On the implementation of exact solution of a beam on elastic foundation in numerical calculations

Jiří Vala

Abstract—Advanced numerical analysis of structures forces the massive implementation of numerical algorithms, whose convergence can be verified just for the norms of decompositions to finite elements, etc., tending (in some reasonable sense) to zero, not for real discrete calculations. This brings non-physical discontinuity and imperfectness to all numerical results. This paper shows a possible remedy utilizing, at least for sufficiently smooth loads and material characteristics, some knowledge of analytical solutions in standard variational formulations.

As a model example, a beam on elastic medium is considered. The variational formulation of its response to statical loads is adopted to involve the knowledge of classical solution, in the simplest case for the whole beam, in the more realistic one for its particular elements. Finally, we obtain only the sparse system of linear algebraic equations, without any duty of numerical differentiation or integration. Some natural generalization to response to dynamical loads, to plates replacing beams, etc. are sketched, together with the brief references to practical applications.

Keywords—Beam on elastic foundation, weak and strong solutions, numerical analysis.

I. Introduction

PROGRESS in both hardware and software development, together with new achievements in mechanics of structures, building physics, computational and numerical mathematics and related research areas leads to extensive, frequently parallel numerical computations, handling discretized versions of variational principles of continuum mechanics, typically using some finite element techniques. However, their convergence (usually in a weak sense) relies on the infinitesimally small norms of families of decompositions of such elements, whereas particular steps with such finite norms suffer from non-physical discontinuity and imperfectness of solution, even if high smoothness of solution can be verified theoretically – cf. [3], Chap. 14. This can be suppressed by local mesh refinement techniques, together with domain decomposition tricks, in some degree – cf. [23], Appendix B. However, in this paper we shall introduce another approach, making use of some (at least partial) knowledge of exact solution. As a model problem we shall take a statical analysis of a beam of elastic foundation, generalizable in several directions in the natural way.

The frequently applied model (cf. [2], [12] and many other rather new papers) relies on the historical one-parameter formulation by [29] and [22]; the overview of later achievements in the development of more-parameter models by [6], [9], [18], [19], [10], [14], [27], etc., is contained in [15], more recent results like [28] and [8] are discussed in [4] in details. Most these approaches implement the Green functions, with expectable numerical difficulties. An alternative approach is presented in [16], referring to certain analogy to the nonlocal beam theory by [5].

In this paper we shall work namely with a slight modification of the model by [16]. Our aim is to prove that, under some additional smoothness assumptions on applied loads (and, clearly, also on material characteristics), the evaluation of general solution of a strong problem can be helpful to force some better continuity during the numerical analysis of a weak (variational) problem.

II. Formulation of a model interaction problem

Following [16] (including illustrative schemes), let us consider a beam lying on an elastic foundation. In the standard 3-dimensional real Euclidean space R^3, supplied with the Cartesian coordinate system (x, y, z), we are able to localize such beam using $x \in [0, l]$ where l denotes its positive length. The deformation of a beam has 3 possible causes: i) the vertical load $q_z(x)$ on $[0, l]$, ii) some single vertical forces $\bar{T}_z(x)$, iii) some single (bending) moments $\bar{M}_y(x)$ (in the plane perpendicular to y), both in ii) and iii) for a finite number of discrete points x. To simplify the notation, without any loss of generality, we are allowed to consider non-zero $\bar{T}_z(0)$, $\bar{T}_z(l)$, $\bar{M}_y(0)$ and $\bar{M}_y(l)$ only.

As the geometrical and material characteristics we shall introduce the sectional area A, the bending inertia J_y and the Young moduli of elasticity for bending E and for torsion G; for simplicity, we shall use the composite characteristics GA and EJ_y everywhere. Let the vector (u, v, w) characterize the change of position of particular points of a beam, respecting the small strain simplification and the Kirchhoff bending theory, i. e. $u = z\varphi_y$, $v = 0$ (due to the symmetry), $\varepsilon_{xx} = z\,\mathrm{d}\varphi_y/\mathrm{d}x$ for the normal strain and $\gamma_{xz} = \mathrm{d}u/\mathrm{d}z + \mathrm{d}w/\mathrm{d}x = \varphi_y + \mathrm{d}w/\mathrm{d}x$ for the tangential strain, consequently

$$\sigma_{xx} = E\varepsilon_{xx} = Ez\,\mathrm{d}\varphi_y/\mathrm{d}x \quad (1)$$

for the normal stress and

$$\tau_{xz} = G\gamma_{xz} = G(\varphi_y + \mathrm{d}w/\mathrm{d}x) \quad (2)$$

for the tangential stress other strain and stress components in R^3 are neglected. Using the usual notation for the bending moments M_y, i. e. the 2-dimensional integrals of $z\sigma_{xx}$ over any vertical beam section, and for the shear forces T_z, i. e. the integrals of τ_{xz} over the same section, we have $M_y = EJ_y\mathrm{d}\varphi/\mathrm{d}x$ and $T_z = GA(\varphi_y + \mathrm{d}w/\mathrm{d}x)$. In the following

J. Vala is with the Institute of Mathematics and Descritpive Geometry, Faculty of Civil Engineering, Brno University of Technology, 602 00 Brno, Veveří 331/95, Czech Republic, e-mail vala.j@fce.vutbr.cz.

Manuscript received June 14, 2018; revised July 11, 2018.

text we shall omit all above mentioned indices and apply the simple prime symbols instead of d/dx: so we can write briefly

$$M = EJ\varphi' \,, \qquad T = GA(\varphi + w')$$

instead of (1) and (2).

Let (ϖ, ω) denote the integrals of $\varpi(x)\omega(x)$ over $[0, l]$, i. e. the scalar products of $\varpi, \omega \in L^2[0, l]$ in the standard notation of Lebesgue spaces (of square integrable functions) by [21], or certain dualities in the corresponding function spaces in more general cases. Similarly we can take $[\varpi, \omega] = \varpi(l)\omega(l) - \varpi(0)\omega(0)$. Moreover, let $(\!(\varpi, \omega)\!)$ refer to the 3-dimensional integrals introduced as integrals of $(\!(\varpi, \omega)\!)$ over all vertical beam sections. Finally, let us introduce the elastic foundation characteristic β. The real deformation of a beam then corresponds to the minimum of energy

$$\Phi(w, \varphi) = \frac{1}{2}(\!(\sigma, \varepsilon)\!) + \frac{1}{2}(\!(\tau, \gamma)\!) + \frac{1}{2}(w, \beta w) - (w, q) - [\varphi, \bar{M}] - [w, \bar{T}] \qquad (3)$$

where σ must be taken from (1) and τ from (2). Thus (3) can be converted to the form

$$\Phi(w, \varphi) = \frac{1}{2}(\varphi', EJ\varphi') + \frac{1}{2}(\varphi + w', GA(\varphi + w')) + \frac{1}{2}(w, \beta w) - (w, q) - [\varphi, \bar{M}] - [w, \bar{T}] \qquad (4)$$

III. Theoretical and numerical analysis

Our aim is to minimize (4) effectively. From practical reasons, let us suppose $EJ > 0$, $GA > 0$ and $\beta \geq 0$ everywhere. Using the first and Gâteaux differentials of $\Phi(w, \varphi)$ in any admissible direction $(\tilde{w}, \tilde{\varphi})$, assuming that EJ, GA and β are independent of (w, φ) (the dependence on x is still allowed), we receive

$$D\Phi(w, \varphi; \tilde{w}, \tilde{\varphi}) = (\tilde{\varphi}', EJ\varphi) + (\tilde{\varphi} + \tilde{w}', GA(\varphi + w')) + (\tilde{w}, \beta w) - (\tilde{w}, q) - [\tilde{\varphi}, \bar{M}] - [\tilde{w}, \bar{T}] \,, \qquad (5)$$

$$D^2\Phi(w, \varphi; \tilde{w}, \tilde{\varphi}; \tilde{w}, \tilde{\varphi}) = (\tilde{\varphi}', EJ\tilde{\varphi}) + (\tilde{\varphi} + \tilde{w}', GA(\tilde{\varphi} + \tilde{w}')) + (\tilde{w}, \beta\tilde{w}) \,. \qquad (6)$$

The Taylor expansion for a real parameter t

$$\Phi(w + t\tilde{w}, \varphi + t\tilde{\varphi}) = \Phi(w, \varphi) + tD\Phi(w, \varphi; \tilde{w}, \tilde{\varphi}) + \frac{t^2}{2} D^2\Phi(w, \varphi; \tilde{w}, \tilde{\varphi}; \tilde{w}, \tilde{\varphi})$$

demonstrates that a minimum of (4) corresponds to

$$D\Phi(w, \varphi; \tilde{w}, \tilde{\varphi}) = 0 \,, \qquad (7)$$

inserting (5), whereas the uniqueness of such minimum needs positive $D^2\Phi(w, \varphi; \tilde{w}, \tilde{\varphi}; \tilde{w}, \tilde{\varphi})$ by (6). However, forcing $D^2\Phi(w, \varphi; \tilde{w}, \tilde{\varphi}; \tilde{w}, \tilde{\varphi}) = 0$ (independent of (w, φ) here), we have $\tilde{\varphi} = 0$ and $\tilde{w}' = -\tilde{\varphi} = 0$ everywhere and also $\tilde{w} = 0$ in the case of positive β, which leads to the contrary. Otherwise, admitting $\beta = 0$ identically, i. e. no elastic foundation is present, a linear function $w(x)$ is allowed, which corresponds to the classical case of an insufficiently supported beam (not solvable in the statical context properly).

Let us remind that the appropriate choice for (w, φ) and $(\tilde{w}, \tilde{\varphi})$ is (due to the boundary conditions) a subspace V of the Sobolev space $W^{1,2}[0, l]$ (cf. [21] again); for more general cases see [25]. The direct numerical computations rely on some reasonable finite-dimensional approximation of V, e. g. using the finite element approach with Hermitean cubic splines and derivation of such approximations from the sparse systems of linear algebraic equations in the case of constant EJ, GA and β, supplied by some iteration procedure in the more general cases. However, in addition to such variational formulation, in the following considerations we shall apply the integration by parts by the Green - Ostrogradskii theorem (performable at least in the sense of distributions).

IV. Special classes of analytical solutions

Using the above sketched approach, from (5) and (7) we obtain

$$-(\tilde{\varphi}, EJ\varphi'') + (\tilde{\varphi} + \tilde{w}') - (\tilde{w}, GA(\tilde{\varphi}' + \tilde{w}'')) + (\tilde{w}, \beta w) = (\tilde{w}, q) + [\tilde{\varphi}, \bar{M} - EJ\varphi''] + [\tilde{w}, \bar{T} - GA(\varphi + w')] \,. \qquad (8)$$

Utilizing the notation from the previous section, (8) yields

$$\beta w - T' = q \,, \; M' = T \,, \; M = EJ\varphi' \,, \; T = GA(\varphi + w') \qquad (9)$$

on $[0, l]$, as well as $T = \bar{T}$ and $M = \bar{M}$ in its boundary points. In particular, the limit case $GA \to \infty$ forces $\varphi = -w'$ and (9) degenerates substantially: this is the case of the classical Winkler support. More generally, introducing the new notation, compatible with [16], $\alpha = \sqrt{\beta/(EJ)}$ and $\mu = EJ/(GA)$, and assuming constant values of EJ, α and μ (or EJ, GA and β, alternatively) on $[0, l]$, from (9) we obtain, step by step,

$$T = EJ\varphi'' = EJ(\gamma'' - w''') = \mu T'' - EJw''' \,,$$

$$-EJw'''' = T' - \mu T''' = \beta w - q - \mu\beta w'' + \mu q'' \,,$$

thus finally

$$w'''' - \mu\alpha^2 w'' + \alpha^2 w = \bar{q} - \mu\bar{q}'' \qquad (10)$$

where, for simplicity, we take $\bar{q} = q/(EJ)$. Moreover, for the evaluation of M, T and φ from w we can derive the formulae

$$M = EJ\varphi' = EJ(\gamma' - w'') = \mu\beta w - \mu q - EJw'' = \mu\alpha^2 EJw - EJ\mu w'' - EJ\bar{q} \,, \qquad (11)$$

$$T = M' = \mu\alpha^2 EJw' - EJ\mu w''' - EJ\bar{q}' \,, \qquad (12)$$

$$\varphi = \gamma - w' = (\mu^2\alpha^2 - 1)w' - \mu w'''' - \mu^2\bar{q}' \,. \qquad (13)$$

The crucial differential equation (10) is linear, moreover with constant coefficients, thus its fundamental solution (at least theoretically, for sufficiently and smooth $\bar{q}$, appropriate for formal integration steps) could be available. At first, let us study the homogeneous case with $\bar{q} = 0$ identically. The characteristic equation corresponding to (10), containing the (in general complex) eigenvalues λ, is

$$\lambda^4 - \mu\alpha^2\lambda^2 - \alpha^2 = 0 \,. \qquad (14)$$

Let us consider 4 different (linearly independent) fundamental solutions h_j on $[0, l]$ with $j \in \{1, 2, 3, 4\}$. Let us distinguish between the following qualitative cases:

a) If $\alpha = 0$ then all 4 roots of (14) degenerate to zero. Therefore the required system of fundamental solutions is $\phi_1(x) = 1$, $\phi_2(x) = x$, $\phi_3(x) = x^2$, $\phi_4(x) = x^3$. In all remaining cases we shall suppose $\alpha > 0$ and work with the discriminant of (14) $\delta = \alpha\sqrt{\mu^2\alpha^2 - 4}$.
b) If $\mu\alpha \neq 2$ (thus $\delta \neq 0$) then 4 roots of (14) are $\lambda_1 = \sqrt{(\mu\alpha^2 + \delta)/2}$, $\lambda_2 = \sqrt{(\mu\alpha^2 - \delta)/2}$, as well as $-\lambda_1$, $-\lambda_2$. Therefore the required system of fundamental solutions is $\phi_1(x) = \exp(\lambda_1 x)$, $\phi_2(x) = \exp(-\lambda_1 x)$, $\phi_3(x) = \exp(\lambda_2 x)$, $\phi_4(x) = \exp(-\lambda_2 x)$. However, for $\mu\alpha < 2$ (thus $\delta < 0$) such approach is not optimal for practical calculations because of the presence of complex-valued functions; thus this case will be handled by d) separately.
c) If $\mu\alpha = 2$ (thus $\delta = 0$) then $\lambda = \lambda_1 = \lambda_2$ by b) coincide. Therefore the required system of fundamental solutions is $\phi_1(x) = \exp(\lambda x)$, $\phi_2(x) = \exp(-\lambda x)$, $\phi_3(x) = x\exp(\lambda x)$, $\phi_4(x) = x\exp(-\lambda x)$.
d) If $\mu\alpha < 2$ (thus $\delta < 0$) the much better approach than b) is the following: taking $\psi_1 = \sqrt{2\alpha + \mu\alpha^2}/2$ and $\psi_2 = \sqrt{2\alpha - \mu\alpha^2}/2$, we can rewrite all roots of (14) by a) also in the form $\lambda_1 = \psi_1 + i\psi_2$, $\lambda_2 = \psi_1 - i\psi_2$, etc. Therefore the required system of fundamental solutions (real-valued here fortunately) is $\phi_1(x) = \exp(\psi_1 x)\cos(\psi_2 x)$, $\phi_2(x) = \exp(\psi_1 x)\cos(\psi_2 x)$, $\phi_3(x) = \exp(-\psi_1 x)\cos(\psi_2 x)$, $\phi_4(x) = \exp(-\psi_1 x)\cos(\psi_2 x)$.

Let us now consider the vector $h_0 = (\phi_1, \phi_2, \phi_3, \phi_4)$ and the corresponding vectors of the 1st, 2nd and 3rd derivatives (with respect to x) h_1, h_2 and h_3. Then W containing 4 lines h^0, h^1, h^2 and h^3 is the well-known Wronski matrix. Moreover, being motivated by (11), (12) and (12), we can introduce $h_M = \mu\alpha^2\phi_0 - \phi_2$, $h_T = h'_M$ and $h_\varphi = (\mu\alpha^2 - 1)\phi_1 - \mu\phi_3$, too. To handle the nonhomogeneous case (with $\bar{q}$ other than zero), we can use the additional decomposition $w = w_h + w_q$ with $w_h = h_0 C$ with some column vector C of real constants, which represents the general solution in the homogeneous case, and any particular solution w_q derived for the nonhomogeneous case. However, for rather complicated q an easy derivation of w_q is not available: [16] pays attention namely to linear functions q (which removes q'' at all), the method of undetermined coefficients (without additional tricks using infinite series) needs to have q just as a solution of another ordinary differential equation with constant coefficients, the method of variation of parameters (our constants C) requires the general inversion of W and leads to complicated integrals typically; for more considerations of this type cf. [24]. The best choice here seems to be the exploitation of the Cauchy method (less frequent in the literature): using the basic facts from theory of parametric integrals, one can verify easily that

$$w_q(x) = \int_0^x v(x-t)(\bar{q}(t) - \mu\bar{q}''(t))\,\mathrm{d}t\,, \qquad (15)$$

taking $v(x) = h_0(x)W^{-1}(0)(0,0,0,1)^T$, satisfies (10) together with the Cauchy initial conditions $w_q(0) = 0$, $w'_q(0) = 0$, $w''_q(0) = 0$ and $w'''_q(0) = 1$. Clearly, to obtain w_q, we need only the integrability of q and q'' in (15) (thanks to all remaining bounded multiplicative terms), in the best case in the simple analytical way. However, even the numerical quadrature in (15) is possible because we shall work namely with $w(0)$, $w(l)$, $\varphi(0)$ and $\varphi(l)$.

Consequently we have $w = w_q + h_0 C$ where C should be determined from some boundary conditions. Since such C have (in general) no reasonable physical interpretation, it is useful to take $U = (w(0), \varphi(0), w(l), \varphi(l))^T$ instead of C. The construction of the transformation matrix $\mathcal{A}$, using the additive decomposition $U = U_h + U_q$, analogous to $w = w_h + w_q$,

$$C = \mathcal{A}(U - U_q)\,, \quad U - U_q = \mathcal{A}^{-1}C \qquad (16)$$

is easy and not expensive: its 4 lines are just $h_0(0)$, $h_\varphi(0)$, $h_0(l)$ and $h_\varphi(l)$; moreover, the practical construction of $\mathcal{A}^{-1}$ can be avoided using any method of solution of 4 linear algebraic equations with 4 unknowns. Similarly, introducing the notation $F = (-T(0), -M(0), T(l), M(l))^T$ and $\bar{F} = (-\bar{T}(0), -\bar{M}(0), \bar{T}(l), \bar{M}(l))^T$ (preserving the usual convention for the orientation of shear forces and bending moments), searching for the transformation matrix $\mathcal{B}$ satisfying

$$F - \bar{F} = \mathcal{B}C = \mathcal{K}(U - U_q) \qquad (17)$$

with $\mathcal{K} = \mathcal{B}\mathcal{A}^{-1}$ ($\mathcal{B}^{-1}$ is not needed anywhere), in accordance with (11), (12) and (13), we come to the matrix $\mathcal{B}$, whose 4 lines are just $-h_T(0)$, $-h_M(0)$, $h_T(l)$ and $h_M(l)$, multiplied by EJ.

For an illustration, in the simplest case a) we receive

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -6\mu & 0 & -1 & 0 \\ l^3 & l^2 & l & 1 \\ -3l^2 - 6\mu & -2l & -1 & 0 \end{bmatrix},$$

$$\mathcal{B} = EJ \begin{bmatrix} 6 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ -6 & l^2 & l & 1 \\ -6l & -2 & 0 & 0 \end{bmatrix},$$

We can also notice that the approximation of w by cubic polynomials using the Hermitean approach (and splines if more intervals are considered, because of the presence of some loads like $\bar{T}$ and $\bar{M}$ on their interfaces) is quite exact in this case, thus no approximation error in (7) can be expected. Nevertheless, the direct integration will be then the most effective. In all other cases b), c) and d) such conclusions are not true: W contains other functions than polynomials.

V. Analytical solutions in numerical calculations

The good idea, how to implement the results from the preceding section in the numerical analysis of (5), is seemingly to take $w = h_0\mathcal{A}^{-1}U + w_q$ and insert this into (5) directly. This manifests the symmetry of the corresponding system of linear equations where some prescribed boundary values $w(0)$, $\varphi(l)$, $w(l)$ or $\varphi(l)$ (or their analogies for more intervals, as discussed above), but leads to rather complicated formulae. However, the MATLAB-based symbolic computations (referring to the core of MAPLE), as well as numerical experiments in MATLAB, demonstrate the preservation of such symmetry even in the case if an intuitive, physically motivated approach of [16], using the system (stiffness) matrix $\mathcal{K}$.

The original program in MATLAB has been prepared to generate the special code for the MATLAB interpreter for various configurations, to be able to check and analyse results both at the symbolic and at the numerical level. However, in the following illustrative example we take (for simplicity) $l = 1$ and $EJ = 1$ everywhere, varying the setting of μ and α only. The formal evaluation of $\mathcal{K}^{-1}$ is (except the case a)) rather expensive and its results may be confused, thus the comparison of numerical outputs is preferable. The choice $\mu = 1$ and $\alpha = 0$ (the case a), cf. the presentation of $\mathcal{A}$ and $\mathcal{B}$ above) gives

$$\mathcal{K} = \begin{bmatrix} 0.923077 & -0.461538 & -0.923077 & -0.461538 \\ -0.461538 & 1.230769 & 0.461538 & -0.769231 \\ -0.923077 & 0.461538 & 0.923077 & 0.461538 \\ -0.461538 & -0.769231 & 0.461538 & 1.230769 \end{bmatrix}.$$

The choice $\mu = 1$ and $\alpha = 1$ (the case d)) gives

$$\mathcal{K} = \begin{bmatrix} 1.238331 & -0.500000 & -0.776778 & -0.424356 \\ -0.500000 & 1.238331 & 0.424356 & -0.776778 \\ -0.776778 & 0.424356 & 1.238331 & 0.500000 \\ -0.424356 & -0.776778 & 0.500000 & 1.238331 \end{bmatrix}.$$

The choice $\mu = 2$ and $\alpha = 1$ (the case c)) gives

$$\mathcal{K} = \begin{bmatrix} 0.776962 & -0.275007 & -0.346902 & -0.205637 \\ -0.275007 & 1.126922 & 0.205637 & -0.886918 \\ -0.346902 & 0.205637 & 0.776962 & 0.275007 \\ -0.205637 & -0.886918 & 0.275007 & 1.126922 \end{bmatrix}.$$

The choice $\mu = 3$ and $\alpha = 1$ (the case b)) gives

$$\mathcal{K} = \begin{bmatrix} 0.606271 & -0.194408 & -0.202896 & -0.130340 \\ -0.194408 & 1.087466 & 0.130340 & -0.925302 \\ -0.202896 & 0.130340 & 0.606271 & 0.194408 \\ -0.130340 & -0.925302 & 0.194408 & 1.087466 \end{bmatrix}.$$

The largest absolute error in asymmetry in these 4 matrices K is of order 10^{-14}, as well as the biggest absolute value of the left-hand side of (10) with $w = h_j$ with some $j \in \{1, 2, 3, 4\}$; for this check the 4th derivatives of h_j, not included in W, must be evaluated, too. This results corresponds to the floating-point representation of real numbers in double precision, both in MATLAB and in Fortran or C++, used for the development of the finite-element based structural analysis software RFEM, as discussed in [17], which has been also the first motivation for this study.

The implementation of the above sketched computational approach into the RFEM environment is still in development. The theoretical analysis exhibits the removal of non-physical phenomena, mentioned in *Introduction*, in the same sense as in the case of Timoshenko beam, studied in [16] in great details. The experience with more complex computational tools, including the more-dimensional, dynamical, multiphysical, etc. ones, cannot be summarized yet. However, the careful usage of this approach is necessary: for less smooth problems the implementation of a finite element basis coming from fundamental solutions of a similar problem (even as the starting estimate for iterations) instead of the standard (and more simple) one can be counterproductive.

VI. Conclusions and generalizations

We have demonstrated the rationality of the utilization of finite element bases coming from the (at least partially) known fundamental solutions on one linearized one-dimensional model problem of a beam of elastic foundations. Similarly to the "exact" methods using integral transforms, Green functions, etc., we receive the sparse systems of algebraic equations instead of the system of ordinary differential equations in the case of response of the building structure to statical loads, but no infinite, singular and similar integrals occur in the computational algorithm. The natural generalization is to analyze response to dynamical loads, consequently we have some ordinary differential equations instead of partial ones. Analogous considerations can be done for plates instead of beams, too.

The development of the RFEM-based software, implementing (among others) the results presented in this paper, including the above sketched generalizations, is documented by [20] and [26]. This computational experience shows that the extension of this approach to various problems of the real world, as indicated (in very different sense and directions) by [1], [7], [11] or [13], is desirable. However, some unclosed problems concerning both the existence and uniqueness of solutions and the convergence of the sequences of approximate solutions in some practically transparent sense can be seen as the research challenge for the near future, primarily as a part of both projects of application research mentioned in *Acknowledgement*.

Acknowledgment

The author appreciates the financial support from the projects i) of specific university research at Brno University of Technology FAST-S-18-5184, ii) of Ministry of Education, Youth and Sports of the Czech Republic AdMaS UP (Advanced Materials, Structures and Technologies), National Sustainability Programme I, LO1408.

References

[1] A. S. Adeoye and T. Awodola, Dynamic response to moving distributed Masses of pre-stressed uniform Rayleigh beam resting on variable elastic Pasternak foundation. *Edelweiss Applied Science and Technology* 2 (2018), 46–55.

[2] A. A. Al-Azzawi, Analysis of Timoshenko beam resting on nonlinear compressional and frictional Winkler foundation. *ARPN Journal of Engineering and Applied Sciences* 6 (2011), 100–113.

[3] S. C. Brenner and L. Ridgway Scott, *The Mathematical Theory of Finite Element Methods*. Springer, New York, 2002.

[4] D. Dinev, Analytical solution of beam on elastic foundation by singularity functions. *Engineering Mechanics* 19 (2012), 381–392.

[5] A. C. Eringen, *Nonlocal Continuum Field Theories*. Springer, New York, 2002.

[6] M. M. Filonenko-Borodich, Some approximate theories of elastic foundations. *Uchenyie zapiski Moskovskogo Gosudarstvennogo Universiteta: Mekhanika* 46 (1940), 3–18.

[7] D. V. Griffith, J. Paiboon, J. Huang and G. A. Fenton, Reliability analysis of beams on random elastic foundations. *Géotechnique* 63 (2013), 180–188.

[8] E. Gülkan and B. M. Alemdar, An exact finite element for a beam on a two-parameter elastic foundation: a revisit. *Structural Engineering and Mechanics* 7 (1999), 259–276.

[9] M. Hetényi, A general solution for the bending of beams on an elastic foundation of arbitrary continuity. *Journal of Applied Physics* 21 (1950), 55–58.

[10] J. Jensen, On the shear coefficient in Timoshenko beam theory. *Journal of Sound and Vibration* 87 (1963), 621–635.

[11] W.-Y. Jung, W.-T. Park and S.-Ch. Han, Bending and vibration analysis of S-FGM microplates embedded in Pasternak elastic medium using the modified couple stress theory. *International Journal of Mechanical Sciences* 87 (2014), 150–162.

[12] A. Kacar, H. Tugba Tan and M. O. Kaya, Free vibration analysis of beams on variable Winkler elastic foundation by using the differential transform method, *Mathematical and Computational Applications* 16 (2011), 773–783.

[13] A. Karasin, Vibration of rectangular plates on elastic foundations by finite grid solution. *International Journal of Mathematical and Computational Methods* 1 (2016), 140–145.

[14] A. D. Kerr, Elastic and viscoelastic foundation models. *Journal of Applied Mechanics* 31 (1964), 491–498.

[15] V. Kolář and I. Němec, *Modeling of Soil-Structure Interaction*. Academia, Prague, 1989.

[16] G. P. Lignola, F. R. Spena, A. Prota and G. Manfredi, Exact stiffness-matrix of two nodes Timoshenko beam on elastic medium – an analogy with Eringen model of nonlocal Euler-Bernoulli nanobeams. *Computers and Structures* 182 (2017), 556–572.

[17] I. Němec and J. Buček, Nonlinear analysis of structure-soil interaction using surface subsoil model. *Mechanical Structures and Foundation Engineering* in Ostrava (2010), VSB-TU, Ostrava, 2010, paper #17 / 1–16.

[18] P. L. Pasternak, *Osnovy novogo metoda rasczota fundamentov na uprugom osnovanii pro pomoszczi dvukh koefficientov posteli*. Gosstrojizdat, Moscow, 1954. (In Russian.)

[19] E. Reissner, A Note on deflection of plates on a viscoelastic foundation. *Journal of Applied Mechanics* 25 (1958), 144–145.

[20] V. Rek, Explicit integration of equations of motion solved on computer cluster. 18th *Programs and Algorithms of Numerical Mathematics (PANM)* in Hejnice (Czech Republic, 2018), Institute of Mathematics AS CR, submitted.

[21] T. Roubíček, *Nonlinear Partial Differential Equations with Applications*. Birkhäuser, Basel, 2005.

[22] S. P. Timoshenko, On the transverse vibration of bars of uniform cross-section. *Philosophical Magazine* 43 (1922), 125–131.

[23] A. Toselli and O. Widlund, *Domain Decomposition Methods – Algorithms and Theory*. Springer, Berlin, 2005.

[24] W. F. Trench, *Elementary Differential Equations*. Trinity University, San Antonio, 2013.

[25] J. Vala, Computational approaches to some inverse problems from engineering practice. 17th *Programs and Algorithms of Numerical Mathematics (PANM)* in Dolní Maxov (Czech Republic, 2014), Institute of Mathematics AS CR, Prague, 2015, 215–230.

[26] A. Vaněčková, M. Trcala and I. Němec, Dynamic damping – comparison of various concepts from the point of view of physical nature and influence on building constructions. 18th *Programs and Algorithms of Numerical Mathematics (PANM)* in Hejnice (Czech Republic, 2018), Institute of Mathematics AS CR, submitted.

[27] V. Z. Vlasov and N. N. Leontiev, *(Balki, plity i obolochki na uprugom osnovanii*. Fizmatgiz, Moscow, 1960. (In Russian.)

[28] C. M. Wang, K. Y. Lam and X. Q. He, Exact solutions for Timoshenko beams on elastic foundations using Green's functions. *Mechanics of Structures and Machines* 26 (1998), 101–113.

[29] E. Winkler, *Die Lehre von der Elastizität und Festigkeit*. Dominicus, Prague, 1867. (In German.)

Jiří Vala was born on December 21th, 1956, in Brno (now Czech Republic). Professional and research experience from ZPA (Industrial Automation Works) in Prague and from Institute of Physics of Materials AS CR in Brno. Interested namely in numerical and computational analysis of physical processes in deformable bodies and media. Professor at Brno University of Technology since 2009.