Abstract - We study the kernels in the contour integral representation of the remainder term of Gauss-Lobatto quadrature formulae, in particular the location of their maxima on circular and elliptic contours. Quadrature rules with Chebyshev weight functions of all four kinds receive special attention. We also study a general Gauss-Chebyshev-Stancu quadrature with double fixed nodes.

Key Words - quadrature, kernels, contour integral representation, remainder term, Gauss-Lobatto, Gauss-Chebyshev-Stancu

I. INTRODUCTION

Let $\Gamma$ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and $D$ be its interior. Let $f$ be analytic in $D$ and continuous on $\bar{D}$. We consider an interpolatory quadrature rule

$$\int_{-1}^{1} f(t) w(t) dt = \sum_{i=1}^{N} \lambda_i f(\tau_i) + R_N(f)$$

with

$$-1 \leq \tau_1 < \tau_2 < \ldots < \tau_N \leq 1$$

and

$$\omega_N(z) = \omega_N(z,w) = \prod_{i=1}^{N} (z - \tau_i), z \in \mathbb{C}$$

denote its node polynomial (which in general depends on $w$), and define:

$$\rho_N(z,w) = \int_{-1}^{1} \frac{\omega_N(t,w)}{z-t} w(t) dt, z \in \mathbb{C} \setminus [-1,1]$$

then, as is well known, the remainder term $R_N$ in (1) admits contour integral representation

$$R_N(f) = \frac{1}{2\pi i} \oint_{\Gamma} K_N(z,w) f(z) dz$$

where the "kernel" $K_N$ can be expressed, e.g., in the form (see [1]):

$$K_N(z,w) = \frac{\rho_N(z,w)}{\omega_N(z,w)}, z \in \Gamma$$

Note that $\omega_N$ in (3) and (4) may be multiplied by any constant $C \neq 0$ without affecting the validity of (6). It is also evident from (6) that

$$K_N(z,w) = \frac{\rho_N(z,w)}{\omega_N(z,w)}$$

In order to estimate the error in (1) by means of

$$|R_N(f)| \leq (2\pi)^{-1} l(\Gamma) \max_{z \in \Gamma} |K_N(z,w)| \max_{z \in \Gamma} |f(z)|$$

where $l(\Gamma)$ is the length of the contour $\Gamma$, it becomes necessary to study the magnitude of $|K_N|$ on $\Gamma$. This has been done in a number of papers (see [1]) for Gauss-type and other quadrature formulae, and for contours $\Gamma$ that are either concentric circles centered at the origin or confocal ellipses with focal points at $\pm 1$. The thrust of this work has been directed towards upper bounds, or asymptotic estimates, for the maximum of $|K_N|$ in (8). In attempt to remove uncertainties inherent in such estimates, W. Gautschi [2] for Gauss, Gauss-Lobatto and Gauss-Radau formulae, the precise location on $\Gamma$ where $|K_N|$ attains its maximum, and we suggested simple recursive techniques to evaluate $K_N(z,w)$ for any $z \in \mathbb{C} \setminus [-1,1]$. Here we investigate, in the same spirit quadrature rules of Gauss-Stancu type, especially for any of the four Chebyshev weight functions:

$$w_1(t) = (1-t^2)^{-\frac{1}{2}}$$

$$w_2(t) = (1-t^2)^{\frac{1}{2}}$$

$$w_3(t) = (1-t)^{\frac{1}{2}} (1+t)^{\frac{1}{2}}$$

$$w_4(t) = (1-t)^{\frac{1}{2}} (1+t)^{\frac{1}{2}}$$

(9)
II. SOME GENERAL RESULTS FOR CIRCULAR CONTOURS

In this sections, \( \Gamma = C_r, \ C_r = \{ z \in \mathbb{C} : |z| = r \} \), where \( r > 1 \). For positive weight functions \( w \) and quadrature rules of Gaussian type, with \( N = n \) it is known from [2] that:

\[
\max_{z \in C_r} |K_n(z, w)| =
\begin{cases}
K_n(r, w), & \text{if } w(t) \text{ is nondecreasing on } (-1,1) \\
K_n(-r, w), & \text{if } w(t) \text{ is nonincreasing on } (-1,1)
\end{cases}
\tag{10}
\]

We now explore the implications of this results to Gauss-Lobatto formulae (subsection 2.1.) and Gauss – Radau formulae (subsection 2.2.)

2.1. Gauss-Lobatto formulae

These are the quadrature rules (1) with \( N=n+2, \tau_N = -1, \tau_1 = 1 \) and \( R_N(f) = 0 \), whenever \( f \in P_{2n+1} \) (the class of polynomials of degree \( 2n+1 \)). They are clearly interpolatory. We denote \( w^L(t) = (1-t^2)w(t) \) and write \( \Pi_n(\cdot;w^L) \) for the polynomial of degree \( n \) (suitably normalized) orthogonal with respect to the weight function \( w^L \). It is well known that:

\[
\omega_{n+2}^L(z,w) = (1-z^2)\Pi_n(z;w^L)
\tag{11}
\]

from which then follows:

\[
\rho_{n+2}(z,w) = \int_{-1}^{1} \frac{(1-t^2)\Pi_n(t;w^L)}{z-t} w(t) dt =
\int_{-1}^{1} \frac{\Pi_n(t;w^L)}{z-t} w^2(t) dt = \rho_N(z,w)
\]

and, therefore by (6)

\[
K_{n+2}(z,w) = \frac{K_n(z;w^L)}{1-z^2}
\tag{12}
\]

Here \( K_n(\cdot;w^L) \) is the Kernel for \( n \)-point Gauss formula relative to the weight function \( w^L \). Since \( |1-z^2| \) attains its minimum of \( C_r \) at \( z = r \) and \( z = -r \), and since

\[
\frac{w^L(t)}{w^L(-t)} = \frac{w(t)}{w(-t)},
\]

we have as an immediate consequence of (10)

\[
\max_{z \in C_r} |K_{n+2}(z, w)| =
\begin{cases}
\frac{1}{r^2-1}K_n(r;w^L), & \text{if } \alpha > 1, \beta > 1 \\
\frac{1}{r^2-1}K_n(-r;w^L)
\end{cases}
\tag{13}
\]

depending on whether \( w(t) \), is nondecreasing or nonincreasing respectively. In particular, for the Jacobi weight functions \( w(t) = (1-t)^\alpha (1+t)^\beta, \alpha > -1, \beta > -1 \) the first relation in (13) holds if \( \alpha \leq \beta \) and the second if \( \alpha > \beta \).

2.2. Gauss – Radau formulae

These are pairs of such formulae namely (1) with \( N=n+2, \tau_N = -1, \) and (1) with \( N=n+1, \tau_1 = 1 \), both having \( R_n(f)=0 \) for \( f \in P_{2n} \). It suffices to consider one of them, say the former since the kernels of the two formulae are simply related. If we denote \( w(-t)=w^*(t) \) and write \( K_N(\cdot;w) \) for the Kernel of the Radau formula with \( \tau_1 = -1 \) and \( \tau_1 = 1 \), respectively, a simple computation indeed will show that

\[
K_N^+(z,w) = -K_N(-z,w^*)
\]

i.e., the modulus of \( K_N^+(z,w) \) for the weight function at the point \( z \) has the same value as the modulus of \( K_N(-z,w^*) \) for the weight function \( w^* \) at the point \( -z \), the mirror image of \( z \) with respect to the imaginary axis.

For the Radau formula with \( \tau_N = -1 \), we write \( w_\mathbb{R}(t) = (1+t)w(t) \) and have, as is well known,

\[
W_{n+1}(z,w) = (1+z)\pi_n(z;w_\mathbb{R})
\tag{15}
\]

There follows, similar to the case of Lobatto formulae,
Where $K_N(z, w)$ is the Kernel for the n-point Gauss formula relative to the weight function $w \Re$. Since $|1 + z|$ on $\Re$ attains its minimum at $z = -r$, we can now apply the second results in (10), giving

$$\max |K_{n+1}(z, w)| = \frac{|K_n(-r, w \Re)|}{r - 1}$$ (17),

provided $\frac{w \Re(t)}{w \Re(-t)}$ is no increasing on (-1,1). Unfortunately, this condition is not satisfied for the Chebyshev weights $w_1, w_2, w_3$, (cf. (9)). We conjecture, in fact, that the maximum in (17) obtained at $z = r$, rather than $z = -r$, when $w = w_3$.

III. REMAINDER KERNELS FOR CHEBYSHEV WEIGHT FUNCTIONS

In this section, after some preliminaries on orthogonal polynomials, we provide explicit formulae for Gauss-Lobatto, and Gauss-Stancu type rules, of $K_n(\cdot, w)$ when $w = w_i, \{i = 1, 2\}$ (cf (9)).

3.1 Preliminaries

We shall need some facts about Jacobi polynomials with half-integer parameters. They are given here in a form general enough to be applicable (if need be) to a Lobatto formulae with multiple fixed nodes.

**Lemma 3.1.** The polynomial of degree $n$ orthogonal on $(-1, 1)$ with respect to the weight functions

$$(1-t^2)^{\frac{1}{2}+k}, k \geq 0 \text{ an integer, is given by } T_n^{(k)}(t)$$

where $T_n$ denotes the n-th degree Chebyshev polynomial of the first kind.

**Proof:** See equations (6.21.7) in [4] and the paragraph following that equations.

The following lemma is also known, but are stated here in a form were suitable, for our purposes. We recall that Chebyshev polynomials $U_n, V_n$ of the second and third kind (orthogonal relative to weight functions $\left(1-t^2\right)^{\frac{3}{2}}$ and $\left(1-t\right)^{\frac{1}{2}} (1+t)^{\frac{1}{2}}$, respectively, are given by>

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}$$

$$V_n(\cos \theta) = \frac{\cos \left(\frac{n+1}{2}\theta\right)}{\cos \frac{\theta}{2}}$$ (18)

**Lemma 3.2.** Let $U_{n,k}$ be a polynomial of degree $n$ orthogonal on $(-1, 1)$ with respect to the weight functions

$$(1-t)^{\frac{1}{2}}(1+t)^{\frac{1}{2}+k}, k \geq 0 \text{ an integer. Then}>$$

$$U_{n,0}(t) = U_n(t)$$

$$U_{n,k}(t) = \frac{1}{1+t} U_{n+1,k-1}(t) +$$

$$\left(n+k+\frac{1}{2}\right)\left(n+k+1\right)U_{n,k-1}(t)$$ (19)

**Proof:** Define

$$U_{n,k}(t) =$$

$$\left[\frac{n!(n+k+1)!\sqrt{\pi}}{2^{\frac{n+k+1}{2}}\Gamma\left(n+k+\frac{3}{2}\right)}\right] \left[\left(1+t^2\right)^{\frac{1}{2}+k}\right](t)$$

and use the second relation in [4], eq (6.5.4) with $\alpha = \frac{1}{2}$,

$$\beta = -\frac{1}{2} + k$$.

**Lemma 3.3.**

Let $V_{n,k}$ be the polynomial of degree $n$ orthogonal on $(-1, 1)$ with respect to the weight function $(1-t)^{1/2} (1+t)^{1/2+k}$, $k \geq 0$ are integer. Then

$$V_{n,0}, V_{n,0}(t)$$
\[ V_{n,k}(t) = \frac{1}{1+t} \left[ V_{n+1,k-1}(t) + \frac{(n+k)(n+k+1)}{(n+1/2)(n+1/2+1/2)} V_{n,k-1}(t) \right] \]

(20)

**Proof:** Define

\[
V_{n,k}(t) = \left[ \frac{n!(n+k)!\sqrt{\pi}}{\Gamma(n+k+1/2)\Gamma(n+k+1)} \right] P_n^{\left(-1/2,k\right)}(t),
\]

and use the second relation in [4] with \( \alpha = -\frac{1}{2}, \beta = -\frac{1}{2} + k \)

### 3.2 Gauss-Chebyshev-Lobatto formulae

We begin with the weight functions \( w_1 \) and consider (1) with \( w = w_1, \quad N = n + 2, \quad \tau_N = -1, \quad \tau_1 = 1, \quad R_N(f) = 0, \text{ for } f \in P_{2n+1}. \)

Since the nodes \( \tau_\nu, 2 \leq \nu \leq N - 1, \) are the zeros of \( \Pi_n(\cdot; (1-t^2)w_1) = \Pi_n(\cdot; w_2), \) we may take

\[
\omega_{n+2}(z,w_1) = (1-z^2)U_n(z)
\]

(21)

giving

\[
\rho_{n+2}(z,w_1) = \int_{-1}^{1} \frac{(1-t^2)U_n(t)}{z-t}w_1(t)\,dt = \int_{-1}^{1} \frac{U_n(t)}{z-t}w_2(t)\,dt
\]

(22)

\[
\int_{-1}^{1} \frac{U_n(t)}{z-t}w_2(t)\,dt = \frac{\pi}{u^{n+1}}
\]

(23)

where \( t \) and \( u \) are related by the familiar conformal maps

\[
z = \frac{1}{2}(u + u^{-1}), \quad |u| > 1
\]

(24)

which transforms the exterior of the unit circle \( \{u \in \mathbb{C} : |u| > 1\} \), into the whole \( z \)-plane cut along \([-1,1] \).

Concentric circles \( |u| = \rho, \quad \rho > 1 \) there by are mapped into confocal ellipses

\[
\rho_p = \left\{ z \in \mathbb{C} : z = \frac{1}{2} \left( \rho e^{i\theta} + \rho^{-1} e^{-i\theta} \right), 0 \leq \theta \leq 2\pi \right\}
\]

(25)

with foci at \( \pm 1 \) and sum of semiarcs equal to \( \rho \).

Substituting (19) in (17) and (18), and noting that

\[
z^2 - 1 = \frac{(u-u^{-1})^2}{4}, \quad \text{one obtains}
\]

\[
K_{n+2}(z,w_1) = -\frac{4\pi}{u^{n+1}(u-u^{-1})(u^{n+1} - u^{-(n+1)})}
\]

(26)

Proceeding to the weight function \( W_2 \), we recall that the node \( \tau_\nu, 2 \leq \nu \leq N - 1, \) are now the zeros of

\[
\Pi_n(\cdot; (1-t^2)w_2) = \Pi_n(\cdot; w_2), \text{ hence, by Lemma 3.1. (with k=2), the zeros of } T_{n+2}^*.
\]

Therefore,

\[
\omega_{n+2}(z,w_2) = (1-z^2)T_{n+2}^*(z)
\]

(27)

With the help of

\[
T_{n+2}^*(z) = \frac{1}{2} [U_{n+2}(z) - U_n(z)], T_{n+2} = (n+2)U_{n+1}(z)
\]

one then gets

\[
\omega_{n+2}(z,w_2) = \frac{n+2}{2} \left[ -(n+2)(U_{n+2}(z) - U_n(z)) + 2zU_{n+1}(z) \right]
\]

(28)

which can be simplified, using the recurrence relation

\[
2zU_{n+1} = U_{n+2} + U_n, \text{ to}
\]

\[
(27) \quad \omega_{n+2}(z,w_2) = -\frac{(n+1)(n+2)}{2} \left( U_{n+2}(z) - \frac{n+3}{n+1} U_n(z) \right)
\]

(29)

In terms of the variable \( u \), cf.(20), using the first relation in (19), this can be written as:

\[
\omega_{n+2}(z,w_2) = \frac{(n+1)(n+2)}{2} \left[ u^{n+3} - u^{-n+3} - \frac{n+3}{n+1} (u^{n+1} - u^{-(n+1)}) \right]
\]

(30)
From (27), and the second relation in (19), we find
\[
\rho_{n+2}(z,w) = \frac{(n+1)(n+2)}{2} \int_{-1}^{1} \frac{U_{n+2}(t)}{1-t} \frac{U_{n+1}(t)}{1-t} w_{n+1}(t) dt - \frac{(n+1)(n+2)}{2} \int_{-1}^{1} \frac{U_{n+1}(t)}{1-t} \frac{U_{n+1}(t)}{1-t} w_{n+1}(t) dt
\]
\[
= - \frac{(n+1)(n+2)}{2} \pi \left[ u^{n+1} - u^{n+2} - u^{n+1} - u^{n+2} \right] / u^{n+1} - u^{n+2}
\]

Therefore, finally,
\[
K_{n+2}(z,w_{2}) = \frac{\pi}{u^{n+1}} \left( u^{1-n} - u^{3-n} - u^{3-n} u^{n+1} - u^{n+1} \right)
\]  

(28)

* In the case \( w = w_{3} \) we have

\[
\omega_{n+2}(t,w_{3}) = \left( 1-t^{2} \right) \pi n \left( t; 1-t \right) \left( 1+t \right) \frac{1}{2} \left( 1+t \right)
\]

hence, by Lemma 3.2, (with \( k=1 \) and (16),

\[
\omega_{n+2}(z,w_{3}) = \left( 1-z^{2} \right) \cdot U_{n+1}(z) \left( 1-z \right) \left( U_{n+1}(z) + \frac{n+2}{n+1} U_{n}(z) \right)
\]

Using (19) togheter with \( 1-z = -\frac{(u-1)^{2}}{2u} \) yields

\[
\omega_{n+2}(z,w_{3}) = -\frac{1}{2} \frac{u-1}{u+2} u^{n+1} - u^{n+2} - u^{n+1} - u^{n+2} \left( u^{n+1} - u^{n+1} \right)
\]

Furthermore,

\[
\rho_{n+2}(z,w_{3}) = \int_{-1}^{1} \omega_{n+2}(t,w_{3}) \frac{w_{n+1}(t)}{z-t} dt
\]

\[
= \int_{-1}^{1} \frac{u^{n+1} + n+2}{u^{n+1} + n+1} \frac{w_{n+1}(t)}{z-t} dt = \pi \frac{n+2}{n} \left( u^{n+1} + n+1 \right)
\]

giving

\[
K_{n+2}(z,w_{3}) = -\frac{\pi}{u^{n+1}} \left( u^{1-n} - u^{3-n} - u^{3-n} u^{n+1} - u^{n+1} \right)
\]  

(29)

* In the case \( w = w_{4} \) is easily transformed to the previous case, since \( w_{4}(t) = w_{3}(1-t) \) implies

\[
\omega_{n+2}(z,w_{4}) = (-1)^{n} \omega_{n+2}(-z;w_{3})
\]

\[
\rho_{n+2}(z,w_{4}) = (-1)^{n+1} \rho_{n+2}(-z;w_{3})
\]

Therefore, \( K_{n+2}(z,w_{4}) = -K_{n+2}(-z;w_{3}) \) or equivalently,

\[
K_{n+2}(z,w_{4}) = -K_{n+2}(-z;w_{3})
\]  

(30)

The Kernel for \( w = w_{4} \) is thus obtained from that for \( w = w_{3} \) essentially by reflection on the imaginary axis.

### 3.3 Main results. Gauss-Chebyshev-Stancu type formulae

3.3.1. We consider a quadrature formulae with double fixed nodes \( \pm 1 \) and the weight function \( w = w_{1} \), \( N = n + 4 \), \( \tau_{N} = -1 \), \( \tau_{N}' = 1 \), \( \tau_{1} = \tau_{1}' = 1 \), and \( R_{N}(f) = 0 \) for \( f \in P_{2n+3} \) [3]. The nodes fixed polyom's is the form

\[
\left( 1-t^{2} \right)^{2}.
\]

From here the nodes \( \tau_{\nu}, 2 \leq \nu \leq N - 1 \) are the zeros of polyom

\[
\Pi_{n}(\nu; \left( 1-t^{2} \right)^{2} w_{1}) = \Pi_{n}(\nu; \left( 1-t^{2} \right)^{2} w_{1}) = T_{n+2}^{(2)}(\nu)
\]

In the last equality we are based on lemma 1. Results that the nodes polyom's is:

\[
\omega_{n+4}(z,w_{1}) = \left( 1-z^{2} \right)^{2} T_{n+2}^{(2)}(z)
\]  

(31)

which, by the differential equations satisfied by \( T_{n+2} \), becomes

\[
\omega_{n+2}(z,w_{2}) = z T_{n+2}'(z) - (n+2)^{2} T_{n+2}(z)
\]

and with the help of:

\[
T_{n+2}'(z) = \frac{1}{2} \left[ U_{n+2}(z) - U_{n}(z) \right]
\]

\[
T_{n+2}(z) = (n+2) U_{n}(z)
\]

one then gets:

\[
\omega_{n+4}(z,w_{1}) = \left( 1-z^{2} \right)^{2} \left[ z T_{n+2}'(z) - (n+2)^{2} T_{n+2}(z) \right]
\]

or

\[
\omega_{n+4}(z,w_{1}) = \frac{(n+1)(n+2)}{2} \left( 1-z^{2} \right)^{2} (n+2)^{2} U_{n}(z)
\]  

(32)

and with the relations (6) and (32) we have
\[ \rho_{n+4}(z, \omega_1) = \int_{-1}^{1} \frac{(1-t)^2}{z-t} T_{n+2}^{(2)}(t) w(t) \, dt = -\frac{(n+1)(n+2)}{2u^{n+4}} \left[ u^{-2} - \frac{n+3}{n+2} \right] \]

We recall (11) and operate \( 1-z^2 = \left( \frac{u-u^{-1}}{2} \right)^2 \) we obtain
\[ \omega_{n+4}(z, \omega_1) = \frac{(n+1)(n+2)}{4} \left( u-u^{-1} \right)^2 \cdot \frac{u^{n+3} - u^{-(n+3)}}{u-u^{-1}} - \frac{n+3}{n+1} \left( u^{n+1} - u^{-(n+1)} \right) \]

Therefore finally
\[ K_{n+4}(z, \omega_1) = \frac{\rho_{n+4}(z, \omega_1)}{\omega_{n+4}(z, \omega_1)} = \frac{4\pi}{u^{n+1} \left( u-u^{-1} \right)^2} \cdot \frac{u^{-1} - u^{-3} - \frac{n+3}{n+1} \left( u-u^{-1} \right)}{u^{n+3} - u^{-(n+3)} - \frac{n+3}{n+1} \left( u^{n+1} - u^{-(n+1)} \right)} \]  \hspace{1cm} (33)

**Remark 1** The kernel of Gauss-Chebyshev-Stancu formulae verify:
\[ K_{n+4}(z, \omega_1) = \frac{4}{(u-u^{-1})^2} K_{n+2}(z, \omega_1) \]

where \( K_{n+2}(z, \omega_2) \) represented the kernel of Gauss-Lobatto quadrature formulae with simple fixed nodes to the Chebyshev function of second kind.

This last kernel it was determined in relation [28] by the W. Gautschi.

3.3.2. We also consider a quadrature formulae with fixed nodes \( \tau_N = -1 \) (simple) and \( \tau_1 = 1 = \tau_1 \) (double) follow that the polynom of fixed nodes is \( (1-t)(1+t)^2 \) and the weight function \( w = w_1 \).

\( N = n+3 \) and \( \Re(f) = 0 \), for any polynom \( f \in P_{2n+2} \).

From here the nodes \( \tau_{\nu}, 2 \leq \nu \leq N-1 \) are the zeros of polynom:
\[ \Pi_n \left( (1-t)(1+t)^2 w \right) = \Pi_n \left( (1-t)^{1+2} \right) = \Pi_n \left( (1-t)^2 \right) \]  

and from here we have:
\[ \omega_{n+3}(z, \omega_1) = (1-z)^{1+2} U_{n,1}(z) \]  \hspace{1cm} (34)

But cf. Lemma 3.2. we have:
\[ U_{n,1}(t) = \frac{1}{1+t} \left( U_{n+1,0}(t) + \frac{(n+3/2)(n+2)}{(n+1)(n+3/2)} U_{n,1}(t) \right) \]

or
\[ U_n(t) = \frac{1}{1+t} U_{n+1}(t) + \frac{n+1}{n+1} U_n(t) \]

Thus:
\[ \rho_{n+3}(z, \omega_1) = \int_{-1}^{1} \frac{(1-t)(1+t)^2 U_{n,1}(t)}{z-t} w(t) \, dt = \]
\[ \int_{-1}^{1} \frac{(1-t)(1+t)^2}{z-t} \left[ U_{n+1}(t) + \frac{n+2}{n+1} U_n(t) \right] w(t) \, dt = \]
\[ \int_{-1}^{1} \frac{1}{z-t} \left[ U_{n+1}(t) + \frac{n+2}{n+1} U_n(t) \right] w_2(t) \, dt = \]
\[ \int_{-1}^{1} \frac{1}{z-t} U_{n+1}(t) w_2(t) \, dt + \frac{n+2}{n+1} \int_{-1}^{1} \frac{1}{z-t} U_n(t) w_2(t) \, dt = \]
\[ = \frac{\Pi}{u^{n+2}} + \frac{n+2}{n+1} \frac{\Pi}{u^{n+1}} \]

Result that:
\[ \rho_{n+3}(z, \omega_1) = \frac{\Pi}{u^{n+1}} \left( \frac{n+2}{n+1} + \frac{1}{u} \right) \]  \hspace{1cm} (35)

We recall (34) and operate \( 1-z^2 = -\left( \frac{u-u^{-1}}{2} \right)^2 \) we obtain:
\[ w_{n+3}(z, \omega_1) = -\frac{(u-u^{-1})^2}{4} \left[ \frac{u^{n+2} - u^{-(n+2)}}{u-u^{-1}} + \frac{n+2}{n+1} \frac{u^{n+1} - u^{-(n+1)}}{u-u^{-1}} \right] = \]
\[ = \left( \frac{u-u^{-1}}{4} \right) \left[ u^{n+2} - u^{-(n+2)} + \frac{n+2}{n+1} \frac{u^{n+1} - u^{-(n+1)}}{u-u^{-1}} \right] \]

For Kernel, with wellknown formulae, we have
\[ K_{n+3}(z; w_1) = \frac{\rho_{n+3}(z; w_1)}{\omega_{n+3}(z; w_1)} = \frac{1}{u^{n+1}} \left[ \frac{n+2}{n+1} \left( \frac{u+1}{u} \right)^{(n+1)} - 1 \right] \]

\[ \frac{u-u^{-1}}{4} \left[ u^{n+2} - u^{-1} - (n+2) + \frac{n+2}{n+1} \left( u^{n+1} - u^{-1} - (n+1) \right) \right] \]

and

\[ K_{n+3}(z; w_1) = \frac{4\pi}{u^{n+1}(u-u^{-1})} \left( u^{n+2} - u^{-1} - (n+2) + \frac{n+2}{n+1} \left( u^{n+1} - u^{-1} - (n+1) \right) \right) \]

**Remark 2:** The Kernel \( K_{n+3}(z; w_1) \) verify:

\[ K_{n+3}(z; w_1) = \frac{2n}{(n+1)^2} K_{n+2}(z; w_3) \]

Where \( K_{n+2}(z; w_3) \) represented the Kernel of Gauss-Lobatto quadrature formulae with simple fixed nodes to the Chebyshev weight function of third kind.

### 3.4. Chebyshev-Radau formulae

In analogy to (19) one has

\[ V_n(z) = \frac{u^{n+1} + u^{-n}}{u+1}, \]

\[ \frac{1}{z-t} \int_{-1}^{1} \frac{V_n(t)}{z-t} w_3(t) dt = \frac{2\pi}{(u-1)u^n} \]

(37)

The first relation follows from the second relation in (14) by writing all cosines in exponential form, using Euler’s formula, and then putting \( u = e^{i\theta} \). To prove the second relation, substitute \( t = \cos \theta \) to obtain:

\[ \frac{1}{z-t} \int_{-1}^{1} \frac{V_n(t)}{z-t} w_3(t) dt = 2 \int_{0}^{\pi} \frac{\cos \left( \frac{n+1}{2} \right) \theta \cdot \cos \frac{1}{2} \theta}{z - \cos \theta} d\theta = \]

\[ \frac{\pi}{2} \cos \left( \frac{n+1}{2} \right) \theta \cdot \cos \theta \]

\[ \int_{0}^{\pi} \frac{\cos \left( \frac{n+1}{2} \right) \theta \cdot \cos n\theta}{z - \cos \theta} \]

And then use Equation (5.3) in [1] and the equation immediately following it to evaluate the last integral. For reasons indicated in Subsection 2.2, we consider only Radau formulae with the fixed point at -1. Thus, \( N = n+1, \tau_n = -1 \) in (1), and \( \Re_N(f) = 0 \) for \( f \in P_{2n} \). We treat in turn the four weight functions \( w_i, i = 1, 2, 3, 4 \) (cf. (9))

* For \( w = w_1 \), in view of \( \Pi_n(\cdot(1+t)w_1) = \Pi_n(\cdot w_3) \), we can take

\[ \omega_{n+1}(z; w_1) = (1+z)V_n(z), \]

which, by the first relation in (37) and \( 1+z = \left( \frac{u+1}{2} \right)^2 \), gives

\[ \omega_{n+1}(z; w_1) = \frac{1}{2} \left( \frac{u+1}{2} \right)^2 \left( \frac{u^n}{2} + u^{-1} - (n+1) \right) \]

And by the second relation in (37)

\[ \rho_{n+1}(z; w_1) = \frac{2\pi}{(u-1)u^n}, \]

hence

\[ K_{n+1}(z; w_1) = \frac{4\pi u}{(u^2-1)(u^{2n+1}+1)} \]

(38)

* In the case \( w = w_2 \), we are led to

\[ \Pi_n(\cdot(1+t)w_2) = \Pi_n \left( \cdot \left( 1-t \right)^2 \right)^{\frac{3}{2}} \]

And may apply Lemma 3.2. (K = 1) to obtain

\[ \omega_{n+1}(z; w_2) = (1+z)U_{n,1}(z) = \]

\[ U_{n+1}(z) + \frac{n+2}{n+1} U_n(z) \]

Using (19) we find

\[ \omega_{n+1}(z; w_2) = \]

\[ \frac{1}{u-u^{-1}} \left\{ u^{n+2} - u^{-1} - (n+2) + \frac{n+2}{n+1} \left( u^{n+1} - u^{-1} - (n+1) \right) \right\} \]

And

\[ \rho_{n+1}(z; w_2) = \]

\[ \frac{\pi}{u^{n+1}} \left( u^{-1} + \frac{n+2}{n+1} \right) \]
Giving
\[ K_{n+1}(z,w_2) = \frac{1}{u^{n+1}} \cdot \frac{1 - u^{-2} + \frac{n+2}{n+1} u - u^{-1}}{u^{n+2} - u^{-(n+2)} + \frac{n+2}{n+1} (u^{n+1} - u^{-(n+1)})} \]  
(39)

* For \( w = w_3 \) since
\[ \Pi_n \left( \cdot ; (1+t) w_3 \right) = \Pi_n \left( \cdot ; (1-t)^2 \left( 1 + t \right)^2 \right) \]

we can appeal to Lemma 3.3 (with \( K=1 \)) and obtain, similarly as above, using (37), that
\[ K_{n+1}(z,w_3) = \frac{2\pi n+1}{u^n n-1 u^{n+2} + u^{-(n+1)} + \frac{2n+3}{2n+1} (u^{n+1} + u^{-n})} \]  
(40)

* Finally, when \( w = w_4 \), we have \((1+t) w_4 = w_2\) so that
\[ \omega_{n+1}(z,w_4) = (1+z) U_n(z) \]
and we find using (19) that
\[ K_{n+1}(z,w_4) = \frac{2\pi}{u^{n+1}} \cdot \frac{u-1}{(u+1) (u^{n+1} - u^{-(n+1)})} \]  
(41)

* We also consider a quadrature formulae with double fix node \(-1\) and the weight function \( w = w_1 \), \( N = n + 2 \), \( \tau_N = \tau'_N = -1 \) and \( \Re(f) = 0 \) for \( f \in P_{2n+1} \) [3].

The nodes fixed polynom’s is in the form \((1+t)^2\). From here the nodes \( \tau_N, 2 \leq v \leq N-1 \) are the zeros of the polynom
\[ \Pi_n \left( \cdot ; (1+t)^2 w_1 \right) = \Pi_n \left( \cdot ; (1-t)^2 \left( 1 + t \right)^2 \right) \]

From Lemma 3.3. (\( K=1 \)) results for nodes polynom’s that:
\[ \omega_{n+2}(z,w_1) = (1+z)^2 V_{n,1}(z) = \left[ \frac{n+3}{2} V_{n+1}(z) + \frac{n+1}{2} V_n(z) \right] \]

\[ (1+z) V_{n+1}(z) + \frac{n+3}{1+2} V_n(z) \]

Than, from relation (4) we have:
\[ \rho_{n+2}(z,w_1) = \int_{-1}^{0} \frac{\omega_{n+2}(t,w_1) w_1(t) dt}{z-t} \]

\[ \left( 1+t \right)^2 V_{n+1}(t) + \frac{n+3}{1+2} V_n(t) \]
\[ \int_{-1}^{0} \frac{1}{z-t} \left( 1-t \right)^{\frac{1}{2}} \left( 1+t \right)^{\frac{1}{2}} dt \]

\[ \int_{-1}^{0} \frac{V_{n+1}(t) + \frac{n+3}{1+2} V_n(t)}{z-t} \int_{-1}^{0} \left( 1-t \right)^{\frac{1}{2}} \left( 1+t \right)^{\frac{1}{2}} dt \]

\[ \frac{2\pi}{(u-1) u^{n+1}} + \frac{n+3}{1+2} \left( 1-t \right)^{\frac{1}{2}} \left( 1+t \right)^{\frac{1}{2}} \]

\[ \frac{2\pi}{(u-1) u^{n}} \left( 1+ \frac{n+3}{1+2} \right) \]  
(42)

IV. THE MAXIMUM OF THE KERNEL FOR CHEBYSHEV WEIGHT FUNCTIONS

4.1. For \( w = w_2 \) we have only empirical and asymptotic results. The computation shows that \( K_{n+2}(z,w_2) \), \( z \in \mathbb{C} \), attains its maximum on the real axis if \( n = 1 \) or \( n = 2 \). If \( n \) is odd and \( n \geq 3 \), the maximum is attained on the real axis if \( 1 < \rho < \rho_n \), and on the imaging axis if \( \rho_n < \rho \) (at either place if \( \rho = \rho_n \)).

If \( n \geq 4 \) is even, the behavior is more complicated: we have a maximum on the real axis if \( 1 < \rho < \rho'_n \), on the imaging axis if \( \rho_n < \rho \), and in between if \( \rho'_n < \rho < \rho_n \), where \( \rho'_n, \rho_n \) are certain numbers satisfying \( 1 < \rho'_n < \rho_n \).

Numerical value for \( n = 3(1)20 \) have been determined by a bisection procedure in [5] and are shown in Table 1.
Table 1:

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<th>ρ′_n</th>
<th>ρ_n</th>
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The empirical observations above can be verified asymptotically as $\rho \searrow 1$, or as $\rho \to \infty$, for any fixed $n$. In the just case a lengthy calculation reveals that when $\theta = 0$ (i.e., $z = \frac{\rho + \rho^{-1}}{2}$)

$$
\left| K_{n+2} \left( \frac{1}{2} (\rho + \rho^{-1}), w_2 \right) \right| \sim \frac{3\pi}{(n+1)(n+2)(n+3)} (\rho^{-1})^{-2}, \rho \searrow 1
$$

(43)

The maximum of kernel $K_{n+4}(z; w_1)$ is asymptotically determined by the:

$$
\max_{z \in \mathbb{C}_r} \left| K_{n+1}(z, w_1) \right| = \max_{z \in \mathbb{C}_r} \left| K_{n+2}(z, w_2) \right| \sim \frac{4}{(u-u^{-1})^2} \frac{3\pi}{(n+1)(n+2)(n+3)} (\rho^{-1})^{-2} (\rho-1)^2
$$

whereas, for other values of $\theta$, including $\theta = \frac{\pi}{2}$.

4.2. Radau formulae; circular contours. The case $w_1$ again, is amenable to analytic treatment. We now have

$$
z = re^{i\theta}, \ r > 1, \text{ and } \ u = z + \sqrt{z^2 - 1} = e^{i\theta} \left( r + \sqrt{r^2 - e^{-2i\theta}} \right),
$$

where the branch of square root is taken that assigns positive values to positive arguments. There follows

$$
\frac{u}{u^2 - 1} = \frac{1}{u - u^{-1}} = \left( 2e^{i\theta} \sqrt{r^2 - e^{-2i\theta}} \right)^{-1}
$$

Hence

$$
\left| \frac{u}{u^2 - 1} \right| \leq \frac{1}{2\sqrt{r^2 - 1}}
$$

The bound being attained for $\theta = 0$ and $\theta = \pi$. Furthermore,

$$
\left| u^{2n+1} + 1 \right| = \left| \left( r + \sqrt{r^2 - e^{-2i\theta}} \right)^{2n+1} + e^{-i(2n+1)\theta} \right| \geq \left| r + \sqrt{r^2 - e^{-2i\theta}} \right|^{2n+1} - 1 \geq \left( r + \sqrt{r^2 - 1} \right)^{2n+1} - 1
$$

with equality holding for $\theta = \pi$. Consequently,

$$
\max_{z \in \mathbb{C}_r} \left| K_{n+1}(z; w_1) \right| = \frac{4\pi}{R - R^{-1}} \frac{1}{R^{2n+1} - 1}
$$

(44)
where \( R = r + \sqrt{r^2 - 1} \). (45)

**Theorema 1**

The kernel of the \((n+1)\)–point Radau formula (with fixed node at \(-1\)) for the Chebyshev weight function \(w_1\) attains its maximum modulus on \(C_r\) on the negative real axis; the maximum is given by (44), (45).

For \(w = w_2\), we conjecture

\[
\max_{z \in C_r} |K_{n+1}(z, w_2)| = |K_{n+1}(-r, w_2)| = \frac{\pi}{R^{n+2} \left( R^{n+2} - R^{(n+2)} + (R^{n+1} - R^n) \right)}.
\]

Where the denominator is easily shown to be positive for \(R > 1\), and for \(w = w_3\),

\[
\max_{z \in C_r} |K_{n+1}(z, w_3)| = |K_{n+1}(r, w_3)| = \frac{2\pi R + 1}{R^{n+1} R - 1 \left( R^{n+2} - R^{(n+2)} + (R^{n+1} - R^n) \right)}.
\]

When \(w = w_4\), the Kernel is sufficiently simple to be treated analytically. Note, first of all, we have \(|u| \geq R\) (with equality for \(\theta = \pi\)) hence

\[
\left| u^{n+1} - u^{(n+1)} \right| \geq |u|^{n+1} - 1 \geq R^{n+1} - 1.
\]

Again with equality holding for \(\theta = \pi\). Next, there follows

\[
\frac{z - 1}{z + 1} = \left( \frac{u - 1}{u + 1} \right)^2,
\]

so that

\[
\left| \frac{u - 1}{u + 1} \right|^4 = \left( \frac{z - 1}{z + 1} \right)^2 = \frac{r^2 - 2r \cos \theta + 1}{r^2 + 2r \cos \theta + 1} \leq \frac{r + 1}{r - 1} = \frac{R + 1}{R - 1}.
\]

Here again, the bound is attained for \(\theta = \pi\). Consequently,

\[
\max_{z \in C_r} |K_{n+1}(z; w_4)| = |K_{n+1}(-r; w_4)| = 2\pi \frac{R + 1}{R - 1 R^{2n+2} - 1}.
\]

This proves

**Theorema 2**

The kernel of the \((n+1)\)–point Radau formula (with fixed node at \(-1\)) for the Chebyshev weight function \(w_4\) attains its maximum modulus on \(C_r\) on the negative real axis; the maximum is given by (46).

4.3. Radau formulae: elliptic contours.

Putting \(u = \rho e^{i\theta}\) in (38), one obtains, for \(w = w_1\)

\[
|K_{n+1}(z, w_1)| = \frac{4\pi\rho}{\left[ (\rho^4 - 2\rho^2 \cos 2\theta + 1)(\rho^4 + 2\rho^2 + 1) \cos (2n+1) \theta + 1 \right]^2}.
\]

Which clearly takes on its maximum at \(\theta = \pi\). Thus

\[
\max_{z \in \rho} |K_{n+1}(z; w_1)| = |K_{n+1}(-1/2, \rho^{-1})| = \frac{4\pi\rho}{\left( \rho^2 - 1 \right) \left( \rho^{2n+1} - 1 \right)}.
\]

and we have

**Theorema 3**

The kernel of the \((n+1)\)–point Radau formula (with fixed node at \(-1\)) for the Chebyshev weight function \(w_1\) attains its maximum modulus on \(\rho\) on the real axis; the maximum is given by (47).

For \(w = w_2\) and \(w = w_3\), the kernel is found by computation to behave more curiously. In the former case, we have a situation similar to the Lobatto formula for the same weight function, namely, the maximum is attained on the negative real axis, when \(n = 1, 2, 3\), and also when \(n \geq 4\), but
then only if \( 1 < \rho < \rho_n^+ \) or \( \rho_n^- < \rho \), where \( \rho_n^+ \), \( \rho_n^- \) are shown in Table 1; otherwise, the maximum point moves on the ellipse \( \mathcal{E}_\rho \) from somewhere close to the imaginary axis to the negative real axis as \( \rho \) increases.

### Table 1

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \rho_n^+ )</th>
<th>( \rho_n^- )</th>
</tr>
</thead>
<tbody>
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<td>4.7385</td>
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<td>18.533</td>
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</table>

Assymptotically one finds, consistent with the above, that

\[
\left| K_{n+1}(z; w_2) \right| \sim \frac{(n+2)^\pi}{(n+1)^{2n-2}} \rho^{2n+3} \left\{ 1 - 2 \frac{2n+3}{(n+1)(n+2)} \rho^{-1} \cos \vartheta \right\}^{\frac{1}{2}} \rho \to \infty, \tag{48}
\]

and (49)

\[
\left| K_{n+1}\left( \frac{1}{2}(\rho + \rho^{-1}), w_2 \right) \right| \frac{6\pi}{(n+1)(n+2)(2n+3)} \rho^{-2}, \quad \rho \downarrow 1,
\]

the value at the other end approaching the finite limit

\[
\frac{(2n+3)\pi}{2(n+1)(n+2)} \quad \text{when} \quad \rho \downarrow 1,
\]

and there being the familiar peaks of \( O\left( (\rho - 1)^{-1} \right) \) when

\[
(n+1)\sin (n+2) \vartheta + (n+2)\sin (n+1) \vartheta = 0, \quad 0 < \vartheta < \pi
\]

### References


