A simple construction of incoherent finite frames

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Abstract: The frames with low mutual coherence are known to have several applications in compressed sensing. Householder matrices are used in this paper to construct matrices that form finite tight frames and have low mutual coherence. The algorithm also produces finite equiangular tight frames. In cases when the equiangular frames do not exit, it produces tight frames of approximately unit norm. A pair of orthogonal tight frames can be constructed from the resulting frames very easily.
Keywords: Frames, Unit-Norm Tight Frames, Equiangular Frames, Coherence, Orthogonal Frames.

I INTRODUCTION

Frames generalize the basis as they are linearly dependent spanning set for the space they live in. The frames form rectangular matrices as opposed to basis which form a square matrix. Finite frames have been used in varieties of applications. The orthonormal basis provides expansion in terms of unique coefficients as the basis vectors are uncorrelated, in other words the mutual coherence among the basis vectors is zero. Since the frames are overcomplete system of vectors, the coherence can't be zero. However frames can be constructed with low mutual coherence or sometimes even the lowest possible coherence can be achieved. The frames with low mutual coherence are desired in various applications. The frames with minimal dependency of columns are known as the Grassmannian frames. For example the Mercedes Benz frames of three vectors in $\mathbb{R}^2$. These are examples of equiangular frames. The frames with minimal coherence or incoherent frames are known to be useful in compressed sensing where a small sample of data is used to reconstruct a vector of large size i.e. the fewer samples of a signal are sometimes enough to reconstruct it. Mathematically, for a signal $x \in \mathbb{R}^m$, only a few measurements $y \in \mathbb{R}^n$, captured via a linear measurement matrix known as the sampling matrix, are sometimes enough to reconstruct the signal x where $n << m$.

The orthogonality of frames plays an important role in applications [1]. Some examples of pair of orthogonal frames are given in [1, 2, 3]. Authors in [9, 8] have used random sampling matrices to produce incoherent frames. In this paper, similar ideas are used that uses deterministic sampling matrices namely the Householder matrices to produce unit norm finite tight frames with low mutual coherence. Also a pair of orthogonal frames can be constructed from the resulting frames.

A. Frames

A basis $\mathbb{X} = \{\psi_j\}$ $j = 1, 2 \cdots m$ for a Hilbert space of dimension m, is a set of vectors such that each x in the space can be written as,

$$x = \sum_{j=1}^{m} c_j \psi_j. \tag{1}$$

Fewer non zero coefficients c_j in (1) make the basis applicable in signal processing.

Definition I.1. *A frame in a finite dimensional space $\mathbb{H} = \mathbb{R}^m$ or $\mathbb{C}^m$ is a sequence $\mathbb{X} = \{\psi_j\}_{j=1\cdots n}$, $n \geq m$, of vectors in $\mathbb{H}$ for which there exists constants $0 < B \leq A$ such*

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that for all $x \in \mathbb{H}$

$$A\,||x||^2 \leq \sum_{j=1}^{n} |\langle x, \psi_j \rangle|^2 \leq B\,||x||^2.$$

A and B are called frame bounds. A is the greatest lower bound and B is the least upper bound. If $A = B$, then it is called a tight frame (TF). A uniform norm frame is the one in which all vectors have equal norm. If the norm of all the vectors are equal to one, it is called a unit norm tight frame (UNTF). Let $\mathbb{X}$ be a frame. Define linear operator $\mathbb{T}_{\mathbb{X}}^{*}$: $\mathbb{H} \to \mathbb{R}^n$ by

$$\mathbb{T}_{\mathbb{X}}^{*}(x) = \left(\langle x, \psi_j \rangle\right)_j;$$

This operators is known as the analysis operator and the operator $\mathbb{T}_{\mathbb{X}} : \mathbb{R}^n \to \mathbb{H}$, given by

$$\mathbb{T}_{\mathbb{X}}(c)_j = \sum_{j=1}^{n} c_j \psi_j,$$

is known as the synthesis operator. The synthesis operator is adjoint of the analysis operator. The frame operator $S_{\mathbb{X}} : \mathbb{H} \to \mathbb{H}$ is defined as

$$S_{\mathbb{X}}(x) = \mathbb{T}_{\mathbb{X}}^{*}\mathbb{T}_{\mathbb{X}}(x) = \sum_{j=1}^{n} \langle x, \psi_j \rangle\, \psi_j.$$

It is known that the frames for finite dimensional space are precisely the finite spanning sets. Let $\mathbb{X} = \{\psi_k\}_{k=1}^{n}$ be a frame for the vector space $\mathbb{R}^m$, $n \geq m$. Then the analysis operator $\mathbb{T}_{\mathbb{X}}^{*} : \mathbb{R}^m \to \mathbb{R}^n$, is a matrix formed by taking the vectors ψ_k as rows and the synthesis operator $\mathbb{T}_{\mathbb{X}} : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix formed by taking the vectors ψ_k as the columns. Thus they are given by the matrices

$$\mathbb{T}_{\mathbb{X}}^{*} = \begin{pmatrix} - & \psi_1^* & - \\ - & \psi_2^* & - \\ \vdots & \vdots & \vdots \\ - & \psi_n^* - & \end{pmatrix}$$

and

$$\mathbb{T}_{\mathbb{X}} = \begin{pmatrix} | & | & \cdots & | \\ \psi_1 & \psi_2 & \cdots & \psi_n \\ | & | & \cdots & | \end{pmatrix}$$

respectively. In case of $\mathbb{C}^m$, the vectors in the analysis operator need to be conjugated. We note that the analysis operator is injective and the synthesis operator is surjective [4]. If $S_{\mathbb{X}}$ is a frame operator,

$$S_{\mathbb{X}}(x) = \mathbb{T}_{\mathbb{X}}\mathbb{T}_{\mathbb{X}}^{*}(x) = \sum_{j=1}^{n} \langle x, \psi_j \rangle\, \psi_j,$$

and

$$\begin{aligned} \langle S_{\mathbb{X}}x, x \rangle = \langle \mathbb{T}_{\mathbb{X}}^{*}x, \mathbb{T}_{\mathbb{X}}^{*}x \rangle &= ||\mathbb{T}_{\mathbb{X}}^{*}x||^2 \\ &= \sum_{j}^{n} |\langle x, \psi_j \rangle|^2. \end{aligned}$$

It turns out from the frame inequality that, for all $x \in \mathbb{H}$,

$$A\|\|x\|\|^2 \leq \langle S_{\mathbb{X}}x, x \rangle \leq B\|\|x\|\|^2$$

$$\iff A \leq \frac{\langle S_{\mathbb{X}}x, x \rangle}{||x||^2} \leq B.$$

This is the Rayleigh Quotient of the frame operator $S_{\mathbb{X}}$. It follows that A and B are the smallest and largest eigenvalues of $S_{\mathbb{X}}$. If it is a tight frame (also known as A-tight frame), the last equation takes the form

$$S_{\mathbb{X}} = AI_m,$$

where I_m is an identity matrix of size $m \times m$. In this case, the reconstruction formula takes the form

$$x = \frac{1}{A} \sum_{j=1}^{n} \langle x, \psi_j \rangle\, \psi_j. \tag{2}$$

So all the eigenvalues of the operator $S_{\mathbb{X}}$ are equal to A. The frame is said to be normalized tight frame or Parseval frame if $A = B = 1$. So the above equation takes the form

$$x = \sum_{j=1}^{n} \langle x, \psi_j \rangle\, \psi_j = \sum_{j=1}^{n} c_j \psi_j.$$

The matrix

$$G_{\mathbb{X}} = T_{\mathbb{X}}^* T_{\mathbb{X}}$$

is known as the gram matrix.
Let tr denote the trace of a matrix. For a frame with equal norm $a = ||\psi_i||$, we have

$$tr(S_{\mathbb{X}}) = tr(T_{\mathbb{X}} T_{\mathbb{X}}^*) = tr(T_{\mathbb{X}}^* T_{\mathbb{X}})$$

$$= \sum_{i=1}^{n} ||\psi_i||^2 = na^2.$$

Also from (2) $tr(S_{\mathbb{X}}) = Am$. Combining the above two, we have $A = \frac{n}{m}$ if the frame is unit norm. So the unit norm tight frames exist only for $A = \frac{n}{m}$.

B. Frame Potential

The frame potential is the metric used throughout this paper to verify the results. The frame potential of a frame $\mathbb{X}$ is given by [see 6, p. 114-117],

$$FP(\mathbb{X}) = \sum_{i=1}^{n} \sum_{j=1}^{n} |\langle \psi_i, \psi_j \rangle|^2. \quad (3)$$

Let $L = \sum_{i=1}^{n} a_i^2$. It is known that the frame potential satisfies

$$FP(\mathbb{X}) \geq L^2/m,$$

where the equality holds if and only if the frame is a tight frame with bound $\frac{L}{m}$ [6]. Thus $\mathbb{X}$ is a tight frame for the space if and only if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |\langle \psi_i, \psi_j \rangle|^2 = \frac{1}{m} \left(\sum_{i=1}^{n} ||\psi_i||^2 \right)^2 = \frac{L^2}{m}, \quad (4)$$

i.e. the matrix is nonzero and satisfies

$$||G_{\mathbb{X}}||_F = \frac{1}{\sqrt{m}} tr(G_{\mathbb{X}}). \quad (5)$$

In particular, if $a_i = 1$ for all i, then the equality holds if and only if $\mathbb{X}$ is a unit norm (UNTF) tight frame with bound $\frac{n}{m}$. Together with frame potential as a metric used for UNTF or the TF, the following is used to verify the results [8].

Proposition I.2. *A $m \times n$ frame is a $A-$ tight if and only if it satisfies one (hense all) of the following conditions.*
1. The m eigenvalues of $S_{\mathbb{X}}$ are equal to A.
2. The m nonzero eigenvalues of the gram matrix $T_{\mathbb{X}}^* T_{\mathbb{X}}$ are equal to A.
3. The m singular values of $T_{\mathbb{X}}$ are equal to $\sqrt{A}$.
4. The rows of $A^{-1/2} T_{\mathbb{X}}$ is an orthonormal set.

A pair of frames $\mathbb{X}$ and $\mathbb{Y}$ are said to be orthogonal if $T_{\mathbb{X}} T_{\mathbb{Y}}^* = 0$ [2, 3, 4]. This defination requires the number of vectors in both frames to be the same, they counld be the frames of vector sapce of different dimensions. Also the orthogonality of frames can be characterized in terms of their Gram matrix. Two frame sare orthogonal if $G_{\mathbb{X}} G_{\mathbb{Y}}^* = 0$ [2, 3].

Frames are therefore generalization of orthonormal bases. In this paper a frame refers as the analysis matrix. In what follows, we denote $T_{\mathbb{X}}$ simply by $\mathbb{X}$. So $\mathbb{X} = [\psi_1, \psi_2, \cdots \psi_n]$. Frames having orthogonal rows are known as tight frames as in (2). The unit norm tight frames (UNTF), i.e. the tight frames with unit column vectors, are robust against additive noise and are used in communications [9].

Definition I.3. *Let $\mathbb{X}$ be a $m \times n$ frame. The maximal correlation between the columns of $\mathbb{X}$ is given by the mutual coherence, i.e,*

$$\mu(\mathbb{X}) = \max_{\substack{1 \leq i, j \leq n \\ i \neq j}} \frac{|\langle \psi_i, \psi_j \rangle|}{||\psi_i||_2 ||\psi_j||_2}. \quad (6)$$

Frames whose $\mu = \mu(\mathbb{X})$ is lowest are called the incoherent frames. It is known that the lower bound for the mutual coherence is given by the Welch bound [5, 9],

$$\sqrt{\frac{n-m}{m(n-1)}} \leq \mu(\mathbb{X}) \leq 1. \tag{7}$$

The lower bound is achieved for the equiangular unit norm frames. Note that $\mu = \mu(\mathbb{X}) \geq \frac{1}{\sqrt{m}}$ for $n > m$.

Definition I.4. *A sequence of vectors* $\Psi = \{\psi_1, \cdots \psi_n\}$ *in a* $\mathbb{R}^m$ *is said to be Grassmannian frame if it is a frame and if*

$$\mu(\Psi) = inf\{\mu(\Phi)\},$$

where the infimum is taken over all n element unit norm tight frames Φ *of* $\mathbb{R}^m$.

Definition I.5. *An equiangular tight frame (ETF) is a family of vectors* $\psi_1, \psi_2, \cdots \psi_n \in \mathbb{H}^m$*, satisfying the conditions* $||\psi_i|| = 1$ *for* $i = 1, ...n,$ *and* $|\langle\psi_i, \psi_j\rangle| = c,$ *for all* $i \neq j$*, and some constant c, and the condition (2) with* $A = \frac{n}{m}$.

The equiangular tight frames are examples of Grassmannian frames. Such frames are used in sparse signal processing. It is known that the Grassmannian frames always exist although they are difficult to find. If the infimum is the lowest bound given by the Welch bound, the frames obtained are optimal Grassmannian. These are precisely the ETFs. Even though they are known to exist, both Grassmannian and ETF are very difficult to find in general. It is known that the ETF of n vectors in dimension m with coherent parameter μ do not always exist [5]. Let the triplet (m, n, μ) represent frames of n vectors in m dimensional space with coherence parameter μ. The ETFs are known to exist for the triplets $(m, m+1, \frac{1}{\sqrt{m}})$ [5]. The frames for $(3, 6, \frac{1}{\sqrt{5}})$, $(5, 10, \frac{1}{3}) \cdots (7, 28, \frac{1}{3})$ etc are known to exist [5] as provided in the table **??** below. For the equiangular tight frames in $\mathbb{R}^m$, $m \leq n(n+1)/2$ must hold and in $\mathbb{C}^m$, this requirement is $m \leq n^2$ [9]. This paper proposes an algorithm that produces ETFs for the triplets for which they are known to exist. For other triplets the the algorithm produces frames that tight frames close to the UNTFs and the mutual coherence is still low.

C. Reducing the mutual coherence of a frame

Let Ψ be a unit norm tight frame. The gram matrix for this frame is given by

$$G = \begin{pmatrix} \langle\psi_1, \psi_1\rangle & \langle\psi_1, \psi_2\rangle & \cdots & \langle\psi_1, \psi_n\rangle \\ \langle\psi_2, \psi_1\rangle & \langle\psi_2, \psi_2\rangle & \cdots & \langle\psi_2, \psi_n\rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle\psi_n, \psi_1\rangle & \langle\psi_n, \psi_2\rangle & \cdots & \langle\psi_n, \psi_n\rangle \end{pmatrix} \tag{8}$$

The diagonal entries of UNTF are equal to one. The absolute values of the other entries provide the coherence. The mutual coherence is smallest if $\langle\psi_i, \psi_j\rangle = 0$, for $i \neq j$. This is the case of an orthonormal basis. The lower bound for μ is given by the Welch bound. The off diagonal distinct entries provide upper bound. So the coherence can be reduced in two different ways: (i) by having optimally small values of off diagonal entries, and (ii) by reducing the number of distinct off diagonal entries as the following lemma suggests.

Lemma I.6. *Let k be the number of distinct absolute values of the off diagonal entries of the gram matrix. Then,*

$$\mu \leq \sqrt{k}\sqrt{\frac{n-m}{m(n-1)}}.$$

Proof. Similar to [7], let $\alpha_i, \ i = 1, \cdots k$ be the k distinct squared absolute values of the off diagonal entries of the gram matrix. Assume that each value repeated k_i times. The average of these values is given by

$$\frac{1}{k}\sum_{i=1}^{k} k_i\alpha_i = \frac{1}{n(n-1)}\sum_{i\neq j} |\langle\psi_i, \psi_j\rangle|^2 .$$

Since

$$\sum_{i,j} |\langle\psi_i, \psi_j\rangle|^2 = \left(\sum_{i\neq j} |\langle\psi_i, \psi_j\rangle|^2 + n\right).$$

and the left side is the frame potential of the unit norm tight frame, it is given by

$$\sum_{i,j} |\langle \psi_i, \psi_j \rangle|^2 = \frac{n^2}{m}$$

this implies

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\langle \psi_i, \psi_j \rangle|^2 = \frac{n-m}{m(n-1)}.$$

So

$$\mu^2 = \max_i \alpha_i \leq \sum_{i=1}^{k} k_i \alpha_i = k \frac{n-m}{m(n-1)}$$

it follows that

$$\mu \leq \sqrt{k} \sqrt{\frac{n-m}{m(n-1)}}.$$

□

The proposed algorithm reduces the number of distinct entries of the gram matrix. The authors in [9, 8] apply the shrinkage on the off diagonal entries of the gram matrix with a variable shrinkage parameter and using a random sampling matrix F as an initial matrix. Here the Householder matrices are used an initial matrices and a constant shrinkage parameter is used. A few iterations are performed in Matlab. The time required is only a few seconds. The resulting matrices as they satisfy the proposition I.2, can be split on rows to obtain a pair of orthogonal frames. The iterations provided in [9, 8], known as the alternating projections, can be explained as shown in the figure 1 given below. Take a Householder matrix P and apply shrinkage on the off diagonal entries of it, approximate the matrix by a matrix with unit norm columns P_1. Then approximate this matrix by the matrix of low mutual coherence P_2. Again approximate this my the matrix with unit norm columns P_3. Perform few more iterations to obtain the unit norm frames of low coherence.

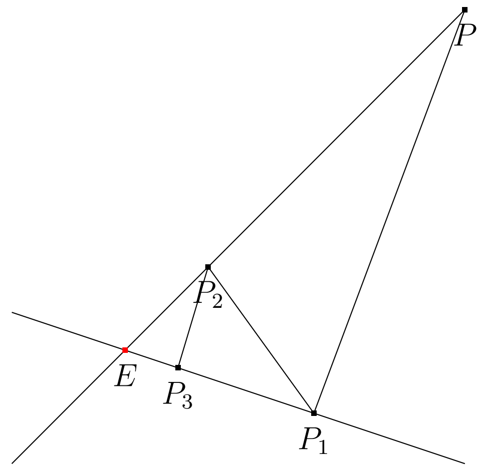

Figure 1: The alternating projection explained

II MAIN RESULT

A simple constructive method for the frames is proposed based on [8, 9]. The initial matrix is a Householder matrix. The running time for the Matlab code is less than a minute.
1. Construct a Householder unitary matrix of size $n \times n$, call it U.
The matlab code used:
x=ones(n,1);
x=x/norm(x);
V=(id-2*x*x'), a Householder matrix;
2. Take only m rows of it, let it be denoted by W.
3. Column normalize W.
4. Let $G = W^*W$. The off diagonal entries of G provide coherence of the columns of W.
5. Reduce the coherence by applying shrinkage method on the entries of $G - I_n$, as in [9], where I_n is an identity matrix of size n. Denote the resulting matrix by $\tilde{G}$. The off diagonal entries are reduced as follows. The tolerance parameter of $\gamma = 1 - \frac{m}{1.5m+n}$ works for small values of m and n used in this paper. A variable γ has been in [9, 8].

$$\tilde{G}(i,j) = \begin{cases} g_{i,j} & \text{if } |g_{i,j}| \leq \frac{\gamma}{\sqrt{m}} \\ \frac{\gamma sign(g_{i,j})}{\sqrt{m}} & \text{if } \frac{\gamma}{\sqrt{m}} < |g_{i,j}| \leq \frac{1}{\sqrt{m}} \\ \frac{g_{i,j}}{\sqrt{m}} & \text{if } \frac{1}{\sqrt{m}} < |g_{i,j}|. \end{cases}$$

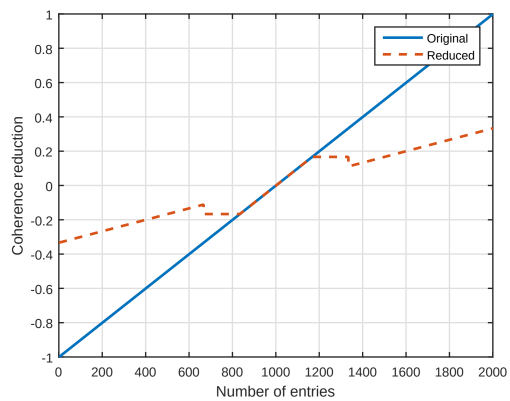

Figure 2: The scheme of coherence reduction

6. The procedure may increases the rank of the Gram matrix. Obtain the Singular Value Decomposition on $\tilde{G} = UDV^t$ and update D by keeping only m nonzero singular values. This makes sure that the rank of D equals m.
7. Construct S_2 as $S_2 = UD^{1/2}$.
8. Let $S_1 = S_2^T$
9. The nearest $\frac{n}{m}$ tight frame S_1' to S_1 in Frobenios norm is obtained as $S_1' = \sqrt{\frac{n}{m}}(S_1S_1^*)^{-1/2}S_1$ [8].
10. Update $W = S_1'$ and perform few iterations. The convergence of the procedure is not proven but it produces the tight frames or ETFs in few iterations.

In case if the Grassmanian frames exist [5], the proposed iterations converge to the equialgular tight frames. In other cases the iterations provide TFs with low mutual convergence. Although the convergence is an issue as in but with Householder matrices, a few iterations provide the ETS or TFs. A partial list of such triplets (m, n, μ) for which the ETFs exists is provided in table **??** below. The algorithm works for other triplets for which the ETFs do not exist as well. For those triplets the convergence of the algorithm is still unclear but it provides frames that are close to UNTFs as the frame potential (4) suggests.

III EXAMPLES

Some of the triplets (m, n, μ) from [5] are shown in the table **??**, for which the algorithm provides ETF. For other triplets (m, n, μ) the frames obtained are tight and close to UNTF as the frame potential (4) suggests. For $m = n$, the algorithm produces orthogonal basis. For the triplet $(m, m+1, \frac{1}{\sqrt{m}})$, the ETSs are known to exist and the algorithm always produces ETFs.
Example 1: If $m = 2$ and $n = 3$, one gets the well known Mercedes Benz frame. $T_{\mathbb{X}}$, and $S_{\mathbb{X}}$ are respectively given by,

$$T_{\mathbb{X}} \begin{pmatrix} 0 & 0.866 & 0.866 \\ -1 & 0.500 & -0.500 \end{pmatrix},$$

$$S_{\mathbb{X}} = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}.$$

and the gram matrix is given by

$$G_{\mathbb{X}} = \begin{pmatrix} 1.0 & -0.5 & 0.5 \\ -0.5 & 1.0 & 0.5 \\ 0.5 & 0.5 & 1.0 \end{pmatrix}.$$

Example 2: If $m = 3$, $n = 4$, the following ETF is obtained.

$$S_{\mathbb{X}} = \begin{pmatrix} 1.333 & 0 & 0 \\ 0 & 1.333 & 0 \\ 0 & 0 & 1.333 \end{pmatrix},$$

where, $T_{\mathbb{X}}$ and $G_{\mathbb{X}}$ are given by

$$\begin{pmatrix} -.7353 & .7648 & -.3367 & -.3073 \\ 0.0000 & .5486 & .3898 & .9383 \\ 0.6777 & .3379 & -.8572 & .1585 \end{pmatrix},$$

and

$$\begin{pmatrix} 1.0 & -0.3333 & -0.3333 & 0.3333 \\ -0.3333 & 1.0 & -0.3333 & 0.3333 \\ -0.3333 & -0.3333 & 1.0 & 0.3333 \\ 0.3333 & 0.3333 & 0.3333 & 1.0 \end{pmatrix}$$

respectively.
Example 3: For $m = 3$, $n = 6$, the following

ETF is obtained. The $T_{\mathbb{X}}$, and $S_{\mathbb{X}} =$ are given by

$$\begin{pmatrix} 0 & 0.778 & -0.889 & 0.179 & 0.66 & -0.37 \\ 0 & 0.441 & 0.1 & -0.876 & -0.604 & -0.814 \\ -1.0 & 0.447 & 0.447 & -0.447 & 0.447 & 0.447 \end{pmatrix}$$

and

$$S_{\mathbb{X}} = \begin{pmatrix} 2.0 & 0 & 0 \\ 0 & 2.0 & 0 \\ 0 & 0 & 2.0 \end{pmatrix}$$

and the gram matrix $G_{\mathbb{X}}$ is given by

$$\begin{pmatrix} 1.0 & -0.4472 & -0.4472 & -0.4472 & -0.4472 & 0.4472 \\ -0.4472 & 1.0 & -0.4472 & 0.4472 & -0.4472 & -0.4472 \\ -0.4472 & -0.4472 & 1.0 & -0.4472 & 0.4472 & -0.4472 \\ -0.4472 & 0.4472 & -0.4472 & 1.0 & 0.4472 & 0.4472 \\ -0.4472 & -0.4472 & 0.4472 & 0.4472 & 1.0 & 0.4472 \\ 0.4472 & -0.4472 & -0.4472 & 0.4472 & 0.4472 & 1.0 \end{pmatrix}$$

Example 4: For $m = 4$, and $n = 6$, the folowing TF is frame obtained. The equiangular frame doesnot exist for this pair [5]. The $T_{\mathbb{X}}$ is given by

$$\begin{pmatrix} -0.5330 & 0.6826 & -0.5330 & 0.6826 & -0.0000 & 0.0000 \\ -0.0000 & -0.6826 & 0.0000 & 0.6826 & -0.5330 & 0.5330 \\ 0.8660 & -0.0000 & -0.8660 & 0.0000 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.8660 & 0.8660 \end{pmatrix},$$

$$S_{\mathbb{X}} = \begin{pmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 1.5 & 0 \\ 0 & 0 & 01.5 & \end{pmatrix},$$

and the gram matrix $G_{\mathbb{X}}$ is given by

$$\begin{pmatrix} 1.0340 & -0.3638 & -0.4660 & -0.3638 & 0.0000 & -0.0000 \\ -0.3638 & 0.9319 & -0.3638 & -0.0000 & 0.3638 & -0.3638 \\ -0.4660 & -0.3638 & 1.0340 & -0.3638 & -0.0000 & 0.0000 \\ -0.3638 & -0.0000 & -0.3638 & 0.9319 & -0.3638 & 0.3638 \\ 0.0000 & 0.3638 & -0.0000 & -0.3638 & 1.0340 & 0.4660 \\ -0.0000 & -0.3638 & 0.0000 & 0.3638 & 0.4660 & 1.0340 \end{pmatrix}$$

Example 5: For $m = 5$, and $n = 10$, an ETF is obtained.

A pair of orthogonal frames: Let $\mathbb{Z}$ and $\mathbb{W}$ be the matrices obtained by taking the first three and the last two rows of $T_{\mathbb{X}}$ respectively. Then the frames $\mathbb{Z}$ and $\mathbb{W}$ satisfy $T_{\mathbb{Z}}T_{\mathbb{Z}}^* = \frac{3}{2}I$, $T_{\mathbb{W}}T_{\mathbb{W}}^* = \frac{3}{2}I$ and $T_{\mathbb{Z}}^*T_{\mathbb{W}} = 0$. So they form a pair of orthogonal frames. In example 5, the frame matrix $T_{\mathbb{X}}$ is a $5 \times 10 matrix$. Taking $\mathbb{Z}$ as the first two rows and $\mathbb{W}$ as the last three rows of $T_{\mathbb{X}}$ produces another pair of orthogonal frames. One can also take $\mathbb{Z}$ as the first four rows and $\mathbb{W}$ as the last row and vice versa to get a pair of orthogonal frame.

The parameter γ used above works for the triplets $(6, 16, \frac{1}{3}), (7, 14, \frac{1}{\sqrt{13}}), (7, 28, \frac{1}{3})$ as well. It provides ETSs.

Example 6: For the triplets for which the equangular frames do not exist [5], viz. $(5, 7, \frac{1}{\sqrt{15}}), (5, 8, \frac{1}{\sqrt{7}}), (5, 9, \frac{1}{\sqrt{10}}), (6, 8, \frac{1}{\sqrt{21}})$ the algorithm provides the TFs as the Proposition I.2 or the frame potential (4) suggests. A pair of orthogonal frame can be obtained.

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