An asymptotic Markovian approach to the portfolio selection problem

Enrico Angelelli, Sergio Ortobelli Lozza, and Gaetano Iaquinta.

***Abstract*—** In this paper we propose a portfolio choice problem under the hypothesis of Markovian returns. In particular, we assume stable Paretian distributed returns which imports a more flexible environment rather than the traditional Gaussian modeling. Therefore under these assumptions we perform an ex-post analysis to investigate the real benefit of our approach and draw some remarkable conclusions.

***Keywords*—** dynamic portfolio selection, stable Paretian distributions, Markov chain, market stochastic bounds.

I. Introduction

IN this paper we propose a methodology to optimize portfolio value in a choice problem framework using a markovian structure to model the asset portfolio returns. A comparison between different portfolio selection strategies is provided. The proposed methodology is tested in an ex-post analysis and the last crisis period data are used to assess the goodness of the method.

A normal distribution of asset returns is a traditional and basic assumption in many theoretical financial studies. However, many empirical studies reject the hypothesis that asset returns are normally distributed (see the fundamental works of Mandelbrot (1963) and Fama (1965) and, among others, Rachev and Mittnik (2000), Rachev et al. (2007) and the references therein). Moreover, many financial events are considered as real witnesses of failure of normal distribution hypothesis in the financial returns (i.e. stock market crash in October 1987, Asian financial crisis in 1997, highly volatile period after September 11, 2001, and the most recent sub-prime mortgage crisis and credit risk crisis (2008-2010)). Therefore a flexibility and statistical reliability in financial model are required to cope with that unrealistic hypothesis. Researchers have spent many efforts to improve methods and propose better models for financial markets. Among the numerous models proposed a fruitful research field appears to be the stable Paretian framework (e.g., Samorodnitsky and Taqqu, 1994) which assumes a financial return distribution more flexible than the traditional one.

Any portfolio dynamic model has to take into account for:

a) Heavy tails and asymmetric shape in returns distribution.
b) A multivariate distribution of underlying asset returns and correlation among asset returns more flexible than the simple Pearson linear correlation.
c) A dynamic portfolio strategy has to be based on the entire sample paths.

In this paper we discuss a portfolio selection model for financial markets based on these three themes with a particular attention to theme c). In order to evaluate and estimate the path dependent portfolio strategies we approximate the return time evolution by using Markovian trees. This approach, originally developed in the option theory (see Cox et al.(1979)), can be efficiently used for portfolio selection problems (as shown by Angelelli and Ortobelli (2009), Iaquinta et al. (2010, 2011) Angelelli et al. (2013)) and in sever other financial fields as discussed by D'Amico et al. (2010). In this framework the evolution of the wealth is derived as a non parametric Markov process. The Markovian approach allows to compute: the statistical distribution of any contingent claim, the distribution of stopping times or first passage time (see Angelelli and Ortobelli (2009) and Angelelli et al. (2011)), and the joint Markov distributions of risky variables. The portfolio selection strategies based on Markovian trees import several results obtained in option theory: path dependent portfolio selection strategies, arbitrage strategies for hedge funds, and strategies based on stopping times of the random wealth process. In order to account of the dependence structure we use the methodology discussed in Ortobelli et al. (2011) and Angelelli et al. (2011). The dependence structure allows us to solve two distinct problems in portfolio choice:

- Account for the common behavior of the returns in the portfolio choices: fundamental in any portfolio choice consistent with investor's preferences, since every investor sorts admissible portfolios with respect to his/her preferences and, doing so, he/she should account for the common behavior of the returns.
- Reduce the dimensionality of the large scale portfolio problem: it is well known that the number of observations necessary in optimization problems increases proportionally with the number of the random variables (see, among others, Papp et al. (2005), Kondor et al. (2007)). Therefore, we deal with the curse of dimensionality. To reduce the dimensionality we use the

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same preselection approach developed from Ortobelli et al. (2010b) which preselects an adequate number of assets considering their forecasted future performance. Then we use a non-Gaussian factor analysis that accounts the joint Markov evolution of returns and their asymptotic behavior.

In the empirical comparison we analyzes the impact of some proposed portfolio selection strategies applied to US market stock returns data. The ex-post analysis provided is based on two different datasets: the last ten years and the last six months. The use of the two different datasets allows to value the impact of the most recent firms on portfolio selection problems. On these assets the Ortobelli et al. (2011) techniques of dimensionality reduction are applied. Then, the optimal portfolios of different reward-risk strategies are determined. Finally, it is evaluated the impact of considering heavy tails comparing the sample paths of the ex-post wealth obtained from the different portfolio strategies. The paper is organized as follows. In Section II we discuss how modeling return series and introduce a set of performance ratios. Section III deals with the ex-post comparison among different portfolio strategies. Finally Section IV draws some remarkable conclusions.

II. Portfolio Selection Problem and Strategies

A. A non parametric Markovian framework

In this section we deal with the returns modelization by Markov process with heavy tailed distributions. We show how to determine the future wealth distribution. Let us introduce some notation. We consider a discrete sequence of investor wealth W_k equally spaced in time $k = 0,1,\ldots,T$ (e.g. days). The initial wealth (i.e. $W_0 = 1$) is invested at time $k = 0$ in *n* risky assets. The gross returns on date $t+1$ of the *n* assets are denoted as $z_{t+1} = [z_{1,t+1},\ldots,z_{n,t+1}]'$. Generally, we assume the standard definition of gross return between time t and time $t+1$ of asset i, as $z_{i,t+1} = \dfrac{s_{i,t+1} + d_{i,[t,t+1]}}{s_{i,t}}$, where $s_{i,t}$ is the price of the *i*-th asset at time *t* and $d_{i,[t,t+1]}$ is the total amount of cash dividends paid by the asset between t and $t+1$. We distinguish the definition of gross return from the definition of return, i.e., $z_{i,t} - 1$ or the alternative definition of log return $r_{i,t} = \log z_{i,t}$. The vector $x = [x_1,\ldots,x_n]'$ indicates the positions taken in the *n* assets. Assuming that no short sales are allowed, the vector *x* of portfolio weights belongs to the $(n-1)$-dimensional simplex $S = \{x \in \Re^n \mid \Sigma_{i=1}^n x_i = 1; x_i \geq 0\}$. The portfolio weight x_i represents the percentage of wealth invested in the *i*-th asset. In a dynamic framework the percentage of wealth invested in each asset could change over time. However, for sake of simplicity, in this paper we study and describe all admissible wealth processes $W(x) = \{W_t(x)\}_{t \geq 0}$ depending on an initial portfolio of weights $x \in S$ that is assumed constant over time. Moreover, we assume that these wealth processes are adapted processes defined on a filtered probability space $(\Omega, \phi, (\phi_t)_{0 \leq t \leq \infty}, \Pr)$. Thus, the gross return of a portfolio x during a period $[t, t+1]$ is given by $z_{(x),t+1} = x' z_{t+1} = \Sigma_{i=1}^n x_i z_{i,t+1}$. From a financial model point of view we assume that the gross returns have a Markovian behavior and can be modeled with an homogeneous Markov chain. Thus, we have to discretize the support of any portfolio. Given a set $\Upsilon(x) = \{z_{(x),[-h]} \mid h = 0,\ldots,H-1\}$ of H past observations of the portfolio gross returns, we define N states denoted as $Z(x) = [z_{(x)}^{(1)},\ldots,z_{(x)}^{(N)}]'$ in the interval $(\min \Upsilon(x); \max \Upsilon(x))$ where w.l.o.g. we assume $z_{(x)}^{(s)} > z_{(x)}^{(s+1)}$ for $s = 1,\ldots,N-1$. In general, the wealth obtained with the portfolio $x \in S$ at time $k = 1,2,\ldots$ is a random variable $W_k(x)$ with a number of possible values increasing as a polynomial of order N in variable k. In order to keep the complexity of the computation reasonable, we first divide the portfolio support $(\min \Upsilon(x); \max \Upsilon(x))$ in N intervals $(a_{(x),i}, a_{(x),i-1})$ where $a_{(x),i}$ (decreasing with index *i*) is given by: $a_{(x),i} = \left(\dfrac{\min \Upsilon(x)}{\max \Upsilon(x)}\right)^{i/N} \cdot \max \Upsilon(x)$ $i = 0,1,\ldots,N$; then, we compute the return associated to each state as the geometric average of the extremes of the interval $(a_{(x),i}, a_{(x),i-1})$ that is

$$z_{(x)}^{(s)} = \sqrt{a_{(x),s}\, a_{(x),s-1}} = \max Y(x) \left(\frac{\max \Upsilon(x)}{\min \Upsilon(x)}\right)^{\frac{1-2s}{2N}}, \; s = 1,2,\ldots,N$$

As a consequence, $z_{(x)}^{(s)} = z_{(x)}^{(1)} u^{1-s}$ where $u = \sqrt[N]{\dfrac{\max \Upsilon(x)}{\min \Upsilon(x)}} > 1$ and the wealth $W_k(x)$ obtained along a path after k steps (i.e. at time k) can only assume $1 + (N-1)k$ distinct values instead of $O(k^N)$. We denote such property as the *recombining effect.*

Thanks to the recombining effect of the wealth $W(x)$, the possible values of $W_k(x)$ up to time T $(k = 1,\ldots,T)$ can be stored in a matrix with T columns and $1 + (N-1)T$ rows resulting in $O(NT^2)$ memory space requirement. The transition matrix $P(x)_k = [p_{i,j;k}(x)]_{1 \leq i,j \leq N}$ valued at time k measures the probabilities $p_{i,j;k}(x)$ (valued at time k) of the transition process from state $z_{(x)}^{(i)}$ at time k to state $z_{(x)}^{(j)}$ at time $k+1$. In this paper we only consider homogeneous Markov chains, so transition matrix does not depend on time and it can be simply denoted by $P(x)$.

In order to simplify the notation, when the choice of the portfolio can be tacitly understood, we omit the reference to the portfolio x. Thus, the transition matrix will be denoted simply as P and similarly we get the probability $p_{i,j}$, the wealth W_k, the state $z^{(s)}$ and so on. Moreover with a little abuse of notation we will use the terms " s-th state" or "state s" of the Markov chain to point both the return $z^{(s)}$ and the index s itself; context will make clear the meaning of the term.

The entries $p_{i,j}$ of matrix P are estimated using the maximum likelihood estimates $\hat{p}_{i,j} = \frac{\pi_{ij}}{\pi_i}$ where π_{ij} is the number of historical observations that transit from the state i to the state j (i.e. from $z^{(i)}$ to $z^{(j)}$) and π_i is the number of historical observations in state i.

The $(N-1)k+1$ values of the wealth $W_k = [w^{(l,k)}]_{1 \le l \le (N-1)k+1}$ after k periods can be computed by the formula:

$$w^{(l,k)} = \left(z^{(1)}\right)^k \cdot u^{(1,l)}, l = 1, \ldots, (N-1)k+1 \qquad (1)$$

thus, the l-th node at time k of the wealth-tree corresponds to wealth $w^{(l,k)}$. The procedure to compute the distribution function of the future gross returns is strictly connected to the recombining feature of the wealth-tree.

Under these assumptions Iaquinta and Ortobelli (2006), have shown how to compute the unconditional and conditional (conditional on the initial state s_0, i.e. $z^{(s_0)}$) probability of each node of the future wealth.

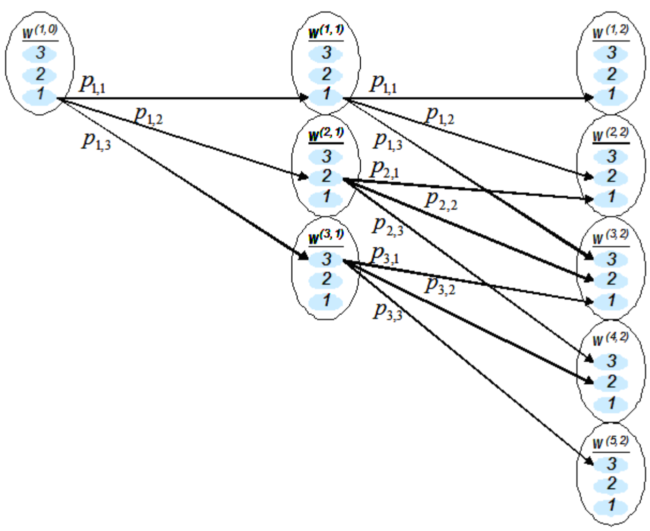

Fig.1: wealth-tree state representation and transition probabilities

In Fig.1 we provide a graphical representation of the wealth-tree and the corresponding probabilities after 2 steps, when we assume the return evolves (starting from state 1) following a simple homogeneous 3-state process. Nodes represent the possible values of wealth $w^{(l,k)}$. Namely, in column $k = 0,1,2$ are represented the possible values of wealth after k steps. The vector of the wealth after two steps is given by $W_2 = [w^{(1,2)}, w^{(2,2)}, w^{(3,2)}, w^{(4,2)}, w^{(5,2)}]'$. In each node of the wealth-tree the 3 states of the Markov chain are emphasized. Arcs connecting nodes represent the transition from a state i to a state j and are labelled with the corresponding probability $p_{i,j}$. Note that some states are not reachable in some nodes of the tree.

B. The asymptotic behaviour of the log returns

Many empirical findings show that log returns present a distribution with heavier tail than distributions with finite variance. Several empirical investigations show that

$$\Pr(|\ln(z_{(x)})| > u) \sim u^{-\alpha} L(u) \text{ as } u \to \infty \qquad (2)$$

where $0 < \alpha < 2$ and $L(u)$ is a slowly varying function at infinity, i.e., $\lim_{u\to\infty} \frac{L(cu)}{L(u)} \to 1$ for all $c > 0$, see, among others, Rachev and Mittnik (2000) and the references therein. The tail behavior of returns implies that the vector of log-returns is in the domain of attraction of a n-dimensional stable law. Moreover, since in all observed data we get $1 < \alpha < 2$, then the relation (2) implies that log returns $r_{(x)} = \ln(z_{(x)})$ admit finite mean and not finite variance. This tail condition also implies that the portfolio log return $r_{(x)}$ is in the domain of attraction of an α-stable law. A simple way to traduce the asymptotic behavior of data consists in assuming the log wealth to be α stable distributed. That is, for each portfolio $x \in S$ the forecasted log wealth ($\ln(W_T(x)) = \Sigma_{t=1}^{T} \ln(z_{(x),t})$) at a given future time T is in the domain of attraction of an $\alpha(x)$ stable distribution. Under this assumption we implicitly assume that all optimal choices are identified by four parameters and the forecasted log wealth of every portfolio can be well approximated by a stable distribution, i.e.: $\ln(W_T(x)) \stackrel{d}{=} S_{\alpha(x)}(\sigma(x), \beta(x), \mu(x))$ where $\alpha(x) \in (0,2]$ is the index of stability, $\sigma(x)$ is the scale parameter, $\mu(x)$ is the location parameter and $\beta(x)$ is the skewness parameter. The estimation of the stable Paretian parameters can be done efficiently in a negligible computational time by applying the consistent quantile McCulloch's method (see McCulloch (1986)). In particular, McCulloch's method requires the knowledge of 5%, 25%, 50%, 75%, 95% quantiles of the log wealth $\ln(W_T(x))$ to obtain these estimates in an acceptable computational time for any portfolio. Then, applying some simple algorithms to compute reward and risk measures with stable distributions we can easily get optimal portfolio strategies that account the Markovian and asymptotic behavior of the final wealth.

C. Portfolio large scale strategies

The classic static portfolio selection problem when no short sales are allowed, can be represented as the maximization of a functional $f:(\Omega,\Im,P)\mapsto\mathbb{R}$ applied to the random portfolio of gross returns $z_{(x),k+1}$ subject to the portfolio weights belonging to the (n-1)-dimensional simplex S, i.e.,

$\max_{x \in S} f(z_{(x)})$

Typically, the functional f(.) is a performance measure or an utility functional. In both cases the functional f(.) should be isotonic with a particular ordering of preference ≽, that is, if X is preferred to Y (X≽Y), then f(X)≥f(Y). The choice of the functional f(.) plays a crucial role in the portfolio strategy. Isotonic utility functionals with non satiable and risk averse preferences have been used in many financial applications. In these cases we have f(*X*)=E(v(*X*)) where *v* is an increasing and concave utility function. However, as suggested in behavioural finance, while all investors prefer more to less they could be neither risk averse nor risk lover. For this reason it makes sense to consider functionals that are monotone, even though they are not consistent with an uncertainty/aggressive order (see, among others, Rachev et al. (2008)). We call *OA performance (utility) functional* any functional computed under the assumption that the gross return of each portfolio follows a Markov chain with N states. In this paper we will use and describe only some OA functionals that consider the forecasted wealth at time T. That is, investors have to periodically (every T periods) compute the portfolio $x \in S$ solution of the problem:

$$\max_{x \in S} f\left(W_T(x)\right) \tag{3}$$

Remark: The vector of weights x solution of the problem (3) represents the percentage of wealth that should be invested in each asset during the period [0,T]. Since the value of the assets change during the period [0,T], then even an OA portfolio strategy generally implies that the wealth could be recalibrated more times during the period [0,T] in order to maintain constant the percentages of the wealth invested in each asset. If T is very large and we do not recalibrate the portfolio periodically (the period should be the same used in the valuation) these percentages invested in the assets could be completely different at the end of investor's temporal horizon. This point has not been explicitly addressed in Angelelli and Ortobelli's analysis (2009) and could have a very big impact in portfolio choices.
In order to determine optimal solution for OA functionals we have to choose different portfolio strategies for non satiable investors which account for asymptotic behaviour of returns. In the following subsection we present some OA functionals used in the empirical comparisons.

In portfolio literature more than one hundred static reward-risk performance measures have been proposed (see Cogneau and Hübner (2009a, 2009b)). Here, we list the Sharpe static strategy and some OA performance functionals isotonic with choices of non satiable investors that will be object of the following empirical analysis. For all the OA portfolio strategies we assume that investors have temporal horizon equal to T.

OA-Sharpe ratio (*OA-SR*). The classic version of the Sharpe ratio (see Sharpe (1994)) values the expected excess return for unity of risk (standard deviation). With the OA-Sharpe ratio we value the expected excess final wealth for unity of risk, i.e.,

$$\mathrm{OA-SR}(W_T(x)) = \frac{E(W_T(x) - W_T(r_b))}{\sigma_{W_T(x) - W_T(r_b)}} \tag{4}$$

where $W_T(r_b)$ is the final wealth at time T we obtain investing in the benchmark r_b. In the Markovian framework we should consider the bivariate evolution of the vector $(W_T(x) - W_T(r_b))$ to value the standard deviation $\sigma_{W_T(x)-W_T(r_b)}$ of $W_T(x) - W_T(r_b)$. Yet, in the following analyses we assume that the riskless asset is not allowed, thus, the OA-Sharpe Ratio is simply given by $\frac{E(W_T(x)-1)}{\sigma_{W_T(x)}}$. When the benchmark r_b is the risk free rate, the Sharpe ratio is isotonic with non-satiable risk averse preferences. However, using Sharpe type measures we generally don't take into account the asymptotic behavior of the wealth (except in the case the optimal portfolios are in the domain of attraction of the Gaussian law).

OA-Asymptotic Sharpe ratio (*OA-ASR*) This performance functional is defined as

$$OA - ASR(W_T(x)) = \\ = \begin{cases} \dfrac{\mu_{\ln(W_T(x))}}{E(|\ln(W_T(x)) - \mu_{\ln(W_T(x))}|^{1.01})} & \text{if } \alpha(\mathrm{x}) > 1.01 \\ 0 & \text{if } \alpha(\mathrm{x}) \le 1.01 \end{cases} \tag{5}$$

where $\mu_{\ln(W_T(x))} = \mu(x)$ is the mean of the stable distribution that better approximates the log final wealth: $\ln(W_T(x)) \overset{d}{=} S_{\alpha(x)}(\sigma(x), \beta(x), \mu(x))$ when $\alpha(x) > 1.01$. We assume $OA - ASR(W_T(x)) = 0$ when $\alpha(x) \le 1.01$ since low indexes of stability imply so heavy tails that the 1.01 moment of the stable distribution is infinite. Observe that when $\alpha = 2$ the final wealth is log normal distributed. Moreover, if $\alpha = 2$ for all the portfolios, the maximization of the OA asymptotic Sharpe ratio is equivalent to the maximization of the Sharpe ratio of the log wealth. As for the Sharpe ratio, this ratio is isotonic with the preferences of non satiable risk averse investors (see Rachev et al. 2008). In order to maximize the *OA-ASR,* we estimate the four stable parameters $(\alpha(x), \sigma(x), \beta(x), \mu(x))$ using the McCulloch's quantile algorithm (see McCulloch (1986)) and then we compute the 1.01 moment of the centered log wealth.

OA-Stable stochastic bounds ratio (*OA-SSBR*) This performance functional is defined as:

$$OA-SSBR(W_T(x)) = \begin{cases} \dfrac{\mathrm{E}(|\Delta_1 - \mu_{\Delta_1}|^{1.01}) + \mu_{\Delta_1}}{\mathrm{E}(|\Delta_2 - \mu_{\Delta_2}|^{1.01}) + \mu_{\Delta_2}} & \text{if } \alpha_1 \text{ and } \alpha_2 > 1.01 \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

where α_1 and α_2 are the indexes of stability respectively of $\Delta_1 = \ln(W_T(x)) - \ln(W_T(\min_i z_i))$, $\Delta_2 = \ln(W_T(\max_i z_i)) - \ln(W_T(x))$, while $W_T(\min_i z_i)$ and $W_T(\max_i z_i)$ are the forecasted wealths at time T obtained respectively by the lower market stochastic bound and the upper market stochastic bound. Moreover $\mu_{\Delta_1} = \mu_1$, $\mu_{\Delta_2} = \mu_2$ are the location parameters of the stable distributions that better approximate respectively $\Delta_1 \stackrel{d}{=} S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ and $\Delta_2 \stackrel{d}{=} S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$. In order to determine the distributions of $\ln(W_T(x)) - \ln(W_Y(\breve{z}_i))$ and $\ln(W_T(\hat{z}_i)) - \ln(W_Y(x))$, where $\breve{z}_i = \min_{i \le n} z_i$ and $\hat{z}_i = \max_{i \le n} z_i$, we have to use the evolution of the bivariate Markov processes $(W_t(x), W_t(\breve{z}_i))$ and $(W_t(x), W_t(\hat{z}_i))$, . Recall that, when no short sales are allowed, the upper and the lower market stochastic bounds among n assets with gross returns z_i $(i = 1, \ldots, n)$ are respectively given by $\hat{z}_i$ and $\breve{z}_i$, since $\breve{z}_i \le z_{(x),k} \le \hat{z}_i$ for any time k and for any vector of portfolio weights $x \in S$ (for further generalizations see Ortobelli et al. (2011) and references therein). This ratio expresses the idea that investors want to maximize the distance between the wealth and the lower market stochastic bound, and to minimize the distance between the wealth and the upper market stochastic bound.

OA-Stable loss ratio (*OA-SLoss*) The OA stable loss ratio values the expected asymptotic log wealth for unity of loss. This ratio can be seen as a particular case of the Starr ratio applied to stable distributions **(**see, among others, Biglova et al. (2004)**)**. Thus, using the asymptotic approximation of log wealth $\ln(W_T(x)) \stackrel{d}{=} S_{\alpha(x)}(\sigma(x), \beta(x), \mu(x))$ we can easily compute

$$OA-SLoss(W_T(x)) = \begin{cases} \dfrac{\mu_{\ln(W_T(x))}}{-E(\ln(W_T(x)) \mid \ln(W_T(x)) \le 0)} & \text{if } \alpha(x) > 1 \\ 0 & \text{if } \alpha(x) \le 1 \end{cases} \quad (7)$$

where $\mu_{\ln(W_T(x))} = \mu(x)$ is the location parameter of the stable distribution that better approximates the final log wealth and $E(\ln(W_T(x)) \mid \ln(W_T(x)) \le 0)$ is obtained using the Stoyanov et al.'s formula for stable distributions (see Stoyanov et al. (2006)). We assume $OA-SLoss(W_T(x)) = 0$ when $\alpha(x) \le 1$ since low indexes of stability imply so heavy tails that the first moment of the stable distribution is infinite. The conditional expected loss $-E(X \mid X \le 0)$ of an α stable random variable $X \stackrel{d}{=} S_a(\sigma, \beta, \mu)$ is given by:

$$-E(X \mid X \le 0) = \left(\frac{2\Gamma((\alpha-1)/\alpha)}{\pi - 2\theta_0} \frac{\cos(\theta_0)}{(\cos(\alpha\theta_0))^{1/\alpha}} \right) \sigma - \mu ,$$

where $\theta_0 = \dfrac{1}{\alpha} \arctan(\beta \tan(\pi\alpha/2))$ (see Stoyanov et al. (2006)).

In order to reduce the dimensionality of the problem we adopt the tecniques developed by Ortobelli et al. (2011) and Angelelli et al. (2011) preselecting no more than 170 assets and then reducing the dimensionality of the preselected assets identifying some common factors to approximate the asset returns.

III. An Empirical Comparison Among Portfolio Strategies

In this section, we evaluate the impact of the proposed modelization on the US stock market. In particular, we consider the stocks traded on the NYSE and on the NASDAQ. Since we want to propose as much as possible a realistic empirical analysis, we have developed a dynamic dataset (described here in the following) that uses all the useful financial data from DataStream.

A. Dynamic Dataset

In this paper we suggest a schema to solve large scale portfolio selection. This means that we expect to extract the lately time series of adjusted prices for a large number of assets from a database (namely DataStream). Dealing with time series with missing data is not an easy task and unfortunately the hundreds of time series available from the database are often spoiled with missing data. In this paragraph we explain how we manage this problem in order to produce a "clean", though large, set of time series to be submitted as input to our portfolio selection framework. The objective is to obtain a set of assets that are reasonably priced on a common set of dates. The time series are first filtered so that the "bad" ones are rejected and the corresponding assets will not compete to enter the portfolio. Then, the promoted series are fixed, if needed, so as all assets are priced on the same set of dates. Once all assets are priced on a common set of dates, data are ready to be passed to the portfolio selection algorithm.

The objective is achieved by a number of steps:

1) each price series available in the database for the chosen exchange is extracted from a fixed date in the past up to the current date in the analysis; a table is built so that assets corresponds to columns and each row correspond to a common date for all prices

2) each row with more than 99% missing data is supposed to correspond to a bank holiday for the exchange (e.g. labour day) and is removed from the table;

3) each asset with at least 3 consecutive missing data is removed from the table;
4) since some pairs of consecutive or sparse missing data can still be present in some columns, we compute the number of missing data for each asset and take the minimum (say m such quantity); afterwards, assets with more than m missing data or at least one pair of consecutive missing data are removed from the table;
5) now only sparse missing data can be present in the table; all missing data are padded forward from the previous date. Naturally, if the missing data correspond to the first date, the padding is made backward from the next date;
Doing so, we have a table with no missing prices and asset returns can be computed.

B. An empirical comparison

In our empirical analysis we use a date set of about two years (500 daily observations) from 15-Sep-2008 till 31-Aug-2010, and assume the following settings:

a) that investors have a temporal horizon of $T = 20$ working days (thus, for each portfolio strategy we should optimize the portfolio every 20 working days for a total of 25 optimizations);
b) that investors cannot invest more than ten percent in a single asset (i.e.: $x_i \in [0,0.1]$);
c) Markov chains have $N = 9$ states;
d) the initial wealth W_0 is equal to 1 at the date 15-Sep-2008.

We perform a comparisons to evaluate the impact of the Stable Paretian approximation by comparing the ex-post performance of different portfolio strategies based on: the OA-Sharpe ratio (4), the OA-Asymptotic Sharpe ratio (5), the OA-Stable loss ratio (7), the OA-Stable stochastic bounds ratio (6). Even in this analysis we preselect assets among all those active either in the last ten years or in the last six months. Then we approximate the returns to reduce the randomness of the problem.
For each strategy, we have to compute the optimal portfolio composition 26 times and at the *k*-th optimization ($k = (0,1,2,\ldots,25)$), two main steps are performed to compute the ex-post final wealth:

Step 1 Determine the market portfolio $x_M^{(k)}$ that maximizes the performance ratio $\rho(W(x))$ associated to the strategy, i.e. the "ideal" solution of the following optimization problem:

$$\max_{x^{(k)}} \rho(W(x^{(k)})$$
$$s.t.$$
$$(x^{(k)})' e = 1,$$
$$x_i^{(k)} \le 0.1; x_i^{(k)} \ge 0$$
$$i = 1,\ldots,n$$

Angelelli and Ortobelli (2009) have observed that the complexity of the portfolio problem is much higher in view of a Markovian evolution of the wealth process. In order to overcome this limit we use the Angelelli and Ortobelli's heuristic algorithm that could be applied to any complex portfolio selection problem that admit more local optima.

Step 2 During the period $[t_k, t_{k+1}]$ (where $t_{k+1} = t_k + T$) we have to recalibrate daily the portfolio maintaining the percentages invested in each asset equal to those of the market portfolio $x^{(k)}$. Thus, the ex-post final wealth is given by:

$$W_{t_{k+1}} = W_{t_k} (\Pi_{i=1}^{T} (x_M^{(k)}{}') z_{(t_k+1)}^{(ex\,post)}) \qquad (8)$$

where $z_{(t_k+1)}^{(ex\,post)}$ is the vector of observed daily gross returns between $(t_k + i - 1)$ and $(t_k + i)$.
Steps 1 and 2 are repeated for all performance ratios until some observations are available.

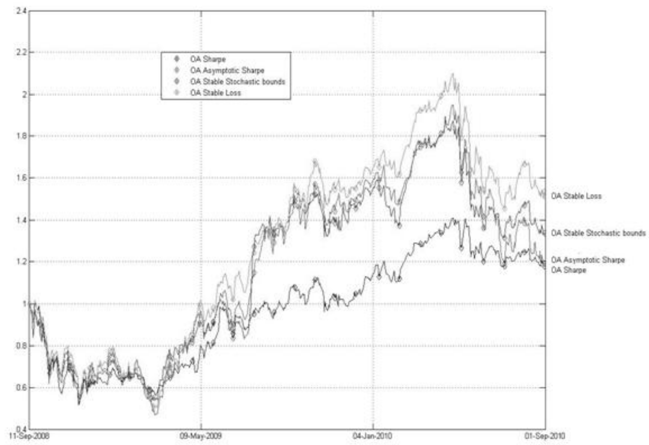

Fig.2: Ex-post comparison of OA Sharpe ratio, OA Asymptotic Sharpe ratio, OA stable loss ratio, OA stochastic bounds ratio applied to preselected assets among all the active in the last 10 years.

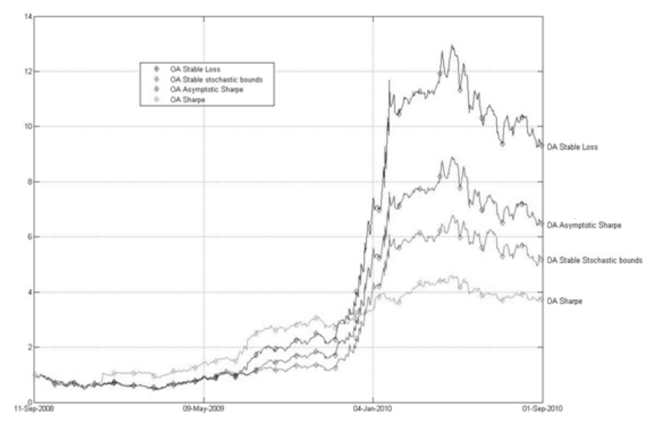

Fig.3: Ex-post comparison of OA Sharpe ratio, OA Asymptotic Sharpe ratio, OA stable loss ratio, OA stochastic bounds ratio applied to preselected assets among all the active in the last 6 months.

The output of this analysis is given in Fig. 2 and 3 and Table 1. Fig. 2 and 3 report the results of all strategies applied to the preselected assets among all the active assets respectively in the last ten years and in the last six months. The comparison between these figures confirms that the recent entries in the market have an important impact in the portfolio choices. As a matter of fact, the results obtained from the stable Paretian strategies (i.e., OA-Asymptotic Sharpe ratio (5), the OA-Stable loss ratio (7), the OA-Stable stochastic bounds ratio (6)) applied to preselected assets among all the active in the last six months present outstanding results considering that we apply the model during a period of global crisis. The OA Stable loss ratio (best strategy) gives more than the 300% for year.

First of all we observe that among the assets selected there are several ones that have

Moreover from this comparison it is still clear that the OA asymptotic Sharpe strategy, as all the other stable Paretian strategies, presents higher final wealth than the OA Sharpe strategy applied to the preselected assets among all active ones (either in the last ten years or in the last six months). However, during some periods of the ex-post comparison the OA Sharpe strategy presents higher wealth than the analogous Stable type strategies. Thus in order to account more precisely these results we have to consider some empirical statistics on the ex-post returns of the portfolio strategies.

Table 1 reports, for all the strategies, the values of:

(1) two reward measures of the ex-post returns (the empirical mean, the AVaR of the opposite random variable i.e., $ETL_{0.05}(-X)$ where

$$ETL_{\omega}(Y) = AVaR_{\omega}(Y) = E(-Y \mid Y \le F_Y^{-1}(\omega));$$

(2) two risk measures of the ex-post returns (the standard deviation σ(X) and the AVaR of the centred random variable $ETL_{0.05}((X-E(X)))$;

(3) all the possible reward risk ratios deriving from these two measures.

Active last 6 months	OA stable loss	OA stable stochastic bounds	OA asymptotic Sharpe	OA Sharpe
mean(X)	0.00554	0.00429	0.004585	0.003104
St.dev(X)	0.04736	0.045429	0.04189	0.031856
ETL(-X)	0.15213	0.140959	0.129619	0.082444
ETL(X-E(X))	0.08327	0.084624	0.075225	0.056342
Sharpe ratio	0.11698	0.09443	0.109455	0.097425
Mean/ ETL(X-E(X))	0.06653	0.050693	0.060951	0.055084
ETL(-X)/ ETL(X-E(X))	1.82685	1.665717	1.723077	1.463283
ETL(-X)/St.dev(X)	3.21197	3.102853	3.094262	2.588051
Active last 10 years	**OA stable loss**	**OA stable stochastic bounds**	**OA asymptotic Sharpe**	**OA Sharpe**
mean(X)	0.00128	0.001082	0.000905	0.00063
St.dev(X)	0.02925	0.032147	0.03321	0.024374
ETL(-X)	0.06753	0.078294	0.076474	0.053801
ETL(X-E(X))	0.06672	0.068351	0.0775	0.060226
Sharpe ratio	0.04389	0.033656	0.027237	0.025844
Mean/ ETL(X-E(X))	0.01924	0.01583	0.011671	0.010459
ETL(-X)/ ETL(X-E(X))	1.01217	1.145477	0.986761	0.893316
ETL(-X)/St.dev(X)	2.30852	2.435485	2.302766	2.207274

Table 1 Empirical mean, standard deviation, $ETL_{0.05}(-X)$ and $ETL_{0.05}(X-E(X))$ on the ex-post returns.

Table 1 suggests that the OA Sharpe strategy is dominated from the stable Paretian type strategies since the ex-post returns obtained by the OA Sharpe strategy presents lower reward/risk performance than almost all the stable Paretian strategies (except for the 6 months stable stochastic bounds strategy). Thus we essentially confirm the results observed in Figs 2 and 3.

The portfolio composition generally changes a lot during the ex-post period. This is confirmed from Fig. 4 that describes the portfolio turnover and its diversification. In particular, it examines how the portfolio composition of the Stable loss strategy changes during the ex-post period. In the first sub-figure (Fig. 4(a)) we have the percentages invested in each assets at each computation of the optimal portfolio.

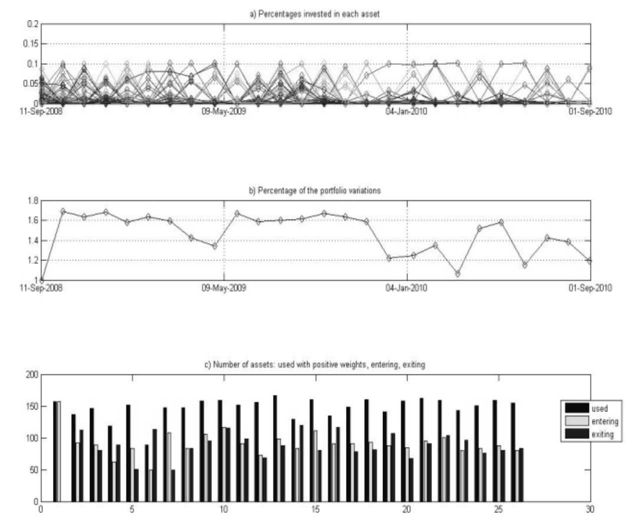

Fig. 4: Portfolio composition and portfolio variations of the OA stable loss strategy applied to preselected assets among all the active in the last 6 months.

The second sub-figure (Fig. 4(b)) points out the percentages Φ_k (k=1,...,25) of the portfolio changed every 20 days obtained by the formula:

$$\Phi_k = \sum_{i=1}^{n} \left| x_{M,i}^{(k)} - x_{M,i}^{(k-1)} \right|$$

In particular Φ_k should belong to the interval [0,2], where the value 0 means that the portfolio composition is not changed during the period $[t_{k-1},t_k]$, while the value 2 corresponds to the case we sell the portfolio and we buy a completely different portfolio. The last sub-figure (Fig. 4(c)) points out the number of:

(1) the quantity of assets used (i.e. those assets whose percentages are greater than zero $x_{M,i}^{(k)}>0$, i=1,...,n);

(2) the quantity of entering assets;

(3) the quantity of exiting assets.

As we observe the portfolio is well diversified among all preselected assets even if there are always some assets in which the strategy suggests to invests the maximum possible (i.e. 10%). Moreover we also observe that the portfolio change a lot every 20 days and even on these changes we should pay the transaction costs. The transaction costs are also paid day by day when we recalibrate the portfolio to maintain constant the percentages of the portfolios. Thus if we assume 0.05% as daily average of transaction costs (these transaction costs are high enough for institutional investors) we should get more than $1.0005^{500} \sim 1.284$ in the 500 days of the ex-post analysis. However, most of the stable Paretian strategies produce some profits since they present a final wealth greater than 1.284 during the last two years of ex post analysis.

However, these results do not consider the transaction costs which must be paid daily in order to maintain constant the percentages invested in each asset. Moreover from this comparison it is still clear that the OA asymptotic Sharpe strategy, as all the other stable Paretian strategies, presents higher final wealth than the OA Sharpe strategy applied to the preselected assets among all active ones (either in the last ten years or in the last six months).

IV. Conclusion

This paper describes a Markovian approach applied to portfolio problems where innovations belong to the stable distributions domain. In particular, we first examine how to approximate the Markovian and asymptotic behavior of wealth. Then, we examine several portfolio strategies under the proposed environment. Finally, we propose an empirical comparison among several strategies that account heavy tails of log return portfolios. The empirical analysis shows that the asymptotic behaviors of the wealth and the recent entries in the market have an important impact in the portfolio choices applied to the US stock market. Moreover, several new questions rise from the proposed methodology and empirical analysis. As a matter of fact, the experiment on the US market suggests that further investigations needed to value: the market efficiency, liquidity constraints, and the impact of portfolio strategies based on the use of proper stopping times.

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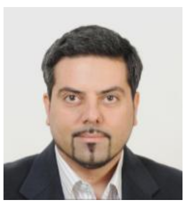

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