Algebraic Design of Controllers for Two-Degree-of-Freedom Control Structure

Radek Matušů, and Roman Prokop

***Abstract*—**The principal aim of this paper is to present an algebraic approach to design of single-input single-output continuous-time controllers under assumption of two-degree-of-freedom (2DOF) closed-loop control system. The method of synthesis is based mainly on description of systems in the ring of proper and Hurwitz-stable rational functions (R_{PS}), Youla-Kučera parameterization of all stabilizing controllers and divisibility conditions in R_{PS}. One of advantages of the proposed approach is the utilization of the single tuning parameter. The work contains the computation and simulation examples which cover the selected cases of only asymptotic tracking of reference signal or also disturbance rejection for first-order and third-order controlled plants.

***Keywords*—**2DOF Control Structure, Algebraic Approach, Controller Design, Linear Systems.

I. Introduction

The control systems with two degrees of freedom (2DOF) contain the controllers with both feedback and feedforward parts. They have substantial advantages in comparison with traditional one-degree-of-freedom (1DOF) configurations, particularly thanks to separation of feedback part (responsible for stabilization and disturbance rejection) and feedforward part (responsible for reference tracking). Thus, the 2DOF structures are very convenient for solving the tasks of disturbance rejection and reference tracking simultaneously [1] – [3].

Quite obviously, the topic of 2DOF control systems and their suitable synthesis has attracted the attention of many researchers during the last years and decades. Among an array of existing methods (see e.g. [1] – [5]), the described algebraic approach [6] – [10] based on the works [11] and [12] represents easy but effective solution with just one tuning parameter influencing the final control response. Furthermore, the problems related to robustness of designed control loops are frequent object of researchers' and engineers' interest [13] – [18].

The main aim of this paper is to present a possible approach based on algebraic tools for designing 2DOF control systems.

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The basic theoretical background is followed by computational and simulation examples where the sets of variously tuned continuous-time controllers are applied to the first and third order controlled plants under assumption of either purely reference tracking problem or reference tracking and load disturbance rejection together. Moreover, the control of the plants with perturbed parameters (gains) was also verified.

The paper is the extended version of the conference contribution [19].

The paper is organized as follows. The key part of the work is the Section 2 which firstly describes a basic theoretical background for an algebraic approach to 2DOF control design then it is divided into three subsections with illustrative examples. They deal successively with controller calculations and various simulations for a first order plant and no disturbances, a first order plant and step-wise load disturbance and a third order plant. Finally, the Section 3 offers some conclusion remarks.

II. Algebraic Approach to 2DOF Control Design

The applied control synthesis is based on the algebraic ideas of Vidyasagar [11] and Kučera [12]. Subsequently, the specific tuning rules has been developed and analyzed e.g. in [6] – [10].

The two-degree-of-freedom (2DOF) closed-loop control system studied in this contribution is shown in fig. 1.

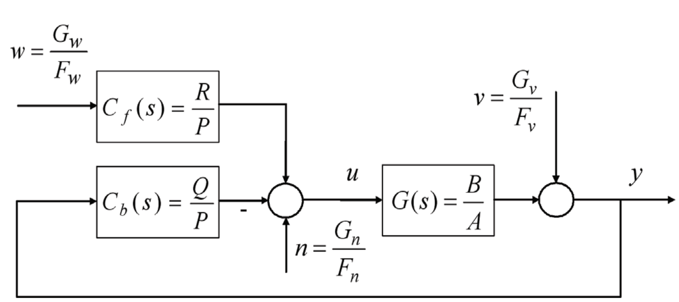

Fig. 1 two-degree-of-freedom control loop

Here, the functions $G(s)$, $C_b(s)$, and $C_f(s)$ represent controlled plant, feedback part of the controller, and feedforward part of the controller, respectively. The signals $w(s)$, $n(s)$, and $v(s)$ then stand for reference, load disturbance, and disturbance signal.

Generally, the control design technique assumes the description of linear systems in fig. 1 not via the common ring of polynomials but in the ring of proper and stable rational functions (R_{PS}). The conversion from the ring of polynomials to R_{PS} can be performed very simply according to:

$$G(s)=\frac{b(s)}{a(s)}=\frac{\dfrac{b(s)}{(s+m)^{\max\{\deg(a),\deg(b)\}}}}{\dfrac{a(s)}{(s+m)^{\max\{\deg(a),\deg(b)\}}}}=\frac{B(s)}{A(s)} \tag{1}$$

$m>0$

The parameter m will be later used as a controller tuning knob.

The fundamental relations resulting from fig. 1 are:

$$\begin{aligned} y(s)&=\frac{B(s)}{A(s)}u(s)+v(s)\\ u(s)&=\frac{R(s)}{P(s)}w(s)-\frac{Q(s)}{P(s)}y(s)+n(s) \end{aligned} \tag{2}$$

Next, the simple adjustment gives:

$$\begin{aligned} y(s)&=\frac{B(s)R(s)}{A(s)P(s)+B(s)Q(s)}\frac{G_w(s)}{F_w(s)}+\\ &+\frac{B(s)P(s)}{A(s)P(s)+B(s)Q(s)}\frac{G_n(s)}{F_n(s)}+\\ &+\frac{A(s)P(s)}{A(s)P(s)+B(s)Q(s)}\frac{G_v(s)}{F_v(s)} \end{aligned} \tag{3}$$

Supposing that no disturbances are affecting the control system (i.e. $n(s)=v(s)=0$), the control error is described by:

$$e(s)=w(s)-y(s)=\left(1-\frac{B(s)R(s)}{A(s)P(s)+B(s)Q(s)}\right)\frac{G_w(s)}{F_w(s)} \tag{4}$$

Assumption of just one-degree-of-freedom (1DOF) control structure, which can be obtained simply by putting $R(s)=Q(s)$, would change the equation (4) to:

$$e(s)=\frac{A(s)P(s)}{A(s)P(s)+B(s)Q(s)}\frac{G_w(s)}{F_w(s)} \tag{5}$$

The primary aim is to guarantee internal stability of the closed-loop system. All stabilizing feedback controllers are given by all solutions of the linear Diophantine equation:

$$A(s)P(s)+B(s)Q(s)=1 \tag{6}$$

with a general solution $P(s)=P_0(s)+B(s)T(s)$, $Q(s)=Q_0(s)-A(s)T(s)$, where $T(s)$ is an arbitrary member of R_{PS} and the pair $P_0(s)$, $Q_0(s)$ represents particular solution of (6). This principle is known as Youla – Kučera parameterization of all stabilizing controllers. More details can be found e.g. in [6], [7], [10]. Consequently, relation (4) takes the form:

$$e(s)=\left(1-B(s)R(s)\right)\frac{G_w(s)}{F_w(s)} \tag{7}$$

and analogically, (5) can be rewritten for 1DOF case as:

$$e(s)=A(s)P(s)\frac{G_w(s)}{F_w(s)} \tag{8}$$

The requirement of zero tracking error:

$$\lim_{t\to\infty}e(t)=\lim_{s\to 0}\left[s\cdot e(s)\right]=0 \tag{9}$$

results in the condition that the expression $F_w(s)$ must disappear from denominator of (7) or (8). Thus, $F_w(s)$ must divide product $A(s)P(s)$ for 1DOF or $F_w(s)$ must divide term $\left(1-B(s)R(s)\right)$ for 2DOF. This condition for 2DOF scenario can be formulated by the second Diophantine equation:

$$F_w(s)Z(s)+B(s)R(s)=1 \tag{10}$$

A. Example 1 – first order plant and no disturbances

The specific process of controller design will be firstly demonstrated on the case of step-wise reference with $F_w(s)=\dfrac{s}{s+m}$, no disturbances, and the first order plant given by transfer function:

$$G(s)=\frac{b_0}{s+a_0} \tag{11}$$

The transposition of the transfer functions in R_{PS} gives the first Diophantine equation (6) as:

$$\frac{s+a_0}{s+m}p_0+\frac{b_0}{s+m}q_0=1 \tag{12}$$

Its particular solution is:

$$p_0=1;\quad q_0=\frac{m-a_0}{b_0} \tag{13}$$

Then, the set of all stabilizing controllers can be expressed by means of Youla-Kučera parameterization:

$$\begin{aligned} P(s)&=p_0+\frac{b_0}{s+m}T(s)\\ Q(s)&=q_0-\frac{s+a_0}{s+m}T(s) \end{aligned} \tag{14}$$

However, the feedback part of the controller in 2DOF configuration calculated only by means of particular solution

(13), which leads to the P controller:

$$C_b(s) = \frac{Q(s)}{P(s)} = \frac{q_0}{p_0} \tag{15}$$

will fulfill the requirement of closed loop stability. Then, the asymptotic tracking of stepwise reference signal can be assured with the assistance of the second Diophantine equation (10), now in the specific form:

$$\frac{s}{s+m} z_0 + \frac{b_0}{s+m} r_0 = 1 \tag{16}$$

with particular solution:

$$r_0 = \frac{m}{b_0}; \quad z_0 = 1 \tag{17}$$

and with general solution (expressed only for the variable $R(s)$ which is useful for control design):

$$R(s) = r_0 + \frac{s}{s+m} \tilde{T}(s) \tag{18}$$

where $\tilde{T}(s)$ is again an arbitrary term in R_{PS}. For example, $\tilde{T}(s) = 0$ leads to the feedforward part of the controller:

$$C_f(s) = \frac{R(s)}{P(s)} = \frac{r_0}{p_0} \tag{19}$$

The final but very important task is the proper choice of the tuning parameter $m > 0$. A possible way of parameter selection for 1DOF configuration based on the requested size of first overshoot of control output is presented e.g. in [8] – [10]. However, this paper will not utilize any exact method for the choice of *m*. It will present more possibilities of *m* with their respective results.

The application of the obtained rules will be demonstrated on the following simulation example. Consider the first order controlled plant given by transfer function:

$$G(s) = \frac{5}{10s+1} = \frac{0.5}{s+0.1} \tag{20}$$

Three controllers were successively tuned by the trio of parameters *m* by using the previously derived rules:

$$m = 0.2 \Rightarrow \quad C_b(s) = 0.2, \quad C_f(s) = 0.4 \tag{21}$$

$$m = 0.5 \Rightarrow \quad C_b(s) = 0.8, \quad C_f(s) = 1 \tag{22}$$

$$m = 1 \Rightarrow \quad C_b(s) = 1.8, \quad C_f(s) = 2 \tag{23}$$

First, the regulators were used for controlling the plant (20) without any disturbances. The results are visualized in fig. 2 which contains the set of output signals and fig. 3 which depicts the set of corresponding control signals (manipulated variables).

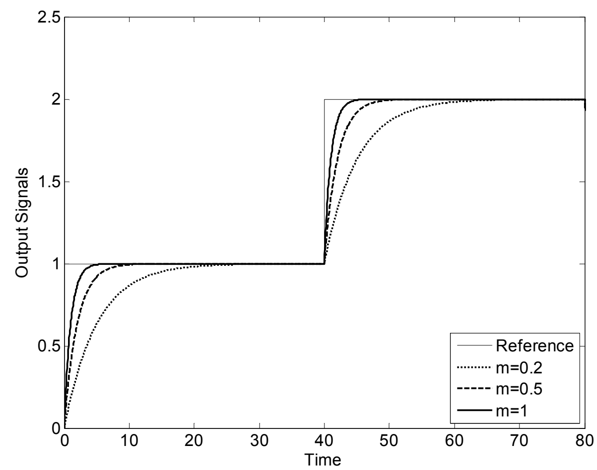

Fig. 2 control of system (20) by controllers (21), (22) and (23) – without disturbances (output signals)

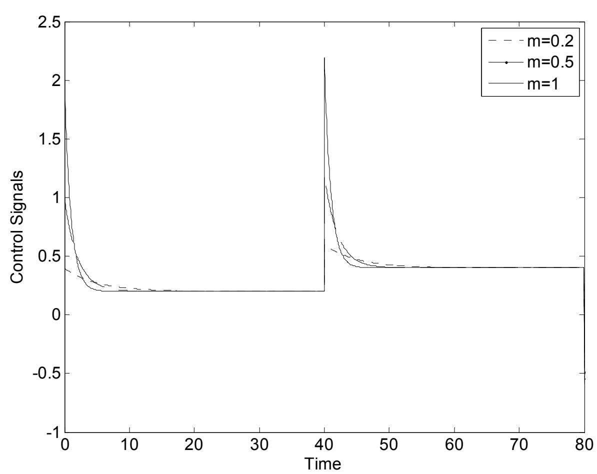

Fig. 3 control of system (20) by controllers (21), (22) and (23) – without disturbances (control signals)

Then, the same controllers were applied, but under assumption of step load disturbance of size -1 which was injected into the output of the controlled plant in 2/3 of simulation time. The results are presented in figs. 4 and 5.

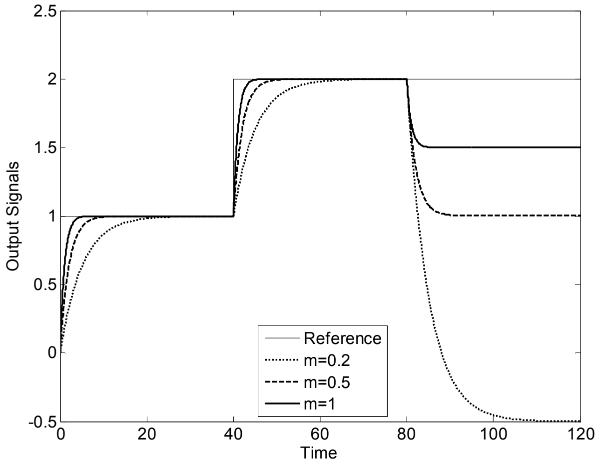

Fig. 4 control of system (20) by controllers (21), (22) and (23) – with step load disturbance -1 (output signals)

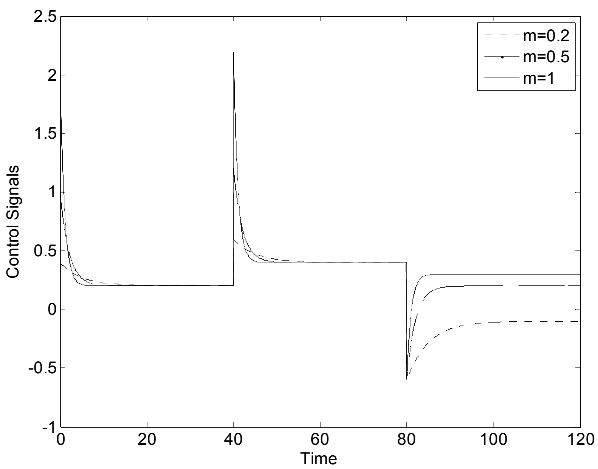

Fig. 5 control of system (20) by controllers (21), (22) and (23) – with step load disturbance -1 (control signals)

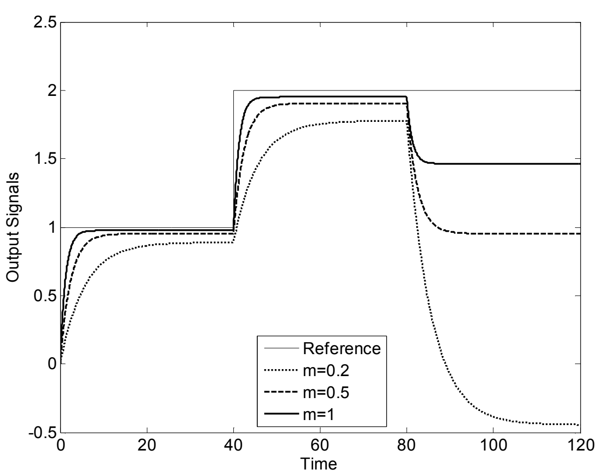

Fig. 6 control of perturbed system (24) by controllers (21), (22) and (23) (output signals)

Finally, the gain of the controlled plant (20) used in the simulation was supposed by 20% lower, i.e.:

$$G_{per}(s)=\frac{4}{10s+1}=\frac{0.4}{s+0.1} \tag{24}$$

The figs. 6 and 7 show the obtained results.

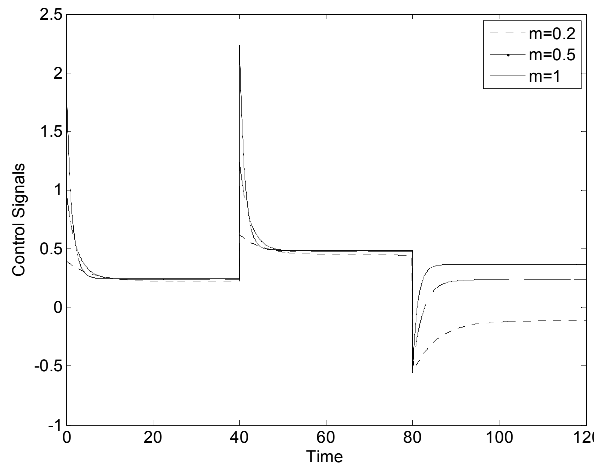

Fig. 7 control of perturbed system (24) by controllers (21), (22) and (23) (control signals)

As it was demonstrated, the application of the controllers (15) and (19) is sufficient for the case of the first order plant (11) with step-wise reference signal but only without any disturbances and without perturbations. However, if some disturbance or change in plant parameters occurs, the controller ability of reference tracking fails. Consequently, the closed control loop remains stable but keeps the permanent control error.

B. Example 2 – first order plant and step-wise load disturbance

Thus, from the practical viewpoint, it is more convenient to design such feedback part of the controller which guarantees not only stability of the circuit in fig. 1, but also rejection of the disturbance (typically e.g. step-wise load disturbance). The appropriate feedback part of the controller from the set (14) can be chosen on the basis of divisibility conditions which are briefly outlined bellow equation (9). Specifically, now the term $F_n(s)=\frac{s}{s+m}$ (for step-wise load disturbance signal) must disappear from the denominator of tracking error:

$$e(s)=B(s)P(s)\frac{G_n(s)}{F_n(s)}+\left(1-B(s)R(s)\right)\frac{G_w(s)}{F_w(s)} \tag{25}$$

In other words, $F_n(s)$ must divide $B(s)P(s)$. So, it has to be found appropriate $T(s)$ in (14). The simple adjustment brings the fact that complying $T(s)$ is the one and only:

$$T(s) = -\frac{m}{b_0} \tag{26}$$

Its substitution into (14) results in the numerator and denominator:

$$P(s) = \frac{s}{s+m}$$
$$Q(s) = \frac{m-a_0}{b_0} + \frac{s+a_0}{s+m}\frac{m}{b_0} = \frac{\frac{2m-a_0}{b_0}s + \frac{m^2}{b_0}}{s+m} \tag{27}$$

which consequently gives the feedback part of the controller in the PI form:

$$C_b(s) = \frac{Q(s)}{P(s)} = \frac{\tilde{q}_1 s + \tilde{q}_0}{s} \tag{28}$$

where

$$\tilde{q}_1 = \frac{2m-a_0}{b_0}$$
$$\tilde{q}_0 = \frac{m^2}{b_0} \tag{29}$$

Then, the corresponding feedforward controller part can be obtained though already derived general solution (18) with r_0 from particular solution (17) and with parameter $\tilde{T}(s) = 0$. The simple calculation leads to:

$$C_f(s) = \frac{R(s)}{P(s)} = \frac{\tilde{r}_1 s + \tilde{r}_0}{s} \tag{30}$$

where

$$\tilde{r}_1 = r_0 = \frac{m}{b_0}$$
$$\tilde{r}_0 = r_0 m = \frac{m^2}{b_0} \tag{31}$$

Notice that all controller parameters again depend on the "tuning knob" $m > 0$, generally in a nonlinear way.

From the practical point of view, the same feedback PI controller (28) with the same parameters (29) would be calculated for only 1DOF configuration under requirement of step-wise reference tracking (without disturbance rejection) because $F_w(s)$ must divide product $A(s)P(s)$.

The next simulation example will show the practical applicability of the upgraded tuning methodology. Again, assume the same first order controlled plant (20). In this case, the three controllers were obtained as follows:

$$m = 0.2 \Rightarrow \quad C_b(s) = \frac{0.6s + 0.08}{s} \quad C_f(s) = \frac{0.4s + 0.08}{s} \tag{32}$$

$$m = 0.5 \Rightarrow \quad C_b(s) = \frac{1.8s + 0.5}{s} \quad C_f(s) = \frac{s + 0.5}{s} \tag{33}$$

$$m = 1 \Rightarrow \quad C_b(s) = \frac{3.8s + 2}{s} \quad C_f(s) = \frac{2s + 2}{s} \tag{34}$$

The corresponding output signals and control signals (manipulated variables) for the scenarios without disturbances, with step-load disturbance -1 affecting the output of the controlled plant, and with system containing perturbed gain are depicted in figs. 8 – 13, respectively.

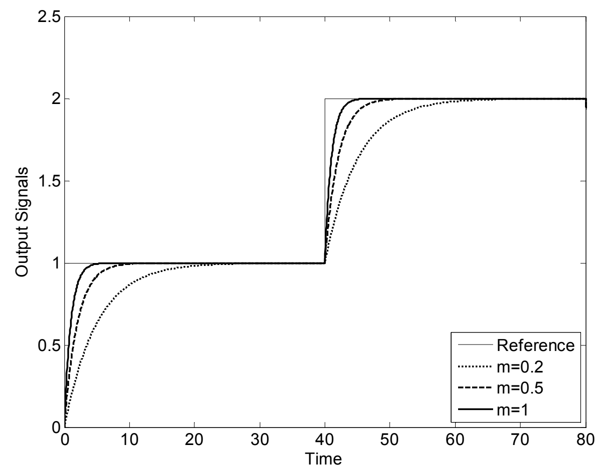

Fig. 8 control of system (20) by controllers (32), (33) and (34) – without disturbances (output signals)

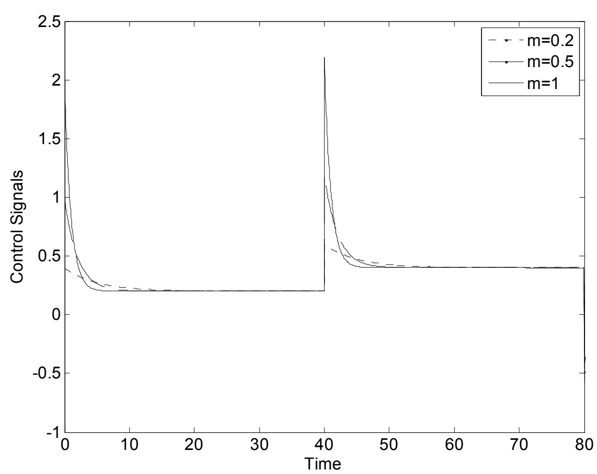

Fig. 9 control of system (20) by controllers (32), (33) and (34) – without disturbances (control signals)

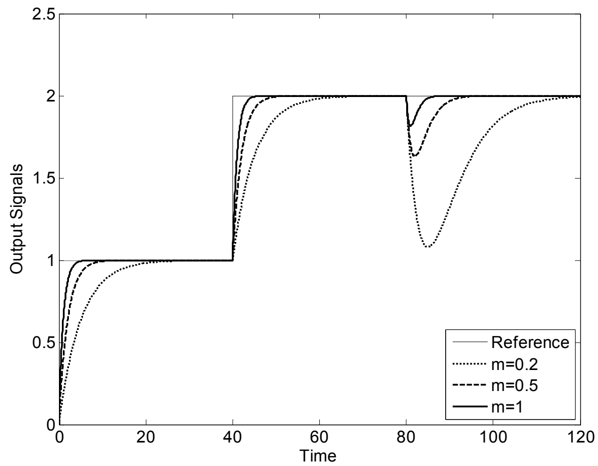

Fig. 10 control of system (20) by controllers (32), (33) and (34) – with step load disturbance -1 (output signals)

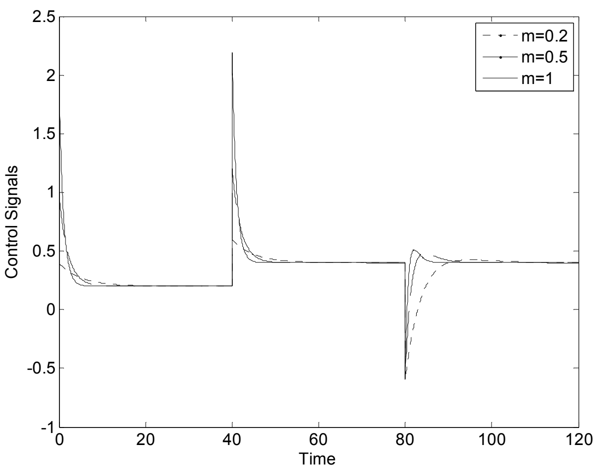

Fig. 11 control of system (20) by controllers (32), (33) and (34) – with step load disturbance -1 (control signals)

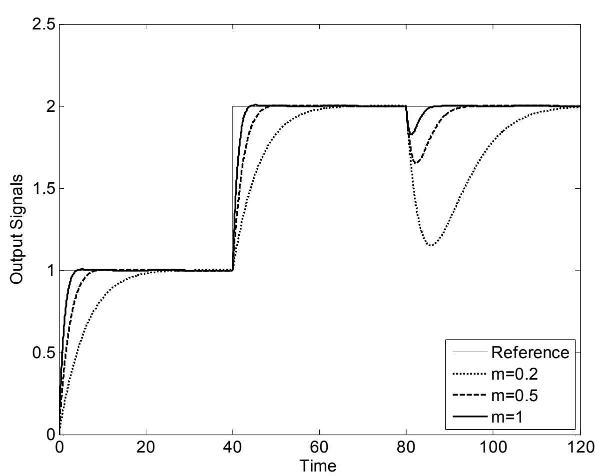

Fig. 12 control of perturbed system (24) by controllers (32), (33) and (34) (output signals)

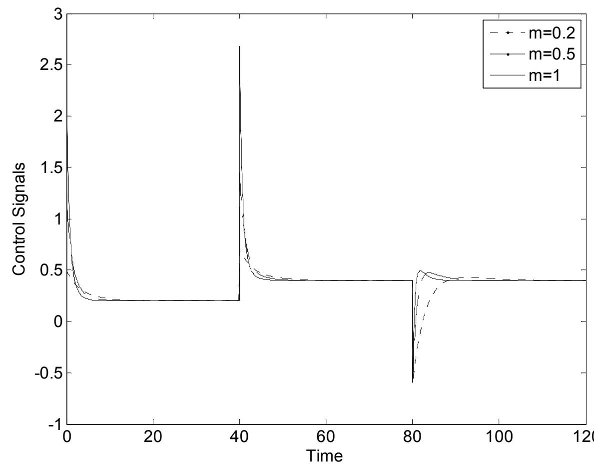

Fig. 13 control of perturbed system (24) by controllers (32), (33) and (34) (control signals)

Note that the control responses from figs. 8 and 9 are exactly the same as the curves from figs. 2 and 3. As can be easily verified, the corresponding closed-loop transfer functions are really identical for this ideal case.

The figs. 10 – 13 clearly prove that the 2DOF controller (28) and (30) is able to deal with step-wise load disturbance and “small” parameter variations in controlled system.

C. *Example 3 – third order plant*

In order to give the outline of a bit more complicated case leading to 2DOF controller with more general transfer functions (not only P or PI), in the final example the same technique is utilized for designing the regulator for the third order plant:

$$G(s)=\frac{2}{(s+1)(2s+1)(5s+1)}=\frac{0.2}{s^3+1.7s^2+0.8s+0.1} \tag{35}$$

The step-wise reference signal and step-wise load disturbance (of size -0.1) are considered. The several choices of tuning parameter lead to:

$$m=0.3 \Rightarrow$$
$$C_b(s)=\frac{-1.43s^3-0.9625s^2-0.1171s+0.0036}{s^3+0.1s^2+0.38s} \tag{36}$$
$$C_f(s)=\frac{0.135s^3+0.1215s^2+0.0365s+0.0036}{s^3+0.1s^2+0.38s}$$

$$m=0.5 \Rightarrow$$
$$C_b(s)=\frac{0.51s^3+1.0775s^2+0.5675s+0.0781}{s^3+1.3s^2+0.74s} \tag{37}$$
$$C_f(s)=\frac{0.625s^3+0.9375s^2+0.4688s+0.0781}{s^3+1.3s^2+0.74s}$$

$$m = 0.7 \Rightarrow$$
$$C_b(s) = \frac{4.25s^3 + 7.5575s^2 + 3.8921s + 0.5882}{s^3 + 2.5s^2 + 2.3} \tag{38}$$
$$C_f(s) = \frac{1.715s^3 + 3.6015s^2 + 2.521s + 0.5882}{s^3 + 2.5s^2 + 2.3}$$

The visualization of output and control signals is provided by fig. 14 and 15.

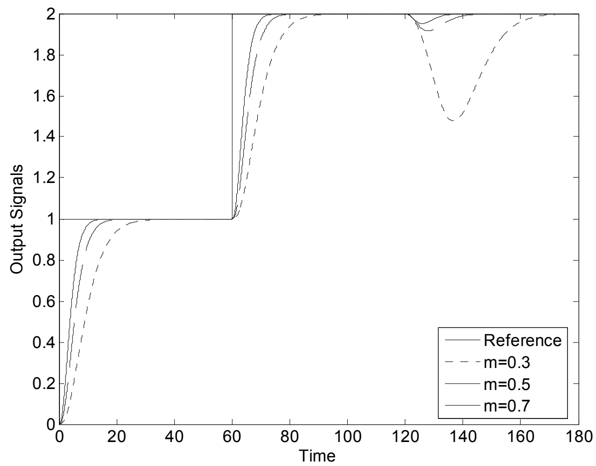

Fig. 14 control of system (35) by controller (36), (37) and (38) (output signals)

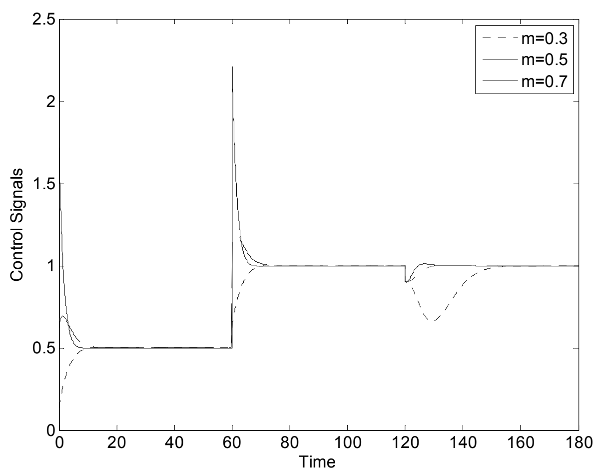

Fig. 15 control of system (35) by controller (36), (37) and (38) (control signals)

Finally, the gain of the controlled plant (21) was supposed by 5% higher, i.e.:

$$G_{per}(s) = \frac{2.1}{(s+1)(2s+1)(5s+1)} = \frac{0.21}{s^3 + 1.7s^2 + 0.8s + 0.1} \tag{39}$$

The figs. 16 and 17 demonstrate the obtained control results.

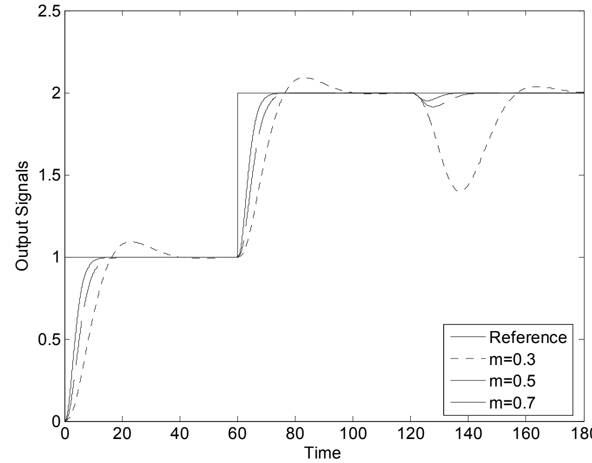

Fig. 16 control of system (39) by controller (36), (37) and (38) (output signals)

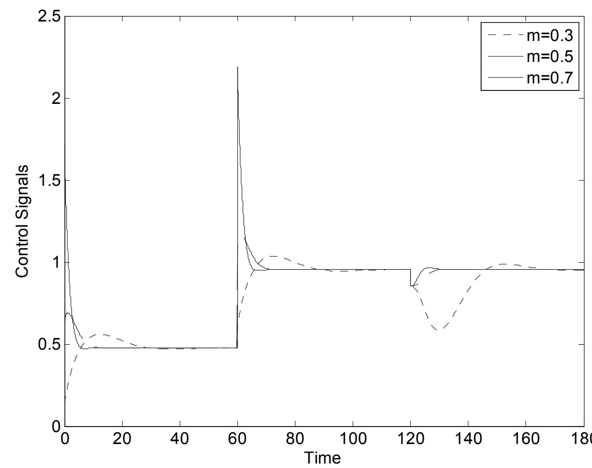

Fig. 17 control of system (39) by controller (36), (37) and (38) (control signals)

III. Conclusion

The paper has been focused on a possible algebraically-based approach for synthesis of 2DOF continuous-time control systems. The easily applicable tuning rules have been fully derived for the first order controlled plant both for just reference tracking or also for disturbance rejection problems. Their effectiveness has been demonstrated on the set of simulation examples for nominal as well as perturbed systems. Moreover, the final example has shown the control experiment for the third order controlled plant using more general third order controllers.

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