The treatment of masonry vaults through the Monge-Ampère Equation

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Abstract—- It is recognized that, apart from a few cases, the No-Tension assumption yields a effective model for structural assessment of masonry structures. The theory is briefly illustrated, and its application to vaults is explained in detail, leading to a Monge-Ampere equation ruling the static regime through a membrane stress surface.

Keywords— Masonry vaults, Stress analysis, No-Tension model, Monge-Ampère equation.

I. Introduction

Masonry is not properly a "material" in the strict sense of the word. It consists in the (generally man-made) assemblage of a basic component (the stones) simply laid on each other or, more often, jointed by mortar. Stones and mortar may have very variable mechanical properties, and the way in which the stones are organized in the masonry volume may (the masonry "texture") may be very different, and is subject to the skill and the creativity of the designer and/or of the builder.

So, "masonry" has not a uniquely defined object, and it is very difficult to set up a mechanical model able to closely reproduce the properties of masonry, fitting all the possible variety of masonry assortment and texture. Anyway, in all structural analyses the engineer is forced to balance the trend to reproduce the material (and consequently the structural behaviour) as closely as possible, with the practical manageability of the analytical tools. Linear theory of structures applied to steel, reinforced concrete and even to masonry, is a successful example of such effort. In all cases the basic theory should include the major features of the behaviour, possibly neglecting many details that poorly influence structural safety assessment, and/or are uncontrollable.

The first step is then to identify the major properties, that are more or less common to all masonry types. The basic knowledge can be achieved through simple experiments. Uni-axial compression/tension tests can be performed on some Representative Volume Element (RVE) of a typical masonry.

After some experiments, it is possible to conclude that: i) the masonry has different elastic moduli in tension (E_t) and compression (E_c); ii) the masonry has different limit stresses in tension (σ_t) and compression (σ_c); iii) the limit stress in tension is much smaller than the limit strength in compression ($\sigma_t << \sigma_c$); iv) the behaviour at failure in compression has some degree of ductility; v) the behaviour at failure in tension is definitely brittle, so tensile strength cannot be recovered absolutely.

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Moreover, surprisingly the limit strength in compression of masonry is larger than the strength of the weak element (the mortar) and is bounded from above by the limit strength of the strong component (the stones); this is due to some complex phenomenon of stress interaction and transverse deformation of mortar with respect to stones. It is also easy to understand that if the axis of the stress is rotated by an angle, say 90°, the results of the experiment may significantly change, in particular as regards the tensile strength. Some similar conclusions can be drawn from biaxial tests (see e.g. [1,2]). Experimental limit strength domains show a high capacity in compression and a very poor limit in tension without ductility.

Summing up, masonry is a non-linear material, strongly hetero-resistant, anisotropic with respect to tensile strength, with compliance coefficients depending on the orientation of the stress axes and different in compression and tension, and with brittle failure at a very low tension threshold.

Therefore, the prevalent feature that characterizes masonry structures, and makes them dissimilar from modern concrete and steel structures, is quite definitely their intrinsic inability to resist tensile stresses. So, it is natural that the material model, that is intended to be an "analogue" of real masonry, in principle cannot resist tensile stress, but, possibly, behaves elastically under pure compression, or plastically if some degree of ductility is ascertained. No-Tension solutions for masonry structures are a very significant reference point and a powerful tool for reliable structural assessment, for many reasons. The first reason is that the NT model is a stable behaviour, poorly subject to uncertainty and aging. Tensile strength is in any case small, uncertain, highly variable in the mass of a structure, not durable in time and so on; anyway neglecting tensile strength leads to a safe assessment. In other words, no doubt that the NT model is a simplified behaviour, that in some cases does not give account of some surprisingly good performance of masonry buildings, but it is also true that if a masonry structure does not pass through a NT check it remains a suspect structure.

The basics for the foundation of a NT material theory are illustrated in [3], where the relevant principles for structural analysis, mainly identified in the classic energy theorems, suitably adapted to the material at hand, are formulated. Limit Analysis theorems and duality principles are discussed in [4].

In (apparently) simple cases, closed-form solutions can be obtained, or, at least, the solution process can be prepared after a preliminary screening of the equilibrium scenario [5].

Wider literature contributed on the subject by the authors and the Naples research group may be found in [6-11].

Masonry elements or components behaving like rigid blocks under dynamic action may be analysed by worst scenario approaches [12-14].

II. Masonry as a No-Tension material

In the previous section it has been recognized that in some cases a tensile capacity along some direction can be attributed to masonry. Anyway, masonry is *brittle* in tension; if in any time in the life of the structure the stress σ exceeds this limit, the bricks crack, and the tensile strength decays to zero. On the other side, there is no doubt that the prevalent feature that characterizes masonry structures, and makes them dissimilar from modern concrete and steel structures, is quite definitely their intrinsic inability to resist tensile stresses. So, it is natural that the material model, that is intended to be an "analogue" of real masonry, cannot resist tensile stress, but, possibly, behaves elastically under pure compression. No-Tension solutions for masonry structures are however a very significant reference point and a powerful tool for reliable structural assessment, for many reasons. The first reason is that the NT model is a stable behaviour, poorly subject to uncertainty and aging. Tensile strength is in any case small, uncertain, highly variable in the mass of a structure, not durable in time and so on; anyway neglecting tensile strength leads to a safe assessment. In other words, no doubt that the NT model is a simplified behaviour, that in some cases does not give account of some surprisingly good performance of masonry buildings, but it is also true that if a masonry structure does not pass through a NT check it remains a suspect structure. In the following the basics for the foundation of a NT material theory are illustrated, and the relevant principles for structural analysis, mainly identified in the classic energy theorems, suitably adapted to the material at hand, are formulated. In (apparently) simple cases, closed-form solutions can be obtained, or, at least, the solution process can be prepared after a preliminary screening of the equilibrium scenario.

A. The standard No-Tension material

In a NT solid the equilibrium against external loads is required to be satisfied by *admissible* stress fields, which imply pure compression everywhere in the solid. Compatibility of the strain field can be ensured by superposing to the elastic strain field an additional *fracture* field, that does not admit contraction in any point and along any direction; that is to say that the stress tensor $\boldsymbol{\sigma}$ must be negative semi-definite everywhere in the solid, while the fracture strain field $\boldsymbol{\varepsilon}_f$ is required to be positive semi-definite.

The material shall, hence, satisfy the following conditions

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_f = \mathbf{C}\boldsymbol{\sigma} + \boldsymbol{\varepsilon}_f \quad ; \quad \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_f \le 0$$
$$Semi-Definite \begin{cases} \boldsymbol{\varepsilon}_f \;\; Positive \\ \boldsymbol{\sigma} \;\; Negative \end{cases} \rightarrow \begin{cases} \varepsilon_{fa} \ge 0 \\ \sigma_a \le 0 \end{cases} \quad \forall a \in r_a \tag{1}$$

where r_a is the set of directions through the generic point in the solid, a is one of such directions, $\boldsymbol{\varepsilon}_f$ is the fracture strain that is assumed to superpose to the elastic strain $\boldsymbol{\varepsilon}_e$ in order to anneal tensile stresses if possible, and $\boldsymbol{C}$ denotes the tensor of elastic constants. Consider moreover that on every elementary surface with normal a, if ε_{fa} is strictly positive σ_a must be zero; by contrast if σ_a is strictly negative, ε_{fa} must be zero. If σ_o is the stress tensor in the point actually associated with fractures $\boldsymbol{\varepsilon}_f$, it follows that

$$\boldsymbol{\sigma}_o \cdot \boldsymbol{\varepsilon}_f = 0 \tag{2}$$

The material admissibility conditions for strain and stress reported in (1) can be synthetically referred to by the set of inequalities $\mathbf{h}_\varepsilon(\boldsymbol{\varepsilon}_f) \ge \mathbf{0}$ and $\mathbf{h}_\sigma(\boldsymbol{\sigma}) \ge \mathbf{0}$ respectively.

As a consequence of (1) and (2), the classical Drucker's rule holds for the fracture strain. With reference to the admissible domains quoted in (1), the normality Drucker's law for no-tension material can thus be written as

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}_o)\cdot\boldsymbol{\varepsilon}_f \le 0 \qquad \forall \boldsymbol{\sigma} \in \Sigma \tag{3}$$

where Σ is the set of admissible stress tensors and $\boldsymbol{\sigma}$ is any admissible stress state other than the effective one $\boldsymbol{\sigma}_o$.

B. Limit Analysis and fundamental theorems

Let consider the body and surface forces, $\mathbf{F}$ acting on volume V and $\mathbf{p}$ acting on the free surface S_p, the displacement field $\mathbf{u}$, the imposed displacement field $\mathbf{u}_o$ characterizing the constrained part of the solid surface S_u, the above mentioned strain field $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_f = \mathbf{C}\,\boldsymbol{\sigma} + \boldsymbol{\varepsilon}_f$, the stress field $\boldsymbol{\sigma}$.

As clear from the above, fracture strains $\boldsymbol{\varepsilon}_f$ can be developed at the considered point only if the stress situation can be represented by a stress tensor $\boldsymbol{\sigma}$ laying on the surface of the material admissibility domain, which is defined for NT bodies by $\mathbf{h}_\sigma(\boldsymbol{\sigma}) \le \mathbf{0}$; obviously if some fracture does exist, it is developed according to the NT material inequalities $\mathbf{h}_\varepsilon(\boldsymbol{\varepsilon}_f) \ge \mathbf{0}$.

B.1.. General setup

Denoting by U the set of possible displacement fields, the class of *fracture admissible mechanisms* is defined by the subset U_f of U containing displacement fields $\mathbf{u}_f$ that are directly compatible with fracture strains $\boldsymbol{\varepsilon}_f$ apart from any elastic strain field

$$\boldsymbol{\varepsilon}_f = \nabla\, \mathbf{u}_f \;\; ; \;\; \mathbf{h}_\varepsilon(\boldsymbol{\varepsilon}_f) \ge \mathbf{0} \tag{4}$$

$$U_f = \{\mathbf{u}_f \in U : \mathbf{h}_\varepsilon(\nabla \mathbf{u}_f) \ge \mathbf{0}\ \} \tag{5}$$

Collapse mechanisms can be defined as fracture admissible mechanisms $\mathbf{u}_f$ such that the mechanical work developed by the applied loads (**p**,**F**) is positive; this condition is analytically expressed by the inequality

$$\int_{S_p} \mathbf{p}\cdot\mathbf{u}_f\, dS + \int_V \mathbf{F}\cdot\mathbf{u}_f\, dV > 0 \tag{6}$$

By the Principle of Virtual Work, a necessary condition for the existence of any admissible stress field $\boldsymbol{\sigma}$ equilibrating the applied loads is that

$$\int_{S_p} \mathbf{p}\cdot\mathbf{u}_f\, dS + \int_V \mathbf{F}\cdot\mathbf{u}_f\, dV = \int_V \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_f\, dV \le 0 \;\forall \mathbf{u}_f \in U_f \tag{7}$$

After (7) one can enounce the "Kinematical Theorem" of Limit Analysis for NT bodies: *if any collapse mechanism exists under the applied loads, no solution can exist for the equilibrium of the NT solid*. In other words: *If any collapse mechanism exists, the solid collapses.*

On the other side, *statically admissible stress fields* $\boldsymbol{\sigma}$ can be defined as tensor fields equilibrating the applied loads and satisfying admissibility conditions, i.e. $\mathbf{h}_\sigma(\boldsymbol{\sigma}) \le \mathbf{0}$ or $\boldsymbol{\sigma}\in\Sigma$, where Σ is the admissible domain, everywhere in the solid. Assuming that under the load pattern (**p**,**F**) a statically admissible stress field $\boldsymbol{\sigma}$ exists for any mechanism $\mathbf{u}_f$, after (7) one gets

$$\int_{S_p} \mathbf{p}\cdot\mathbf{u}_f\, dS + \int_V \mathbf{F}\cdot\mathbf{u}_f\, dV \le 0 \;\; \forall \mathbf{u}_f \in U_f \tag{8}$$

One can, thus, enounce the "Static Theorem" of Limit Analysis for NT bodies: *if under the applied loads any statically admissible stress field* $\boldsymbol{\sigma}$ *exists, no collapse mechanism exists and the structure cannot collapse*.

B.2. The one-multiplier load pattern. The safety factor

Let assume the applied loads as given by the sum of a fixed component ($\mathbf{F}_o$, $\mathbf{p}_o$) and a variable component ($s\mathbf{F}_v$, $s\mathbf{p}_v$) depending on the value assumed by the multiplier s (actually one thus assumes that only the portion $\mathbf{F}_v$, $\mathbf{p}_v$, may be destabilizing and should be controlled)

$$\begin{cases} \mathbf{F} = \mathbf{F}_o + s\mathbf{F}_v = \mathbf{0} & in\ V \\ \mathbf{p} = \mathbf{p}_o + s\mathbf{p}_v & on\ S_p \end{cases} \tag{9}$$

and let define two fundamental classes of load multipliers s for NT bodies: the class of *statically admissible multipliers* β and the class of *kinematically sufficient multipliers* γ. After denoting by $\boldsymbol{\alpha}_n$ the unit outgoing vector normal to the surface S_p, load multipliers β are defined to be statically admissible if the following relations hold

$$\begin{cases} div\, \boldsymbol{\sigma}^\beta + \mathbf{F}_o + \beta\mathbf{F}_v = \mathbf{0} & in\ V \\ \boldsymbol{\sigma}^\beta\, \boldsymbol{\alpha}_n = \mathbf{p}_o + \beta\mathbf{p}_v & on\ S_p \end{cases} \tag{10}$$

$$\mathbf{h}_\sigma(\boldsymbol{\sigma}^\beta) \le \mathbf{0} \tag{11}$$

that is to say, if a stress field $\boldsymbol{\sigma}^\beta$ exists equilibrating the applied loads with s=β and satisfying the NT material admissibility conditions. A stress field satisfying (10) and (11) is qualified as *statically admissible*.

On the other side, load multipliers γ are defined to be *kinematically sufficient* if the following relations hold

$$\begin{cases} \boldsymbol{\varepsilon}_f^\gamma = \nabla\mathbf{u}_f^\gamma & in\ V \\ \mathbf{u}_f^\gamma = \mathbf{0} & on\ S_u \end{cases} \tag{12}$$

$$\mathbf{h}_\varepsilon(\boldsymbol{\varepsilon}_f) \ge \mathbf{0} \tag{13}$$

$$\begin{aligned} &\int_V \mathbf{F}_o\cdot\mathbf{u}_f^\gamma\, dV + \int_{S_p} \mathbf{p}_o\cdot\mathbf{u}_f^\gamma\, dS + \\ &+\gamma\int_V \mathbf{F}_v\cdot\mathbf{u}_f^\gamma\, dV + \gamma\int_{S_p} \mathbf{p}_v\cdot\mathbf{u}_f^\gamma\, dS > 0 \end{aligned} \tag{14}$$

(with ∇ the symmetrical gradient operator), that is to say, if any displacement field $\mathbf{u}_f^\gamma$ exists (a *collapse mechanism*) directly compatible with a NT admissible fracture strain $\boldsymbol{\varepsilon}_f^\gamma$ apart from any elastic strain field, and such that the condition stated by (14) is also satisfied. It is understood that the body is stable under the basic load pattern ($\mathbf{F}_o$, $\mathbf{p}_o$), and that (14) cannot be satisfied by any fracture strain field for $\gamma = 0$.

In other terms it is assumed that the basic loads are suitably chosen in way that they cannot produce collapse.

Extensions to NT continua of the fundamental *static and kinematic theorems* of Limit Analysis allow individuating the value $\bar{s}$ of the load multipliers s, limiting the loading capacity of the body.

On the basis of the *static theorem*, one can state that "*the collapse multiplier* $\bar{s}$ *represents the maximum of the statically admissible multipliers* β"

$$\bar{s} = max\{\beta \in B_o\} \tag{15}$$

where B_o is the class of statically admissible multipliers.

On the basis of the *kinematic theorem*, one can state that "*the collapse multiplier* $\bar{s}$ *represents the minimum of the kinematically sufficient multipliers* γ"

$$\bar{s} = \min\{\gamma \in \Gamma_o\} \tag{16}$$

where Γ_o is the class of kinematically sufficient multipliers.

Thereafter, by means of the static theorem, one can search for the collapse multiplier by implementing the problem

$$\begin{aligned} &Find:\ \max_{\beta,\sigma^\beta}\{\beta\} \\ &Sub \\ &\begin{cases} div\, \boldsymbol{\sigma}^\beta + \mathbf{F}_o + \beta\mathbf{F}_v = \mathbf{0} & in\ V \\ \boldsymbol{\sigma}^\beta\, \boldsymbol{\alpha}_n = \mathbf{p}_o + \beta\mathbf{p}_v & on\ S_p \end{cases} \\ &\mathbf{h}_\sigma(\boldsymbol{\sigma}^\beta) \le \mathbf{0} \end{aligned} \tag{17}$$

Or otherwise, by means of the kinematic theorem, by solving the problem

$$\begin{aligned}
&Find: \min_{\gamma,\varepsilon_f^{\beta},u_f^{\beta}} \{\gamma\} \\
&Sub \\
&\begin{cases} \boldsymbol{\varepsilon}_f^{\gamma} = \nabla \mathbf{u}_f^{\gamma} & in\ V \\ \mathbf{u}_f^{\gamma} = \mathbf{0} & on\ S_u \end{cases} \\
&\begin{cases} \mathbf{h}_{\varepsilon}\left(\boldsymbol{\varepsilon}_f^{\gamma}\right) \geq \mathbf{0} \\ \int_V \mathbf{F}_o \cdot \mathbf{u}_f^{\gamma}\, dV + \int_{S_p} \mathbf{p}_o \cdot \mathbf{u}_f^{\gamma}\, dS + \\ \qquad + \gamma \int_V \mathbf{F}_v \cdot \mathbf{u}_f^{\gamma}\, dV + \gamma \int_{S_p} \mathbf{p}_v \cdot \mathbf{u}_f^{\gamma}\, dS > 0 \end{cases}
\end{aligned} \tag{18}$$

C. Variational principles for the NT equilibrium problem

Analysis of no-tension structures proves that stress, strain and displacement fields obey extremum principles of the basic energy functionals. Therefore the solution displacement and fracture strain fields are found as the constrained minimum of the *Potential Energy* functional, under the condition that the fracture field is positively semi-definite at any point. In other words, if **ε** and **u** are respectively the strain and the displacement fields such that

$$\boldsymbol{\varepsilon} = \nabla \mathbf{u} \tag{19}$$

and **p**, **F** are the surface tractions and the body forces, it is possible to write down the Total Potential Energy (TPE) functional

$$\begin{aligned} \mathsf{E}\left(\mathbf{u},\boldsymbol{\varepsilon}_f\right) &= \frac{1}{2}\int_V \left(\nabla\mathbf{u} - \boldsymbol{\varepsilon}_f\right)\left[\mathbf{D}\left(\nabla\mathbf{u} - \boldsymbol{\varepsilon}_f\right)\right] dV + \\ &- \int_{S_p} \mathbf{p}\cdot\mathbf{u}\, dS - \int_V \mathbf{F}\cdot\mathbf{u}\, dV = \mathrm{L}\left(\mathbf{u},\boldsymbol{\varepsilon}_f\right) + \mathrm{P}(\mathbf{u}) \end{aligned} \tag{20}$$

with **D** the inverse tensor of **C**. The TPE functional $\mathsf{E}(\mathbf{u}, \boldsymbol{\varepsilon}_f)$ is made up by two terms, expressing the energy stored in the body $L(\mathbf{u}, \boldsymbol{\varepsilon}_f)$ and the opposite of the work made by the applied loads $P(\mathbf{u})$. It can be proved that the solution $\mathbf{u}_o$, $\boldsymbol{\varepsilon}_{fo}$ satisfies the following condition

$$\mathsf{E}\left(\mathbf{u}_o,\boldsymbol{\varepsilon}_{fo}\right) = \min_{\substack{\mathbf{u}\in \mathrm{U} \\ \boldsymbol{\varepsilon}_f \in \Phi}} \mathsf{E}\left(\mathbf{u},\boldsymbol{\varepsilon}_f\right) = \mathsf{E}_o \tag{21}$$

which is the minimum of the Potential Energy, conditioned upon admissibility of the fracture strain, with Φ the set of admissible fracture fields. Despite the quadratic functional $L(\mathbf{u}, \boldsymbol{\varepsilon}_f)$ is positive definite, the minimum may be not unique if some mechanism exists such that $P(\mathbf{u}) = 0$.

The stress field can be found, in turn, as the constrained minimum of the Complementary Energy (CE) functional, under the condition that the stress field is in equilibrium with the applied loads and compressive everywhere. In other words, let

$$\begin{aligned} &\mathsf{S}(\boldsymbol{\sigma}) = \frac{1}{2}\int_V \boldsymbol{\sigma}\cdot\mathbf{C}\boldsymbol{\sigma}\, dV - \int_{S_u} \mathbf{T}\cdot\mathbf{u}_o\, dS = L_c(\boldsymbol{\sigma}) + R(\boldsymbol{\sigma}) \\ &\mathbf{T} = \boldsymbol{\sigma}\boldsymbol{\alpha}_n \end{aligned} \tag{22}$$

be the CE functional $\mathsf{S}(\boldsymbol{\sigma})$ defined on the set $\Sigma_{\mathbf{o}}$ of the admissible stress fields ($\Sigma_{\mathbf{o}} \subseteq \Sigma$) in equilibrium with the applied loads, with $L_c(\boldsymbol{\sigma})$ the complementary energy stored in the body and $R(\boldsymbol{\sigma})$ the work by the reactions times the settlements of the constrained points. It can be proved that, if $\boldsymbol{\sigma}_{\mathbf{o}}$ is the solution stress field, the following condition holds

$$\mathsf{S}(\boldsymbol{\sigma}_o) = \min_{\sigma\in\Sigma_o} \mathsf{S}(\boldsymbol{\sigma}) = \mathsf{S}_o \tag{23}$$

Equation (23) expresses the compatibility condition on the solution stress field, i.e. the constrained minimum of $\mathsf{S}(\boldsymbol{\sigma})$ yields the stress field $\boldsymbol{\sigma}_o$ such that the elastic strains $\mathbf{C}\boldsymbol{\sigma}_o$ can be made compatible with a continuous displacement field, by the superposition of a fracture strain field. Since $\boldsymbol{\sigma}\cdot\mathbf{C}\boldsymbol{\sigma}$ is positive definite in Σ, the solution is unique.

D. Convexity of the energy functionals and Limit Analysis as a tool for existence of the solution.

It is easy to prove that the Total Potential Energy and the Complementary Energy functionals are both defined on convex sets. The respective sets of definition are: i) the space of couples $(\mathbf{u},\boldsymbol{\varepsilon}_f)$, with **u** a displacement vector function compatible with the external constraints and $\boldsymbol{\varepsilon}_f$ a semi-positively valued tensor field; ii) the space of semi-negatively valued tensor fields in equilibrium with the applied loads. Because of convexity, both minima exist if the respective definition sets, $U\times\Phi$ on one side and Σ_o on the other side, are not empty. For Σ_o to be not empty it is necessary and sufficient that the structure is under the collapse threshold; in this case a unique minimal point exists for $\mathsf{S}(\boldsymbol{\sigma})$. *U* and Φ are intrinsically not empty, in that the first is the set of three-components vector fields and the second is the space of semi-positive definite 3rd-order tensor fields. The displacement/fracture solution may be not unique if a mechanism exists such that the external work is zero. Anyway, if any collapse mechanism exists, $\mathsf{E}(\mathbf{u},\boldsymbol{\varepsilon}_f)$ diverges, and the minimum does not exist.

It can be concluded that the solution of the NT equilibrium problem exists iff the structure is under the collapse threshold.

III. The NT solution for masonry arches and barrel vaults

NT solutions have been investigated since many years and many results have been produced, also yielding successful comparison with experimental results. As an example, consider the portal arch in Fig. 1a, that has been tested under an horizontal force acting on top of the right pillar, as in Fig. 1b. In Fig. 2 the comparison between the experimental and numerical results is plotted, proving a very good agreement of the NT theory with practice.

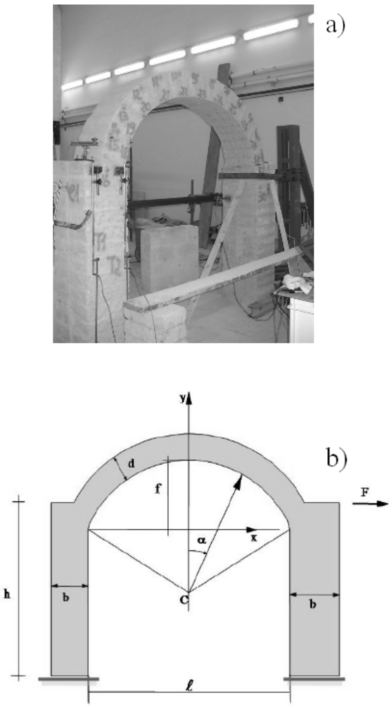

Fig. 1: Experimental and mechanical model: a) The laboratory arch-portal tested under the horizontal force F; b) The mechanical NT model for calculations

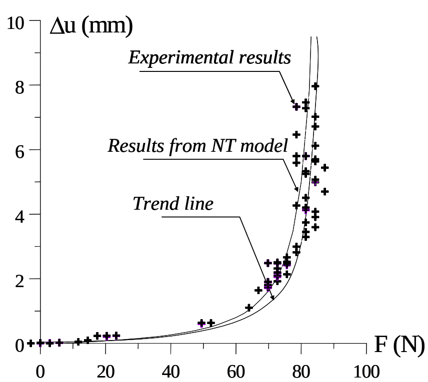

Fig. 2: Experimental and mechanical model: Plot of numerical and experimental results

It is recognized that the NT model is generally capable to find solutions also yielding credible provisions for the fracture distribution, in agreement with technical expectation, as it can be recognized observing the two-span arch-bridge in Fig. 3, subject to a vertical downward settlement of the central pier.

More details on NT arches and vaults theoretical and experimental assessment can be found in [15].

It should be remembered however that in some cases, due to particular masonry texture, a significant degree of tensile capacity can be attributed to masonry, thus explaining some surprisingly good performance of some structural pattern, like e.g. the so called *cantilever stairs* [16-18].

Reinforcement and safe consolidation of structures requiring additional tensile strength to be stable can be provided today by the new technology of Fiber Reinforced Composites (FRP) [19] and the effectiveness can be checked by implementing mechanical models coupling the basic masonry as a NT material with tension-resistant FRP strips [20-22].

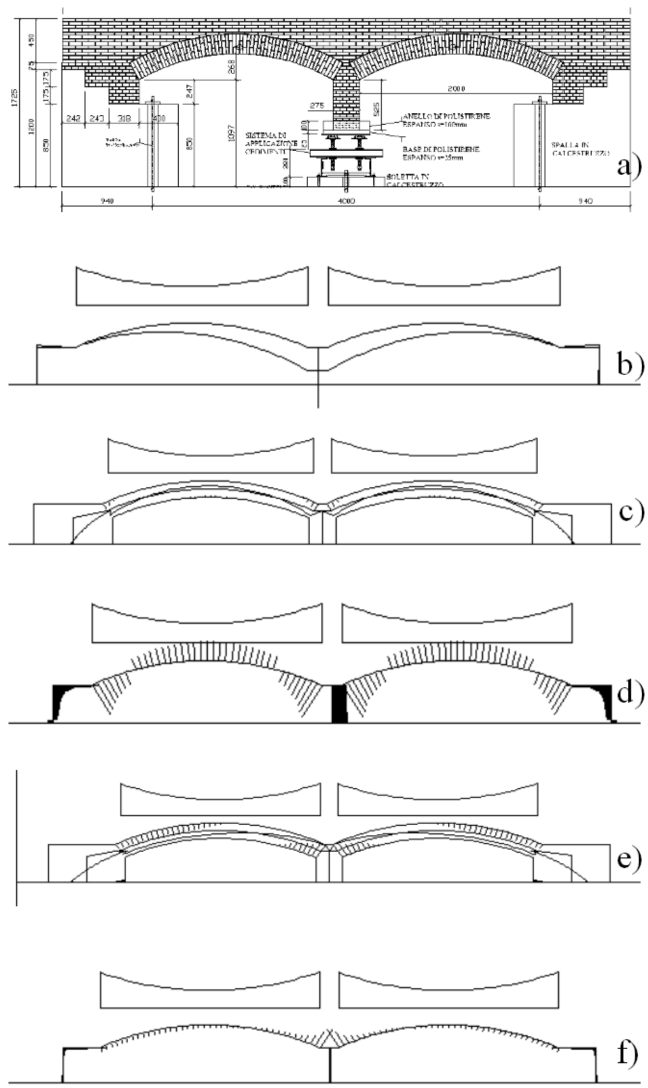

Fig. 3: Sample results from NT model:
a) The arcade and the experimental set up;
b) The arcade in the original configuration and with downward settlement of the central pier;
c) Pressure line and fractures without settlement;
d) Stresses without settlement;
e) Pressure line and fractures with settlement;
f) Stresses with settlement

IV. No-Tension model for masonry-like shells and domes

Equilibrium fields for No-Tension vaults can be built upon the assumption that a membrane stress-surface is considered, included in the profile of the vault, displaying compressive forces along all directions. The idea is not new, it was put forward by Heyman [23], but there are few doubts that it represents a powerful approach to search stresses in masonry vaults, as the 3D direct counterpart of the traditional historical method based on the funicular line of the loads in 2D structural problems (Fig. 4).

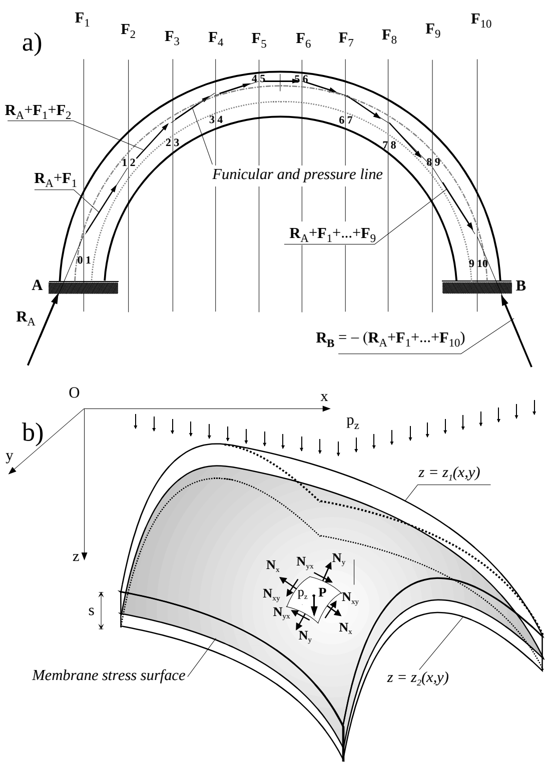

Fig. 4: a) Arch equilibrium analysis: the pressure line;
b) Vault equilibrium analysis: the membrane stress surface

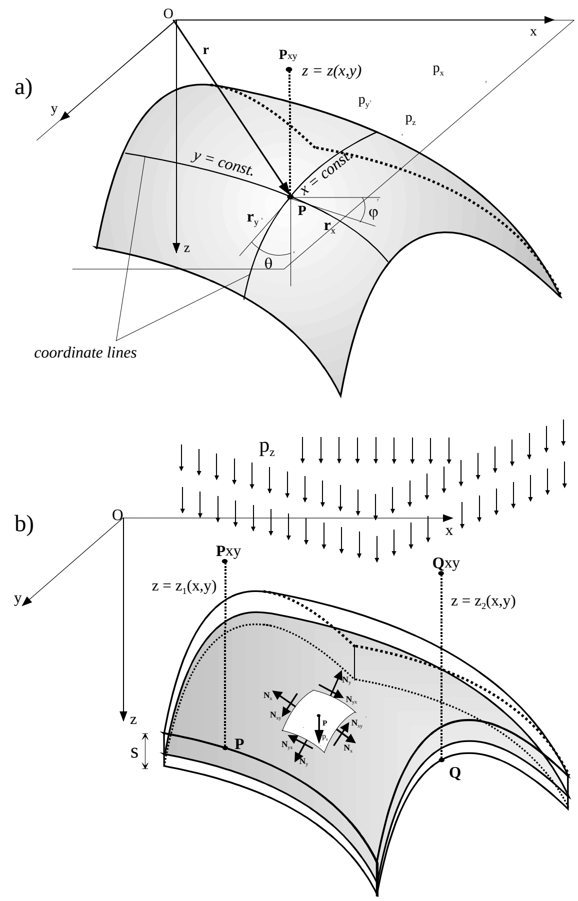

Fig. 5: a) The membrane surface.
b) The vault and its mid-surface under purely vertical loads.

Modelling a large variety of equilibrium stress fields is the preliminary step to find a final solution, yielding a credible pattern.

In the following an approach is outlined to identify membrane stress surfaces both responding to the requirements of stress admissibility and equilibrium with active loads.

It has been proved in [9] that given a vaulted masonry roofing whose extrados and intrados surfaces are respectively

$$z = z_1(x,y) \quad ; \quad z = z_2(x,y) \tag{24}$$

the set of functions z(x,y) producing non-negative output of the Hessian operator

$$\mathcal{H}_z(x,y) = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left[\frac{\partial^2 z}{\partial x\,\partial y}\right]^2 = \frac{\overline{p}_z}{Q} \geq 0 \tag{25}$$

with

$$z_1(x,y) \leq z(x,y) \leq z_2(x,y) \tag{26}$$

yields a set of NT equilibrium vertical load patterns that are admissible for the vault.

The relevant internal forces are as in the classical Pucher approach

$$N_x = \overline{N}_x \frac{\cos\theta}{\cos\varphi}, \quad N_y = \overline{N}_y \frac{\cos\varphi}{\cos\theta}, \quad N_{xy} = \overline{N}_{xy} \quad (27)$$

with the angles θ and φ as in Fig 5a and the internal forces are as in Fig. 5b

$$\begin{aligned} \overline{N}_x &= -\frac{1}{2} Q \frac{\partial^2 z}{\partial y^2} \\ \overline{N}_y &= -\frac{1}{2} Q \frac{\partial^2 z}{\partial x^2} \\ \overline{N}_{xy} = \overline{N}_{yx} &= \frac{1}{2} Q \frac{\partial^2 z}{\partial x\, \partial y}. \end{aligned} \quad (28)$$

where Q is the *thrust factor*, and the admissible load pattern $\overline{p}_z$ is projection onto the plane xy of p_z, i.e. the actual vertical load acting on the vault.

The function z(x,y) is the *membrane surface* yielding the internal forces and equilibrating the applied load p_z.

Note that (25) includes the condition that both second derivatives of the membrane function z(x,y) are jointly positive or negative definite. Any load $\overline{p}_z$ can be reproduced by the function z(x,y) and by its opposite –z(x,y). One of them, say z(x,y), has positive second derivatives, yielding a convex function, the other has negative second derivatives, yielding a concave function. If $z_1(x,y)$ and $z_2(x,y)$ are convex, as is the case if they pertain to a masonry vault, inequalities in (3) select the convex membrane. In this case (28) yields a internal force field that is characterized by pure compression thus producing a NT solution.

$$\left.\begin{aligned} N_x &\le 0 \\ N_y &\le 0 \\ N_{xy}^2 - N_x N_y &\le 0 \end{aligned}\right\rangle \Leftrightarrow \left\langle\begin{aligned} \overline{N}_x &\le 0 \\ \overline{N}_y &\le 0 \\ \overline{N}_{xy}^2 - \overline{N}_x \overline{N}_y &\le 0 \end{aligned}\right. \quad (29)$$

In other words, any equilibrium stress field is intrinsically admissible. This result supports the statement by Thomas Young (reported in [24]) namely: "*The construction of the dome is less difficult than that of an arch since the tendency of each arch to fall is counteracted not only by the pressure of the parts above and below but also by the resistance of those which are situated on each side*…..".

Summing up, solutions of the basic Vault Inequality System (25-26) define convex functions $z(x,y)$ included in the profile of the vault, and enjoy the following property [25]: If $z_i(x,y)$ ($i = 1,\ldots,N$) are N functions, each verifying the possibly homogeneous equation in (25), any convex combination of such functions also yields a member of the solution set of the vault inequality with the strict inequality sign. The homogeneous VIS is the same as (25) with the sign of equality in the first row ($\overline{p}_z = 0$).

V. Equilibrium of vaults through the Monge-Ampère equation

If one aims at finding admissible stress fields in equilibrium with a given load pattern $p_z(x,y)$, one can search for solutions of the equation (2) rewritten in the form

$$Q \cdot \mathcal{H}_z(x,y) - \overline{p}_z(x,y) = 0 \quad (30)$$

under the condition (26), with Q a positive factor. Equation (30) represents the simplest form of the well known Monge-Ampère equation [26], and quite clearly plays a key role in the statics of NT vaults [25].

It is possible to build up a number N of solutions $z_i(x,y)$ ($i = 1,\ldots,N$) of the homogeneous or non-homogeneous VIS, so that solutions of the z-equilibrium equation (30) can be searched in the form

$$z(x,y|c_i) = \sum_{i=1}^{M} c_i z_i(x,y) \quad (31)$$

where each of the basic functions $z_i(x,y)$ is assumed to comply with the VIS (30), in homogeneous or non-homogeneous form. Coefficients c_i yielding, possibly approximate, solutions of (30) correspond to the minimum of the error function

$$\mathcal{E}_o(c_i,Q) = \int_X \left[Q\mathcal{H}_z(x,y \mid c_i) - \overline{p}_z(x,y)\right]^2 dxdy \quad (32)$$

All load patterns $\overline{p}_z(x,y)$ such that coefficients c_i resulting in $\mathcal{E}_o=0$ exist, are called *manageable load patterns* with respect to the assumed form for the function $z(x,y|c_i)$. If the applied load is manageable, equilibrium can be exactly satisfied. Otherwise, equilibrium can be approximately verified, to some extent, depending on the choice of the basic functions $z_i(x,y)$, and the load pattern is qualified as *non-manageable*.

Details and applications are illustrated in [9,25]. Interaction with the reciprocal problem, i.e. No-Compression double curvature structures, can give fruitful contributes to new developments (see e.g. [27]).

VI. Conclusions

Historical masonry vaults and/or cupolas exhibit a large variety of typological assets. Often masonry is well operated, with strong stones and effectively adhesive mortar; in many cases masonry is in worse working order; in other cases a poor masonry is encountered.

Anyway, double-curvature structures can appeal to many equilibrium patterns to sustain at least their own weight plus some light additional loads. So they are, in general, stable systems, provided that their supports are strong and able to contrast thrust forces. Vaults are in general characterized by their shape, and a lot of types can be listed (see e.g. [28]), that have been conceived to be included in any simple or complex architectural design. But the equilibrium paths are also driven by the way masonry is interwoven. In some cases, a masterly design of the masonry tissue and of the vault apparatus may help in improving the structure's stability, and sometimes even in preventing fractures. It should be realized, by contrast, that fractures are almost always a physiological feature of masonry; since almost always it does not possess significant tensile strength, it cannot expand by tension and, when necessary to comply with congruence of the overall deformation, dilatation is provided by fractures.

Anyway, the poor consistency of the tensile resistance of the masonry material, its brittle, desultory and time-aging character, the difficulty in identifying non-zero reliable values, possibly led the ancient builders to introject empirical rules aiming, more or less consciously, at organizing structures in way that they are able to equilibrate loads without needing tensile stresses in the material. A susceptibility that has not been disproved by any analysis performed by modern powerful theoretical, numerical and electronic equipments. After a similar survey, Heyman, in 1966 [29], demonstrated that the failure of the masonry structures was substantially due to the activation of a collapse mechanism, rather than to the probability of crushing in compression.

Many efforts have been devoted to approach masonry structures, both on the side of Anelastic Equilibrium and of Limit Analysis, and many FEM models have been and are currently attempted (see e.g. [30-32]). Actually by referring to the collapse condition, one can just obtain some indications about the safety margins, whilst nothing about the fracture distribution or the behaviour evolution with increasing loads can be predicted.

Under such perspective, even if the NT model still represents an idealization of the real behaviour, one can follow the fracture evolution, assuming the small localized fractures as a phenomenological feature of the masonry material, besides the cases when the crack situation is such to compromise the local material resistance or to activate a collapse mechanism.

Definitely, the solution of the structural problem is based on a suitable re-formulation of the energetic theorems, which, by reflecting the non-linear character of the mechanical model, translates into constrained extremum principles the ordinary conditions of stress equilibrium on one side, and of strain congruence on the other side. In such a way, the final state of the structural solid under different load levels can be identified up to the collapse situation, which can be predicted, as mentioned in the above, by means of the fundamental theorems of Limit Analysis, suitably re-formulated. As far as two-dimensional structural systems are concerned (walls, arches, plane models of masonry bridges and so on), and some their combinations, theory and practice are in a well established state of the art.

Application to the statics of vaults and cupolas is today largely investigated by many authors. The present paper is far from aiming at an exhaustive review, and only a few papers are referenced here, among the manifold that would deserve to be mentioned. It is clear however that the problem is much harder than for plane structures; in this regard it may be enough to consider the intrinsic difficulties to identify collapse mechanisms in applying the kinematic approach of Limit Analysis to double-curvature vaults.

A common feature, however, is that in most cases the main objective is to extend to double-curvature roofing the methods that have historically developed with reference to single curvature arches or analogues, to find admissible stress distributions. The approach illustrated in Secs. 3-4, originally elaborated by the writers, aims at this purpose, on one side providing solution to the collapse problem from the point of view of the static theorem, and on the other side solving the preliminary step to find solutions including compatible strain and fracture fields in agreement with field engineering surveys.

The Monge-Ampere equation, introduced in Sec. 4, is essentially the double-curvature counterpart of the equation of the funicular line. The equation has widely been investigated by mathematicians; nevertheless only a few solutions are available in the literature, so that solutions effective for the problem at hand shall be sought by specific and/or numerical methods. A Ritz-Galerkin type approach, requiring a previous identification of a number of basic functions, has been specifically issued in Sec. 4, proving its manageability in practical examples.

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References

[1] Hegemier G.A., Nunn R.O., Arya S.K.: Behaviour of concrete masonry under biaxial stresses In: *Proc. North American Masonry Conf.*, University of Colorado, Boulder, U.S.A., paper 1(1978).

[2] Page A..W.: The biaxial compressive strength of brick masonry. *Proc. Instn. of Civ. Engrs*, Part 2, vol.71, pp. 893-906(1981).

[3] Baratta A., Corbi O.: An Approach to Masonry Structural Analysis by the No-Tension Assumption - Part I: Material Modeling, Theoretical Setup, and Closed Form Solutions. *Applied Mechanics Reviews*, vol.63(4), pp.040802-1/17, ISSN: 0003-6900, DOI:10.1115/1.4002790 (2010).

[4] Baratta, A., Corbi, O.: Relationships of L.A. Theorems for NRT Structures by Means of Duality, .Intern. *Journal of Theoretical and Applied Fracture Mechanics*, Elsevier Science, vol. 44, pp. 261-274. TAFMEC. doi:10.1016/j.tafmec.2005.09.008 (2005).

[5] Baratta A., Corbi O.: An Approach to Masonry Structural Analysis by the No-Tension Assumption - Part II: Load Singularities, Numerical Implementation and Applications. *Applied Mechanics Reviews*, vol.63(4), pp. 040803-1/21, ISSN 0003-6900, DOI:10.1115/1.4002791 (2010).

[6] Baratta, A., Corbi, O.: On Variational Approaches in NRT Continua. Intern. Journal of Solids and Structures, Elsevier Science, 2005. Vol. 42, pp. 5307-5321. ISSN: 0020-7683. DOI:10.1016/j.ijsolstr.2005.03.075 (2005).

[7] Baratta, A., Corbi, I.: Iterative procedure in no-tension 2D problems: theoretical solution and experimental applications, In: G.C. Sih and L. Nobile Eds., In: Proceedings of the International Conference on Restoration, Recycling and Rejuvenation Technology for Engineering and Architecture Application; Cesena, Code64053, 7-11 June 2004, pp. 67-75. ISBN: 8879997653 (2004).

[8] Baratta, A., Corbi, O.: Duality in non-linear programming for limit analysis of NRT bodies, Structural Engineering and Mechanics, An International Journal, Technopress. vol. 26, no. 1, pp.15-30, 2007.

ISSN: 1225-4568 (2007).

[9] Baratta, A., Corbi, O.: On the equilibrium and admissibility coupling in NT vaults of general shape, Int J Solids and Structures, 47(17), 2276-2284. ISSN: 0020-7683. DOI: 10.1016/j.ijsolstr.2010.02.024 (2010).

[10] Baratta, A., Corbi, I.: Plane of Elastic Non-Resisting Tension Material under Foundation Structures. International Journal for Numerical and Analytical Methods in Geomechanics, vol. 28, pp. 531-542, J. Wiley & Sons Ltd. ISSN 0363-9061, DOI: 10.1002/nag.349 (2004).

[11] Baratta, A., Corbi, I.: Spatial foundation structures over no-tension soil. International Journal for Numerical and Analytical Methods in Geomechanics, vol. 29, pp. 1363-1386, Wiley Ed. ISSN: 03639061, DOI: 10.1002/nag.464 (2005).

[12] Baratta, A., Corbi, I., Corbi O.: Towards a Seismic Worst Scenario Approach for Rocking Systems. Analytical and Experimental Set Up for Dynamic Response, Journal Acta Mechanica, vol.224 (4) , pp. 691-705, ISSN: 0001-5970, DOI:10.1007/s00707-012-0787-9 (2013).

[13] Baratta , A., Corbi, O.: Analysis of the dynamics of rigid blocks using the theory of distributions, Journal of Advances in engineering Software, vol. 44(1), pp.15-25, ISSN: 09659978, DOI: 10.1016/j.advengsoft.2011.07.008 (2012).

[14] Baratta, A., Corbi, I., Corbi, O., Barros, R.C., Bairrão, R.: Shaking Table Experimental Researches Aimed at the Protection of Structures Subject to Dynamic Loading, Open Construction and Building Technology Journal, vol.6, pp.355-360, ISSN: 1874-8368, DOI:10.2174/1874836801206010355 (2012).

[15] Baratta, A., Corbi, I., Corbi, O.: Stress analysis of masonry structures: Arches, walls and vaults, Proceedings of the 6th International Conference on Structural Analysis of Historic Construction: Preserving Safety and Significance, SAHC08;Bath;2 July -2008; Vol.1, pp. 321-329, ISBN: 0415468728;978-041546872-5 (2008).

[16] Baratta, A., Corbi, I.: Statics and Equilibrium Paths of Masonry Stairs, Open Construction and Building Technology Journal, vol.6, pp.368-372, ISSN: 1874-8368, DOI: 10.2174/1874836801206010368 (2012).

[17] Baratta, A., Corbi, I.: Equilibrium models for helicoidal laterally supported staircases, Journal of Computers and Structures, ISSN: 00457949, DOI: 10.1016/j.compstruc.2012.11.007 (2013).

[18] Baratta, A., Corbi, I.: On the Statics of Masonry Helical Staircases, in B.H.V.Topping, Y. Tsompanakis, (Editors), Proceedings of the Thirteenth International Conference on Civil, Structural and Environmental Engineering Computing, Civil-Comp Press, Stirlingshire, UK, Crete;6 -9 September 2011, Paper 59, 16p, ISBN: 978-190508845-4, DOI:10.4203/ccp.96.59 (2011).

[19] Maceri F. (Ed.): *Guide for Design and Construction of Externally Bonded FRP Systems for Strengthening Existing Structures*, Advisory Committee on Technical Recommendations for Construction, National Research Council of Italy (CNR), Roma, 2006, pp. 144

[20] Corbi, I: FRP reinforcement of masonry panels by means of c-fiber strips", *Composites Part B*, vol. 47, pp. 348-356, ISSN: 1359-8368, DOI: 10.1016/j.compositesb.2012.11.005 (2013).

[21] Corbi, I.: FRP Composite Retrofitting for Protection of Monumental and Ancient Constructions, *Open Construction and Building Technology Journal*, vol. 6, pp.361-367, ISSN: 1874-8368, DOI: 10.2174/1874836801206010361 (2012).

[22] Baratta, A., Corbi, O.: An approach to the positioning of FRP provisions in vaulted masonry structures, Composites Part B: Engineering, doi.org/10.1016/j.compositesb.2013.04.043, (2013).

[23] Heyman, J.: *Equilibrium of shell structures*, Oxford University Press, Oxford, pp. 134 (1977).

[24] Cowan H.J.: A history of masonry and concrete domes in building construction. *Building and Environment*. **12**, Pergamon Press, 1977, pp.1-24

[25] Baratta, A., Corbi, O.: On the statics of No-Tension masonry-like vaults and shells: solution domains, operative treatment and numerical validation, *Annals of Solid and Structural Mechanics*, Vol.2(2-4), pp.107-122, DOI: 10.1007/s12356-011-0022-8, ISSN 1867-6936 (Print) 1867-6944 (Online), (2011).

[26] Gilbarg D., Trudinger N. S.: *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, 2nd Edition, 2001, pp. 544

[27] Zhang J.Y., Ohsaki M.: Adaptive force density method for form-finding problem of tensegrity structures. *International Journal of Solids and Structures*, **43**, 2006, pp. 5658–5673

[28] Donghi D.: *Manuale dell' Architetto* (Italian), Vol. I, Part I. Ed. UTET,, Torino (1906).

[29] Heyman, J.: The stone skeleton. *Journal of Solids and Structures*, vol.2, , pp. 269-279 (1966).

[30] Szolomicki, J. P.: Structural behaviour of masonry vaults. In Gürlebeck K. and Könke C. (eds.): *18th Int. Conf. on the Application of Computer Science and Mathematics in Architecture and Civil Engineering*, Weimar, 2009, Germany (2009).

[31] Brencich A., Gambarotta L., Ghia A.: Structural models for the assessment of the masonry dome of the Basilica of S. Maria of Carignano in Genoa. in Lourenço P.B., Roca P. (Eds.): *Proc.3rd International Seminar on Historical Constructions*, , Guimarães, 2001, pp.675-684 (2001).

[32] Lucchesi, M., Padovani, C., Pasquinelli, G., Zani, N.: Static analysis of masonry vaults, constitutive model and numerical analysis. *J. of Mechanics of Materials and Structures*, vol.2(2), 2007, 221-244 (2007).