The Hamiltonian Dynamical System Associated to the Isotropic Harmonic Oscillator - A Numerical Study of Conservation Laws

Romulus Militaru, Florian Munteanu

Abstract—-In this paper we study the interplay between dynamical systems geometrical theory and computational calculus of dynamical systems. The viewpoint is geometric and we also compute and characterize objects of dynamical significance, in order to understanding the mathematical properties observed in numerical computation for dynamical models arising in mathematics, science and engineering. We use appropriate numerical methods for the study of conservation laws of Hamiltonian dynamical system associated to the isotropic harmonic oscillator for one and two dimensional case.

Keywords—-Conservation law, Noether theorem, Runge-Kutta methods, symmetry.

I. INTRODUCTION

Symmetries play a key role in mathematics, physics and very specially in mechanics. The study of those quantities which are conserved by a mechanical system is highly relevant both from a theoretical and from a practical point of view ([8], [28]). Different methods for finding such conserved quantities are known, those based on Noether theory ([26]), which addresses the invariance of the action functional under infinitesimal transformations, have proved powerful and widely used. For theoretical geometrical models, numerical analysis of models and more computation details see also [4], [5], [12], [25], [21], [22].

The inverse problem of Lagrangian mechanics is the question ([1], [30]): given a system of second-order ordinary differential equations, under what circumstances does there exist a regular Lagrangian function, such that the corresponding Lagrange equations are equivalent (i.e. have the same solutions) as the original equations. Locally, the question can be translated immediately into more precise terms as follows: considering a given second-order system in normal form $\ddot{q}^i = f^i(q, \dot{q})$, which (for the time being) we take to be autonomous for simplicity, what are the conditions for the existence of a symmetric, non-singular multiplier matrix $(g_{ij}(q, \dot{q}))$ such that

$$g_{ij}\left(\ddot{q}^j - f^j(q, \dot{q})\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}$$

for some L. Clearly (g_{ij}), if it exists, will become the Hessian of the Lagrangian L. The literature on this problem is extensive ([8]); the conditions for the existence of L are usually referred to as the Helmholtz conditions, but these can take many different forms depending on the mathematical tools one uses and on the feature one focusses on ([27]).

In this paper we will study symmetries, conservation laws and relationship between this in the geometric framework of Classical Mechanics ([1], [2], [8], [23], [24]). Also, we will use appropriate numerical methods for the study of the behavior of dynamical system associated to the isotropic harmonic oscillator for one dimensional case and two dimensional case. For theoretical geometrical models, numerical analysis of models and more computation details see also [4], [5], [12], [21], [22].

There is a very well-known way to obtain conservation laws for a system of differential equations given by a variational principle: the use of the Noether Theorem ([26]) which associates to every symmetry a conservation law and conversely. However, there is a method introduced by G.L. Jones ([19]) and M. Crâşmăreanu ([10]) which can be obtained new kinds of conservation laws, without the help of a Noether's type theorem.

In the second section we recall the basic notions and results for the geometrical study of a Hamiltonian dynamical systems, we present the classical Noether Theorem ([26]), the Theorem of Jones-Crâşmăreanu ([19], [10]) and, finally, we focus on two examples: 1D and 2D isotropic harmonic oscillator ([10], [30]).

In the third section, we will make a numerical study for the isotropic harmonic oscillator. More precisely, constructing a Matlab-based numerical code, we are looking to approximate and characterize different types of invariants and also to extract statistical information on the dynamical behavior and perform some comparisons for different initial conditions associated to the considered problem.

All manifolds are real, paracompact, connected and C^∞. All maps are C^∞. Sum over crossed repeated indices is understood.

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R. Militaru is with Department of Applied Mathematics, University of Craiova, Al. I. Cuza st., no. 13, 200585-Craiova, Romania (corresponding author to provide phone: +40726003426; e-mail: militaruromulus@yahoo.com).

F. Munteanu is with Department of Applied Mathematics, University of Craiova, Al. I. Cuza 13, 200585-Craiova, Romania (munteanufm@central.ucv.ro).

II. CLASSICAL RESULTS

A. Symmetries and Conservation Laws for Dynamical Systems Let M be a smooth, n-dimensional manifold, $C^\infty(M)$ the ring of real-valued smooth functions, $\mathcal{X}(M)$ the Lie algebra of vector fields and $A^p(M)$ the $C^\infty(M)$-module of p-differential forms, $1 \leq p \leq n$. For $X \in \mathcal{X}(M)$ with local expression $X = X^i(x)\frac{\partial}{\partial x^i}$ we consider the system of ordinary differential equations which give the flow $\{\Phi_t\}_t$ of X, locally,

$$\dot{x}^i(t) = \frac{dx^i}{dt}(t) = X^i(x^1(t), \ldots, x^n(t)). \qquad (1)$$

A dynamical system is a couple (M, X), where M is a smooth manifold and $X \in \mathcal{X}(M)$. A dynamical system is denoted by the flow of X, $\{\Phi_t\}_t$ or by the system of differential equations (1).

A function $f \in C^\infty(M)$ is called *conservation law* for dynamical system (M, X) if f is constant along the every integral curves of X, that is

$$L_X f = 0, \qquad (2)$$

where $L_X f$ means the Lie derivative of f with respect to X.

A diffeomorfism $\Phi : M \to M$ is said to be a *symmetry* of (M, X) if Φ maps integral curves of X onto integral curves of X, i.e., $T\Phi(X) = X$.

$Y \in \mathcal{X}(M)$ is called *dynamical symmetry* (or, shortly, *symmetry*) for (M, X) if its flow $\{\Phi_t\}_t$ consists of symmetries of (M, X), or, equivalently, $[Y, X] = L_X Y = 0$.

Let us recall that $\omega \in A^p(M)$ is called *invariant form* for (M, X) if $L_X \omega = 0$.

B. Hamiltonian Systems If (M, ω) is a symplectic manifold then the dynamical system (M, X) is said to be *a dynamical Hamiltonian system* (or, shortly, *Hamiltonian system*) if there exists a function $H \in C^\infty(M)$ (called *the Hamiltonian*) such that

$$i_X \omega = -dH, \qquad (3)$$

where i_X denotes the interior product with respect to X.

The vector field X from (3) is called *the Hamiltonian vector field* associated to H (also denoted by X_H) and

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}$$

with respect to natural local coordinates (x^i, p_i), given by Darboux Theorem, ([23],[8]).

The equation (3) is equivalent, locally, with *Hamilton-Jacobi equations*:

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}\ , \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}\,.$$

A *regular Lagrangian* L on TM is a smooth function $L : TM \to \mathbf{R}$, for which the fundamental metric tensor $g_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j}$ has rank n on $TM \setminus \{0\}$.

Let us remember *the Legendre map* associated to L, $FL : TM \to T^*M$, which is a local diffeomorphism given by $(x^i, y^i) \longrightarrow (x^i, p_i)$, where $p_i = \frac{\partial L}{\partial y^i}$.

Let us consider the Hamiltonian system (TM, S_L) on the symplectic manifold (TM, ω_L), where S_L is the canonical semispray associated to the regular Lagrangian L, locally, given by

$$S_L = y^i \frac{\partial}{\partial x^i} - G^i \frac{\partial}{\partial x^i}\,,$$

where $G^i = g^{ij}\left(\frac{\partial^2 L}{\partial y^j \partial x^k} y^k - \frac{\partial L}{\partial x^j}\right)$ and g^{ij} being the inverse of the fundamental metric of L, g_{ij}, [23].

In fact, the Euler-Lagrange equations associated to L,

$$E_i(L) = \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0$$

are equivalently with the system of ODE's,

$$\frac{dy^i}{dt} = -G^i, \frac{dx^i}{dt} = y^i\,,$$

which give the flow of vector field S_L on TM.

$\omega_L = d\theta_L$ is the Cartan 2-form of L, $\theta_L = J^*(dL)$, J^* being the adjoint of the natural tangent structure J on TM and

$$H = E_L = \frac{\partial L}{\partial y^i} y^i - L$$

is the energy of L or the Hamiltonian of (TM, S_L).

A *Cartan symmetry* for Lagrangian L is a vector field $X \in \mathcal{X}(TM)$ characterized by $L_X \omega_L = 0$ and $L_X H = 0$. It is known that that any Cartan symmetry for Lagrangian L is a symmetry for the canonical semispray S of L.

For each Cartan symmetry X for (M, L) we have $dL_X\theta_L = 0$, which implies that $L_X\theta_L$ is a closed 1-form. If $L_X\theta_L$ is a exact 1-form, then we say that X is *exact Cartan symmetry* for (M, L).

Obviously, the canonical semispray of L is an exact Cartan symmetry for Lagrangian L, [8],[23], [28].

C. Noether Theorem

In the classical case ($k = 1$), we know that Cartan symmetries induce and are induced by constants of motions (conservation laws), and these results are known as Noether Theorem and its converse.

Theorem 1 *([26]) If X is an exact Cartan symmetry with $L_X\theta_L = df$, then*

$$P_X = J(X)L - f$$

is a conservation law for the Euler-Lagrange equations associated to the regular Lagrangian L.

Conversely, if F is a conservation law for the Euler-Lagrange equations associated to the regular Lagrangian L, then the vector field X uniquely defined by

$$i_X\omega_L = -dF$$

is an exact Cartan symmetry.

D. Nonclassical Conservation Laws

If $Z \in \mathcal{X}(M)$ is fixed, then $Y \in \mathcal{X}(M)$ is called Z-*pseudosymmetry* for (M, X) if there exists $f \in C^\infty(M)$ such that $L_X Y = fZ$.
A X-pseudosymmetry for X is called *pseudosymmetry* for (M, X).

Example: ([11],[13]) The Nahm's system from the theory of static SU(2)-monopoles:

$$\frac{dx^1}{dt} = x^2x^3,\ \frac{dx^2}{dt} = x^3x^1,\ \frac{dx^3}{dt} = x^1x^2. \tag{4}$$

The vector field $X = x^2x^3\frac{\partial}{\partial x^1} + x^3x^1\frac{\partial}{\partial x^2} + x^1x^2\frac{\partial}{\partial x^3}$ satisfies $[Y, X] = X$, where $Y = \sum\limits_{i=1}^{3} x^i\frac{\partial}{\partial x^i}$.
This means that Y is a X-pseudosymmetry for (4).

The next theorem gives the association between pseudosymmetries and conservation laws. In this way, we will obtain new kinds of conservation laws, nonclassical, without the help of Noether's type theorem.

Theorem 2 *([10],[19]) Let $X \in \mathcal{X}(M)$ be a fixed vector field and $\omega \in A^p(M)$ be a invariant p-form for X. If $Y \in \mathcal{X}(M)$ is symmetry for X and $S_1, \ldots, S_{p-1} \in \mathcal{X}(M)$ are $(p-1)$ Y-pseudosymmetry for X then*

$$\Phi = \omega(X, S_1, \ldots, S_{p-1}) \tag{5}$$

or, locally,

$$\Phi = S_1^{i_1}\cdots S_{p-1}^{i_{p-1}} Y^{i_p}\omega_{i_1\ldots i_{p-1}i_p} \tag{6}$$

is a conservation laws for (M, X).

Particularly, if Y, $S_1, \ldots, S_{p-1}$ are symmetries for X then Φ given by (5) is conservation laws for (M, X).

Now, we can apply this result to the dynamical Hamiltonian systems.

Proposition 3 *Let be (M, X_H) a Hamiltonian system on the symplectic manifold (M, ω), with the local coordinates (x^i, p_i). If $Y \in \mathcal{X}(M)$ is a symmetry for X_H and $Z \in \mathcal{X}(M)$ is a Y-pseudosymmetry for X_H then*

$$\Phi = \omega(Y, Z) \tag{7}$$

is a conservation law for the Hamiltonian system (M, X_H).

Particularly, if Y and Z are symmetries for X_H then Φ from (7) is a conservation law for (M, X_H).

If $Y = Y^k\frac{\partial}{\partial x^k} + \widetilde{Y}_k\frac{\partial}{\partial p_k}$ and $Z = Z^k\frac{\partial}{\partial x^k} + \widetilde{Z}_k\frac{\partial}{\partial p_k}$ then (7) becomes

$$\Phi = \begin{pmatrix} Y^k & \widetilde{Y}_k \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} Z^k \\ \widetilde{Z}_k \end{pmatrix} = \begin{matrix} \widetilde{Y}_k Z^k - \\ -Y^k\widetilde{Z}_k. \end{matrix} \tag{8}$$

Corollary 4 *If $Y \in \mathcal{X}(M)$ is a X_H-pseudosymmetry for X_H then*

$$\Phi = \omega(X_H, Y) = -L_Y H \tag{9}$$

or

$$\Phi = \frac{\partial H}{\partial x^k}Y^k + \frac{\partial H}{\partial p_k}\widetilde{Y}_k \tag{10}$$

is a conservation law for (M, X_H).

E. Nonclassical Conservation Laws for Regular Lagrangian Systems

If we consider the Hamiltonian system (TM, S_L) on the symplectic manifold (TM, ω_L), where S_L is the canonical semispray and ω_L the Cartan 2-form associated to a regular Lagrangian L on TM, then we have:

Corollary 5 *If $Y = Y^k\frac{\partial}{\partial x^k} + \widetilde{Y}^k\frac{\partial}{\partial y^k} \in \mathcal{X}(TM)$ is a S_L-pseudosymmetry for S_L then*

$$\Phi = \omega_L(S_L, Y) = -L_Y E_L \tag{11}$$

or

$$\Phi = \frac{\partial E_L}{\partial x^k}Y^k + \frac{\partial E_L}{\partial y^k}\widetilde{Y}^k \tag{12}$$

is a conservation law for (TM, S_L).

An immediately consequence of this last result is the following:

Corollary 6 *If the canonical semispray S_L associated to the regular Lagrangian L is 2-positive homogeneous with respect to velocity (S_L is a **spray**) and g_{ij} is the metric tensor of L, then $\Phi = g_{ij}y^i\widetilde{Y}^j$ is a conservation law for (TM, S_L).*

The canonical semispray S_L is a spray if and only if $[S_L, C] = S_L$, that is $L_{S_L}C = S_L$. So, in this case, the canonical vector field $C = y^i\frac{\partial}{\partial y^i}$ is a pseudosymmetry for S_L and using the last corollary we obtain the conservation of the kinetic energy $T = \frac{1}{2}g_{ij}y^iy^j$. This is the case when $L = F^2$, where F is a Finsler function like in [23].

F. Examples

Example: 1D harmonic oscillator

The dynamics of the 1-dimensional harmonic oscillator is given by second-order ordinary differential equation

$$\ddot{q} + \omega^2 = 0\,, \tag{13}$$

where $\omega^2 = \frac{k}{m}$, m is the mass of the oscillator (usualy we take a standard, $m = 1$) and k is the elasticity constant. The period of the oscillations is $T = \frac{2\pi}{\omega}$.

The Lagrangian is $L(q, \dot{q}) = T - V = \frac{1}{2}\left(\dot{q}^2 - \omega^2 q^2\right)$, where T is the kinetic energy $T = \frac{1}{2}\dot{q}^2$ and V is the potential of the elastic force, $V = \frac{1}{2}kq^2$. The total energy of L, $E_L = \dot{q}\frac{\partial L}{\partial \dot{q}} = \frac{1}{2}\left(\dot{q}^2 + \omega^2 q^2\right)$ is a conservation law for (13) and represents the Hamiltonian function of the dynamical system (13). The canonical semispray of L is $S_L = \dot{q}\frac{\partial}{\partial q} - \omega^2 q \frac{\partial}{\partial \dot{q}}$.

If $q(t) = A\cos\omega t + \frac{B}{\omega}\sin\omega t$ is the solution of the Euler-Lagrange equation (13) with initial conditions $q(0) = A$, $\dot{q}(0) = B$, then we have

$$\begin{array}{l} L = -\frac{1}{2}\left(A^2\omega^2 + B^2\right)\cos 2\omega t - AB\sin 2\omega t\,, \\ H = \frac{1}{2}\left(A^2\omega^2 + B^2\right)\,, \\ T = \frac{1}{2}\left(A^2\omega^2\sin^2\omega t + B^2\cos^2\omega t\right)\,. \end{array}$$

Example: 2D isotropic harmonic oscillator

Let the 2-dimensional isotropic harmonic oscillator

$$\begin{array}{rcl} \ddot{q}^1 + \omega^2 q^1 & = & 0 \\ \ddot{q}^2 + \omega^2 q^2 & = & 0 \end{array} \tag{14}$$

a toy model for many methods to finding conservation laws. The Lagrangian is

$$L = \frac{1}{2}\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2\right] - \frac{\omega^2}{2}\left[\left(q^1\right)^2 + \left(q^2\right)^2\right]$$

Next, by applying the conservation of energy we have two conservation laws $\Phi_1 = \left(\dot{q}^1\right)^2 + \omega^2\left(q^1\right)^2$, $\Phi_2 = \left(\dot{q}^2\right)^2 + \omega^2\left(q^2\right)^2$. Let us observe that $\frac{1}{2}\left(\Phi_1 + \Phi_2\right) = H$, the total energy of the system.

A straightforward computation give that the complet lift of $X = q^2\frac{\partial}{\partial q^1} - q^1\frac{\partial}{\partial q^2}$ is an exact Cartan symmetry with $f = 0$ and then the associated classical Noetherian conservation law is

$$\Phi_3 = P_X = J(X)L = X^i\frac{\partial L}{\partial \dot{q}^i} = q^2\dot{q}^1 - q^1\dot{q}^2.$$

Taking into account that the canonical spray of L is

$$S = \dot{q}^1\frac{\partial}{\partial q^1} + \dot{q}^2\frac{\partial}{\partial q^2} - \omega^2 q^1\frac{\partial}{\partial \dot{q}^1} - \omega^2 q^2\frac{\partial}{\partial \dot{q}^2}$$

and

$$Y = \dot{q}^2\frac{\partial}{\partial q^1} + \dot{q}^1\frac{\partial}{\partial q^2} - \omega^2 q^2\frac{\partial}{\partial \dot{q}^1} - \omega^2 q^1\frac{\partial}{\partial \dot{q}^2}$$

is a symmetry for S, it result that

$$Z = q^1\frac{\partial}{\partial q^1} + q^2\frac{\partial}{\partial q^2} + \dot{q}^1\frac{\partial}{\partial \dot{q}^1} + \dot{q}^2\frac{\partial}{\partial \dot{q}^2}$$

is a symmetry for S.

Next, we have $L_Y H = 0$, $L_Z H = 2H$ and then $\Phi = \omega_L(S, Y) = 0$, $\Phi = \omega_L(S, Z) = 2H$, that means that we not have new conservation law applying main Theorem 2. But $\Phi_4 = \omega_L(Y, Z) = \dot{q}^1\dot{q}^2 + \omega^2 q^1 q^2$ is a new conservation law given by the main Theorem 2 or by their corollaries.

We remark that Φ_4 is a nonclassical conservation law, obtained by symmetries, and Φ_4 represent the energy of a new Lagrangian of, $\tilde{L} = \dot{q}^1\dot{q}^2 - \omega^2 q^1 q^2$ ([30]).

Like in the $1D$ case, $q^i(t) = A_i\cos\omega t + \frac{B_i}{\omega}\sin\omega t$, $i = 1, 2$ is the solutions of the Euler-Lagrange equations (14) with initial conditions $q^i(0) = A_i$, $\dot{q}^i(0) = B_i$, $i = 1, 2$, and we have

$$\begin{array}{l} L = -\frac{1}{2}\left(\left(A_1^2 + A_2^2\right)\omega^2 + B_1^2 + B_2^2\right)\cos 2\omega t - \\ \qquad\qquad - \left(A_1 B_1 + A_2 B_2\right)\sin 2\omega t \\ H = \dot{q}^i\frac{\partial L}{\partial \dot{q}^i} - L = \frac{1}{2}\left(\left(A_1^2 + A_2^2\right)\omega^2 + B_1^2 + B_2^2\right) \\ T = \frac{1}{2}\left(\left(A_1^2 + A_2^2\right)\omega^2\sin^2\omega t + \left(B_1^2 + B_2^2\right)\cos^2\omega t\right) \\ \Phi_1 = A_1^2\omega^2 + B_1^2; \\ \Phi_2 = A_2^2\omega^2 + B_2^2 \\ H = \frac{1}{2}\left(\Phi_1 + \Phi_2\right); \\ \Phi_3 = 0 \\ \Phi_4 = H_{\tilde{L}} = \omega^2 A_1 A_2 + B_1 B_2 \\ \tilde{L} = \left(-\omega^2 A_1 A_2 + B_1 B_2\right)\cos 2\omega t - \\ \qquad\qquad -\omega\left(A_1 B_2 + A_2 B_1\right)\sin 2\omega t \end{array}$$

Let us remark that Φ_3 is the module of kinetic momentum, $\vec{L} = \vec{r} \times \vec{p}$.

III. NUMERICAL STUDY

In this section, constructing a Matlab-based numerical code, we are looking to approximate and characterize different types of invariants and also to extract informations on the dynamical behavior and perform comparisons for both different initial conditions associated to the considered problem and for different values of the parameter ω.

In the first stage we focus on the numerical solving of the initial value problem given by the ordinary differential equation with a prescribed initial condition, by appropriate numerical methods, such as Runge-Kutta methods. Numerical integration is a mature subject, but due to the todays high computer efficiency it is still very active especially with regard to algorithms designed for special classes of equations, [17], [29], [18]. Thus we obtain the numerical solution represented by the approximate values of the exact solution for a discrete set of data points.

In the second stage, using this approach we perform a numerical analysis of the conservation laws and main sizes, like the Lagrangian, the Hamiltonian, the kinetic energy, the kinetic momentum.

A. *One Dimensional Harmonic Oscillator*

We consider the initial value problem (IVP) given by:

$$\begin{cases} \ddot{q} + \omega^2 q = 0 \\ q(0) = 1 \\ \dot{q}(0) = 0 \end{cases}$$

(a). $\omega = 0.59$

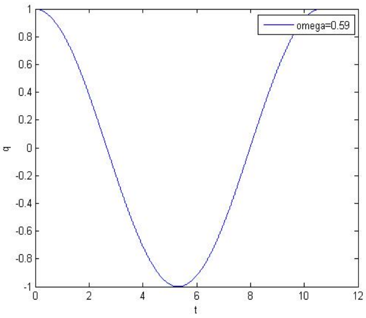

Fig 1: Numerical solution on a period

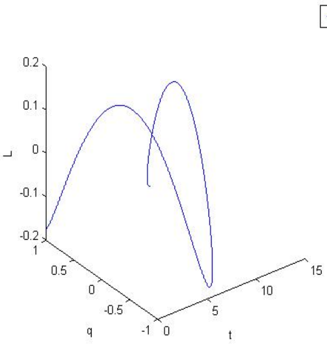

Fig 2: Lagrangian L, function of t and q

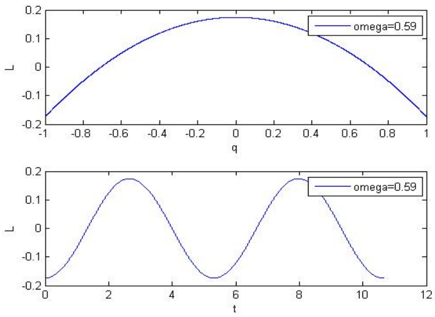

Fig 3: Lagrangian L, as function of q, respectively function of t

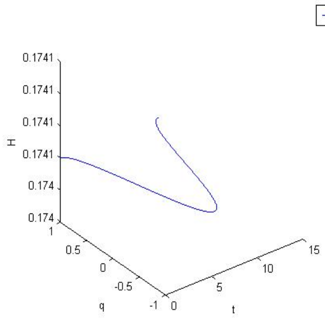

Fig 4: Total energy H, function of t and q

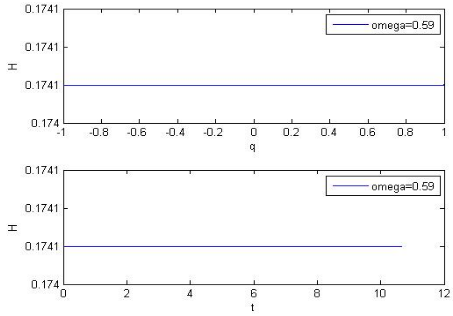

Fig 5: Total energy H, as function of q, respectively as function of t

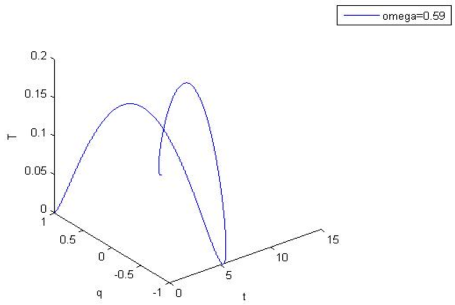

Fig 6: Kinetic energy T, function of t and q

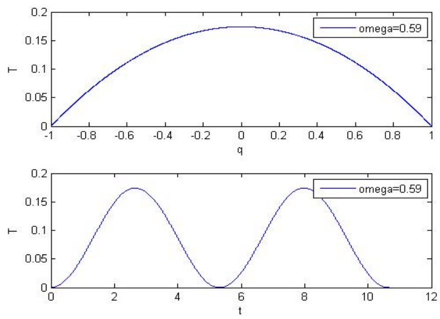

Fig 7: Kinetic energy T, as function of q, respectively as function of t

(b). $\omega = 7.8$

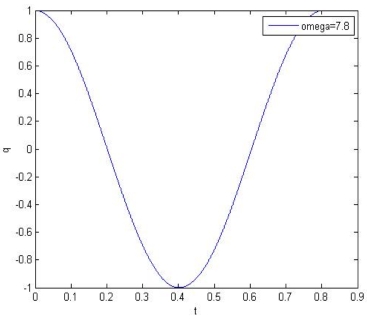

Fig 8: Numerical solution on a period

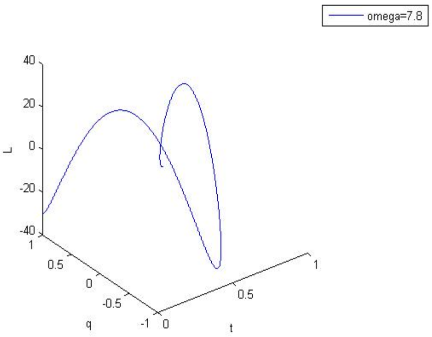

Fig 9: Lagrangian L, function of t and q

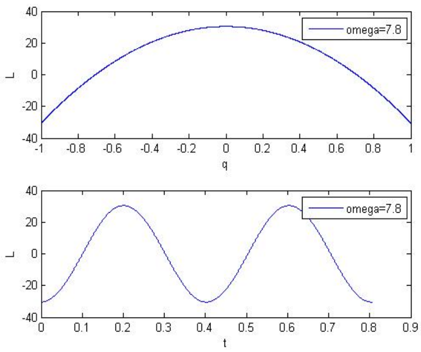

Fig 10: Lagrangian L, as function of q, respectively as function of t

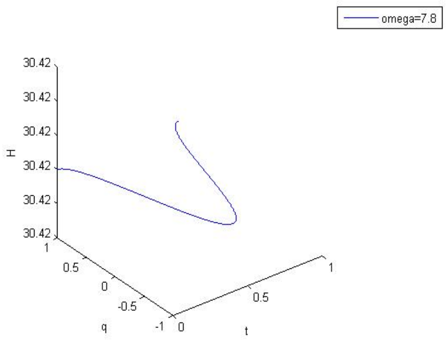

Fig 11: Total energy H, function of t and q

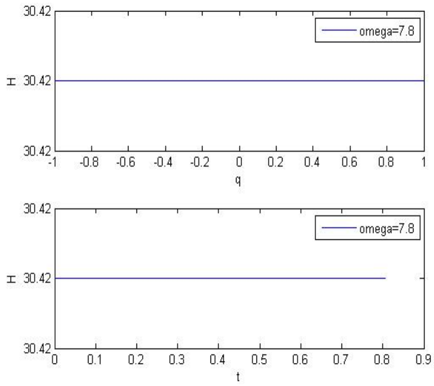

Fig 12: Total energy H, as function of q, respectively as function of t

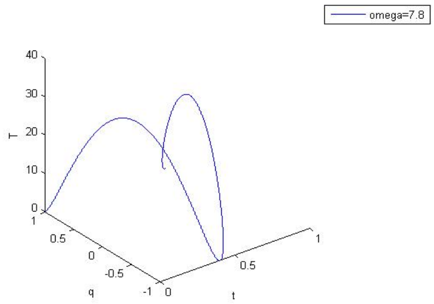

Fig 13: Kinetic energy T, function of t and q

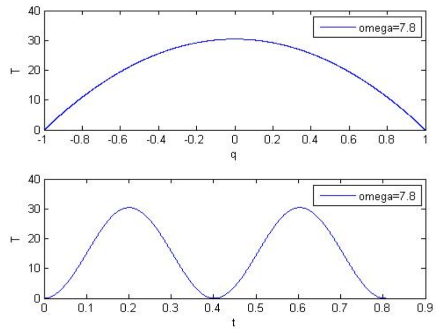

Fig 14: Kinetic energy T, as function of q, respectively as function of t

We present a comparison concerning the kinetic energy T for three different values of the parameter ω :

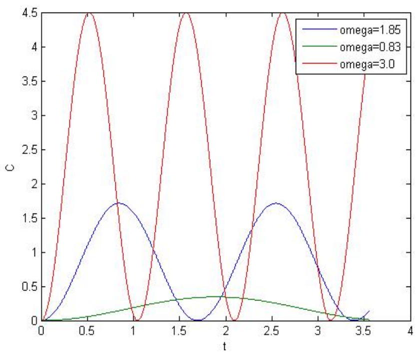

Fig 15: Profiles of the kinetic energy T, for three different values of ω

Next we give the same comparison concerning the comportment of the kinetic energy, but for another initial value problems obtained from the considered second order differential equation characterizing the 1D harmonic oscillator and different initial conditions.
More precisely considering the following IVP:

$$\begin{cases} \ddot{q} + \omega^2 q = 0 \\ q(0) = 1 \\ \dot{q}(0) = 1 \end{cases}$$

the graphic profiles of kinetic energy are:

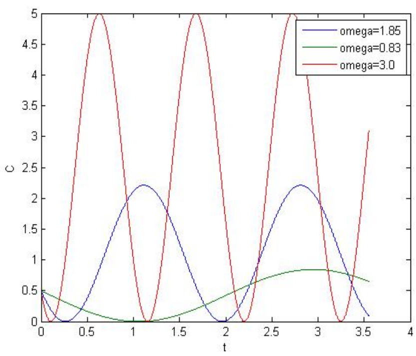

Fig 16: Profiles of the kinetic energy T, for three different values of ω

and also for the following IVP:

$$\begin{cases} \ddot{q} + \omega^2 q = 0 \\ q(0) = 1 \\ \dot{q}(0) = -1 \end{cases}$$

the kinetic energy presents the graphic profiles ploted in the next figure:

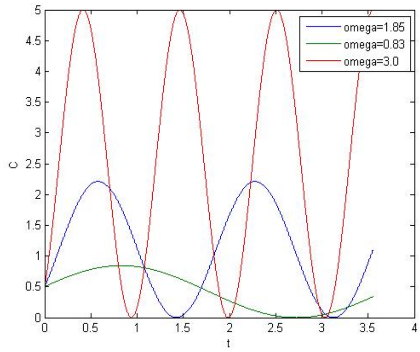

Fig 17: Profiles of the kinetic energy T, for three different values of ω

B. *Two Dimensional Harmonic Oscillator*

We consider the initial value problem given by :

$$\begin{cases} \ddot{q}_1 + \omega^2 q_1 = 0 \\ \ddot{q}_2 + \omega^2 q_2 = 0 \\ q_1(0) = 1 \\ \dot{q}_1(0) = 0 \\ q_2(0) = 1 \\ \dot{q}_2(0) = 0 \end{cases}$$

(a). $\omega = 0.59$

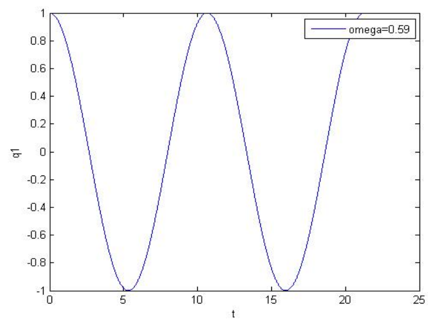

Fig 18: Profile of the numerical solution q_1 on a period

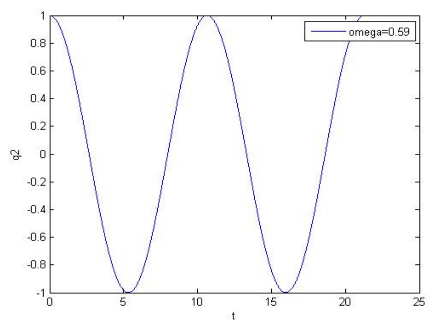

Fig 19: Profile of the numerical solution q_2 on a period

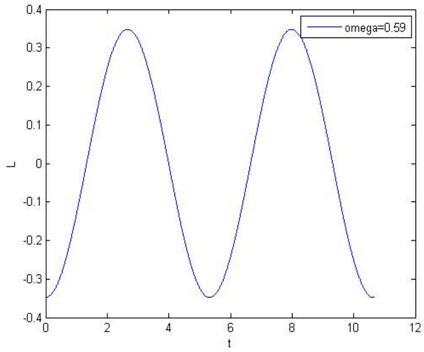

Fig 20: Lagrangian L, function of t

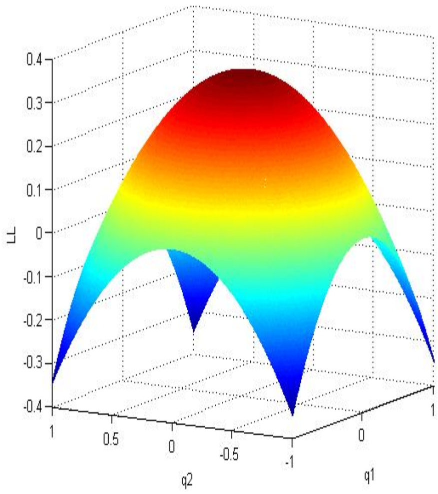

Fig 21: Lagrangian L, function of q_1, q_2

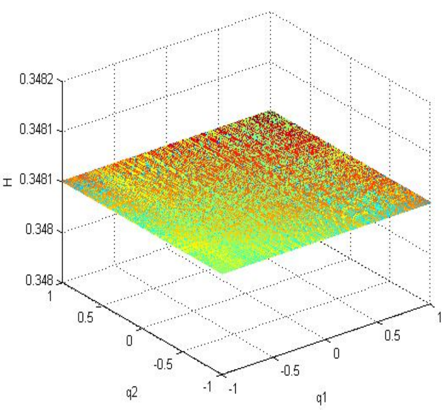

Fig 22: Total energy H, function of q_1, q_2

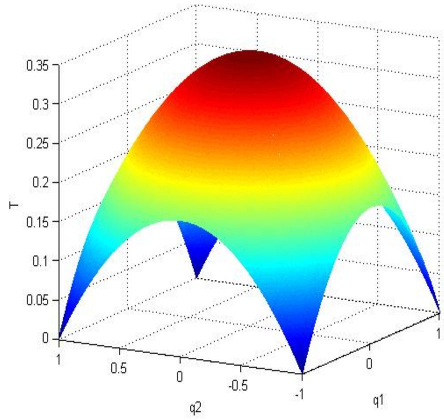

Fig 23: Kinetic energy T, function of q_1, q_2

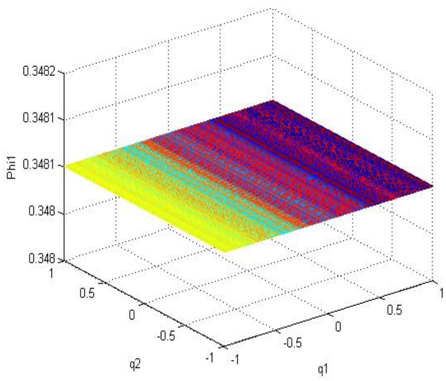

Fig 24: First conservation law Φ_1, function of q_1, q_2

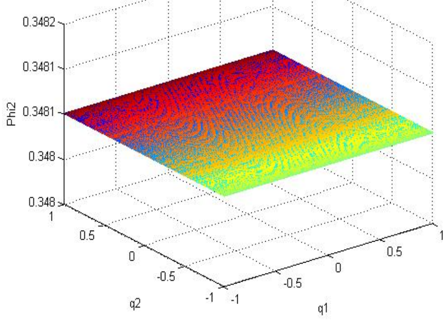

Fig 25: Second conservation law Φ_2, function of q_1, q_2

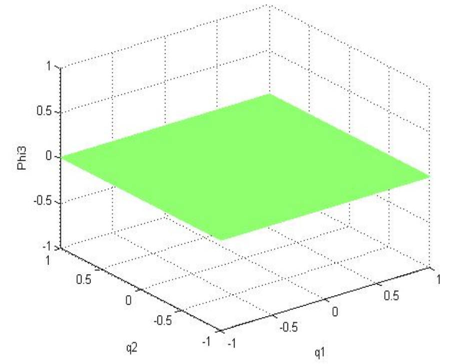

Fig 26: Third conservation law Φ_3, function of $q_1,\ q_2$

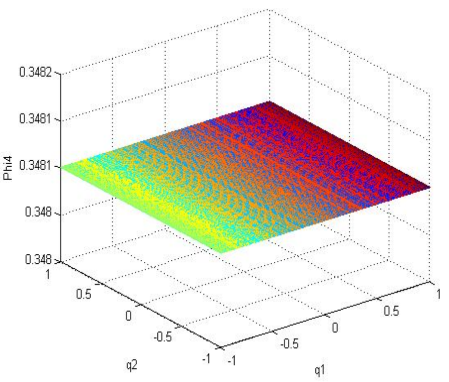

Fig 27: Fourth conservation law Φ_4, function of $q_1,\ q_2$

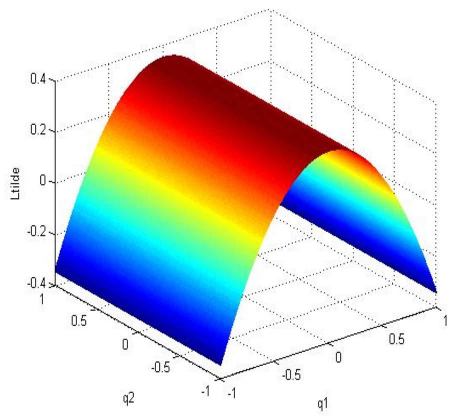

Fig 28: Lagrangian $\tilde{L}$ function of $q_1,\ q_2$

(b). $\omega = 7.8$

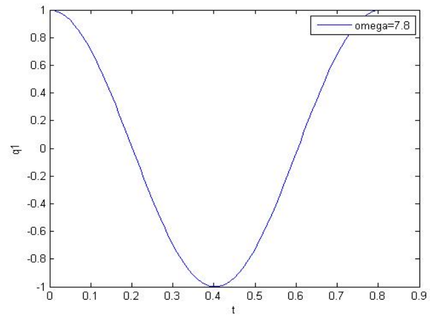

Fig 29: Profile of the numerical solution q_1 on a period

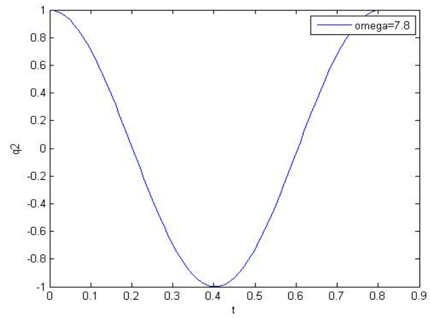

Fig 30: Profile of the numerical solution q_2 on a period

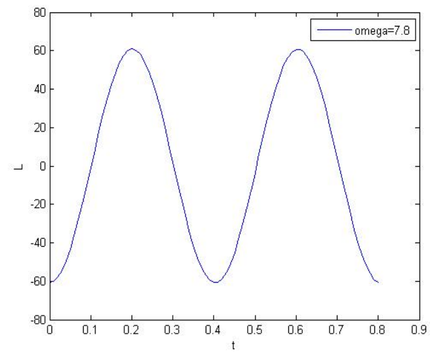

Fig 31: Lagrangian L, function of t

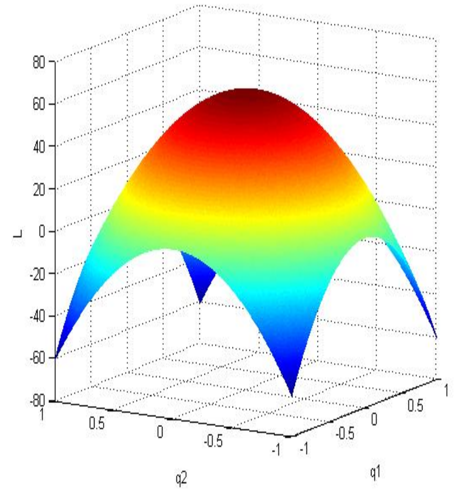

Fig 32: Lagrangian L, function of $q_1,\ q_2$

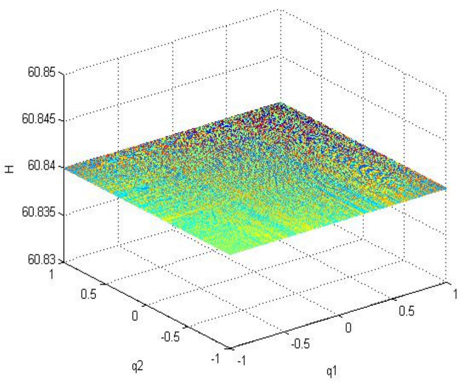

Fig 33: Total energy H, function of $q_1,\ q_2$

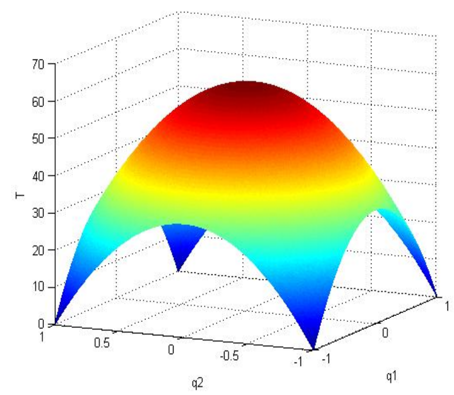

Fig 34: Kinetic energy T, function of $q_1,\ q_2$

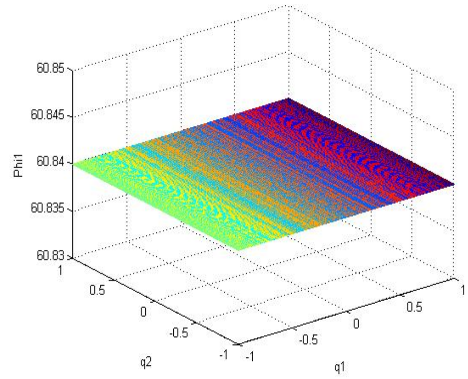

Fig 35: First conservation law Φ_1, function of $q_1,\ q_2$

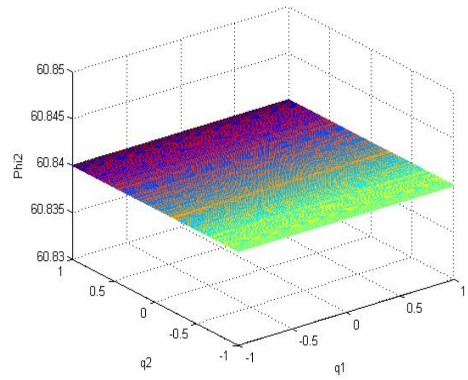

Fig 36: Second conservation law Φ_2, function of $q_1,\ q_2$

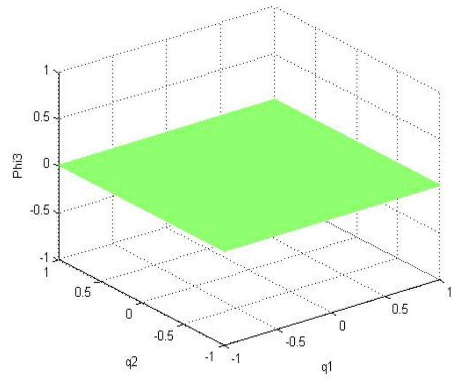

Fig 37: Third conservation law Φ_3, function of $q_1,\ q_2$

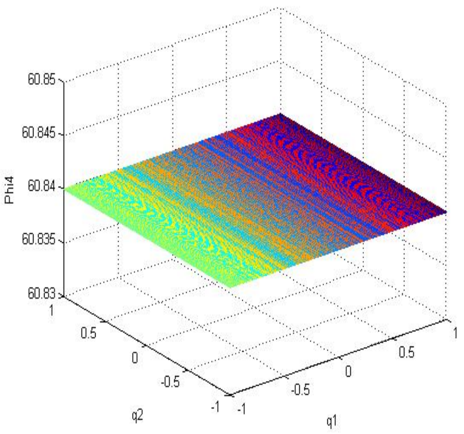

Fig 38: Fourth conservation law Φ_4, function of q_1, q_2

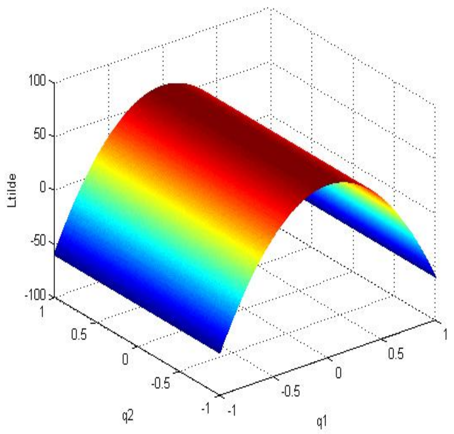

Fig 39: Lagrangian $\tilde{L}$ function of q_1, q_2

Observation 7 *The range of the values of the Lagrangian L is symmetric around zero and the extreme values are directly proportional with the values of parameter ω. This can also be observed for the new Lagrangian $\tilde{L}$.*
The extreme values of the kinetic energy T are also directly proportional with the values of parameter ω. The dependance of the kinetic energy upon the initial conditions, for the case of 1D harmonic oscillator, can be observed in Fig.16 and Fig.17, while the influence of the values of the parameter ω on the comportment of the kinetic energy can be analysed in Fig.15.
The total energy H takes constant values for different values of ω, as we presented in Fig.5, Fig.12, Fig.22 and Fig.33. Thus we are in agreement with the property to be a conservation law.
The others sizes as Φ_1, Φ_2, Φ_3, Φ_4, which are also laws of conservation, are characterized through our numerical study, by constant values, depending on the values of ω, Fig.24-27 and Fig.35-38.

IV. CONCLUSION

We perform a numerical study of this mathematical models, in order to approximate different types of invariants and main sizes, through numerical codes based on appropriate numerical calculus techniques. Thus, starting from certain initial value problems associated to our models, we obtain the numerical solution and we develop the numerical characterization of the main sizes previously analysed from the geometrical point of view. Thus we are able to make different comparisons between these studied quantities for different values of parameters, for different initial conditions etc.

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