^{j+1}\right\|^2+\sum_{i=1}^{n}\eta_i\tau^2\left\|z_{i\bar{t}}^{j+1}\right\|^2.$$

Here

$$\left\|\psi^{j+1}\right\|^2=\sum_{i=1}^{n}\eta_i^{-1}\left\|\psi_i^{j+1}\right\|^2.$$

Using boundedness of $\left\|\psi^{j+1}\right\|$ we find

$$\begin{gathered}\left\|z^{j+1}\right\|^2+\\ \frac{1}{2}\sum_{i=1}^{n}\left\|\tau\sum_{k=1}^{j+1}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2-\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right]\right\|^2\\ \leq\left\|z^j\right\|^2+\\ \frac{1}{2}\sum_{i=1}^{n}\left\|\tau\sum_{k=1}^{j}\left[\left(\frac{\partial U^k}{\partial x_i}\right)^2-\left(\frac{\partial u_i^k}{\partial x_i}\right)^2\right]\right\|^2\\ +2\tau(O(\tau),z^j)+O(\tau^2).\end{gathered}\tag{15}$$

Summing (15) with respect to j from 0 to $m-1$ we get

$$\begin{gathered}\left\|z^m\right\|^2+\frac{\tau}{2}\sum_{i=1}^{n}\left\|\left(\frac{\partial U^m}{\partial x_i}\right)^2-\left(\frac{\partial u_i^m}{\partial x_i}\right)^2\right\|^2\\ \leq 2\tau\sum_{j=0}^{m-1}\left(O(\tau),z^j\right)+C\tau\\ \leq\tau\sum_{j=0}^{m-1}\left[O(\tau^2)+\left\|z^j\right\|^2\right]+O(\tau)\\ \leq\tau\sum_{j=0}^{m-1}\left\|z^j\right\|^2+C\tau.\\ \left\|z^m\right\|^2+\frac{\tau}{2}\sum_{i=1}^{n}\left\|\left(\frac{\partial U^m}{\partial x_i}\right)^2-\left(\frac{\partial u_i^m}{\partial x_i}\right)^2\right\|^2\\ \leq 2\tau\sum_{j=0}^{m-1}\left(O(\tau),z^j\right)+C\tau\\ \leq\tau\sum_{j=0}^{m-1}\left[O(\tau^2)+\left\|z^j\right\|^2\right]+O(\tau)\\ \leq\tau\sum_{j=0}^{m-1}\left\|z^j\right\|^2+C\tau.\end{gathered}\tag{16}$$

The desired result of Theorem 2 now follows from (16) by the standard discrete Gronwall's lemma.

IV. Unique Solvability and Rothe's Scheme for an Average Model

Now let us consider the following first type initial-boundary value problem for an average equation:

$$\frac{\partial U}{\partial t}-\sum_{i=1}^{n}\left(1+\int_{\Omega}\int_{0}^{t}\left|\frac{\partial U}{\partial x_i}\right|^2 dxd\tau\right)\frac{\partial^2 U}{\partial x_i^2}\tag{17}$$

$$=f(x,t),\quad (x,t)\in Q,$$

$$U(x,t)=0,\quad (x,t)\in\partial\Omega\times[0,T],\tag{18}$$

$$U(x,0)=0,\quad x\in\overline{\Omega}.\tag{19}$$

Since problem (17) - (19) similar to problems considered in [16], where investigation of (4) type multi-dimensional scalar equations is given and at first is discussed unique solvability and asymptotic behavior of (17) type models as well, we can follow the same procedure used there. Using modified version of the Galerkin's method and compactness arguments [13], [29] the following statement can be proven.

Theorem 3 *If*

$$f\in W_2^1(Q),\quad f(x,0)=0,$$

then there exists the unique solution U of problem (17) - (19) satisfying properties:

$$U\in L_4\left(0,T;\overset{\circ}{W}{}_4^1(\Omega)\right)\cap L_2\left(0,T;W_2^2(\Omega)\right),$$

$$\frac{\partial U}{\partial t} \in L_2(Q),$$

$$\sqrt{T-t}\,\frac{\partial^2 U}{\partial t \partial x_i} \in L_2(Q), \quad i = 1,...,n.$$

The proof of the formulated theorem is divided into several steps. One of the basic step is to obtain necessary a priori estimates.

Using the scheme of investigation as in, e.g., [16] it is not difficult to get the result of exponentially asymptotic behavior of solution as $t \to \infty$ for the (17) equation with $f(x,t) \equiv 0$ and homogeneous boundary (18) and nonhomogeneous initial (19) conditions.

Coming back to problem (17) - (19) and let us construct additive averaged Rothe's type scheme:

$$\eta_i \frac{u_i^{j+1} - u^j}{\tau} =$$

$$\left(1 + \tau \sum_{k=1}^{j+1} \int_\Omega \left| \frac{\partial u_i^k}{\partial x_i} \right|^2 dx \right) \frac{\partial^2 u_i^{j+1}}{\partial x_i^2} + f_i^{\,j+1}, \qquad (20)$$

$$u_i^0 = u^0 = 0,$$

$$i = 1,...,n, \quad j = 0,1\ldots J-1,$$

with homogeneous boundary conditions, where $u_i^j(x)$, $j = 1,\ldots,J$, is solution of the problem (20) and the following notations are introduced again:

$$u^j(x) = \sum_{i=1}^n \eta_i u_i^j(x), \quad \sum_{i=1}^n \eta_i = 1, \quad \eta_i > 0,$$

$$\sum_{i=1}^n f_i^{\,j+1}(x) = f^{\,j+1}(x) = f(x, t_{j+1}),$$

where u^j denotes approximation of exact solution U of problem (17) - (19) at t_j. We use usual norm $\|\cdot\|$ of the space $L_2(\Omega)$.

Theorem 4 *If problem (17) - (19) has sufficiently smooth solution then the solution of problem (20) converges to the solution of problem (17) - (19) and the following estimate is true*

$$\left\| U^j - u^j \right\| = O(\tau^{1/2}), \quad j = 1,\ldots,J.$$

V. Conclusion

Using early investigated finite difference and finite element schemes for one-dimensional (7) and (17) type models (see, for example, [20] and references therein) now we can reduce numerical resolution of the multi-dimensional integro-differential models (7) and (17) to one-dimensional ones. Carried out various numerical experiments agree with theoretical researches. It is very important to construct and investigate studied in this note type models for more general type nonlinearities and for (7) and (17) type multi-dimensional systems as well.

Partial integro-differential multi-dimensional equations associated with the penetration of a magnetic field in a substance is considered. Existence, uniqueness and long-time behavior of solution of initial-boundary value problem are fixed. The semi-discrete Rothe's type schemes are investigated as well.

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