Study of algorithms for decomposition of a numerical semigroup

Branco, M. B.
Universidade de Évora
Departamento de Matemática
7000 Évora
Portugal
mbb@uevora.pt

Franco, Nuno
Universidade de Évora
Departamento de Matemática
7000 Évora
Portugal
nmf@uevora.pt

Abstract: We study two algorithms to decompose a numerical semigroup S as intersection of irreducible numerical semigroups. We also present a compared study of two algorithms to compute the intersection of two numerical semigroups with embedding dimension two and the same multiplicity.

1 Introduction and basic concepts

Let $\mathbb{N}$ be the set of non negative integers. A **numerical semigroup** S is a subset of $\mathbb{N}$ which contains the zero, is closed under addition and generates $\mathbb{Z}$ as a group (here $\mathbb{Z}$ denotes the set of the integers). From this definition, we can deduce that S admits a unique minimal system of generators $\{s_1 < \cdots < s_p\}$, meaning that $S = \{\sum_{i=1}^{p} a_i s_i \mid a_1, \ldots, a_p \in \mathbb{N}\}$ and no proper subset of $\{s_1, \ldots, s_p\}$ generates S. The integers s_1 and p are known as the **multiplicity** and **embedding dimension** of S. Moreover, $\mathbb{N} \setminus S$ is finite, and the largest integer not belonging to S is known as the **Frobenius number** of S, usually denoted by $\mathrm{g}(S)$.

Given $s_1 \in S \setminus \{0\}$, the **Apéry set** (called so after [1]) of S with respect to s_1 is defined by $\mathrm{Ap}(S, s_1) = \{s \in S \mid s - s_1 \notin S\}$ and it can be proved that if we choose $w(i)$ to be the least element in S congruent with i modulo s_1, then $\mathrm{Ap}(S, s_1) = \{0, w(1), \ldots, w(n-1)\}$. The set $\mathrm{Ap}(S, s_1)$ determines completely the semigroup S, since $S = \langle \mathrm{Ap}(S, s_1) \cup \{s_1\} \rangle$. Moreover, $\mathrm{Ap}(S, s_1)$ contains in general more information that an arbitrary set of generators of S; for instance, $\mathrm{g}(S) = \max(\mathrm{Ap}(S, s_1)) - s_1$.

We say that a numerical semigroup is **irreducible** if it can not be expressed as an intersection of two numerical semigroups containing it properly. From [2] and [3] we can deduce that the class of irreducible numerical semigroups with odd (respectively even) Frobenius number is the same that the class of **symmetric** (respectively **pseudo-symmetric**) numerical semigroups. This kind of numerical semigroups have been widely studied in literature not only from the semigroupist point of view but also by their applications in Ring Theory. In [2] it is show that the semigroup ring associated to an irreducible numerical semigroup is Gorestein or Kunz if the Frobenius number is odd or even, respectively.

The contents of this work are organized as follows. In Section 2, we compare two different algorithms to obtain a numerical semigroup as an intersection of irreducibles numerical semigroups. In Section 3, we present an algorithm to compute the intersection of two numerical semigroups with embedding dimension two and the same multiplicity, and we compare the classical intersection algorithm with this one. In both cases we study the complexity of these algorithms.

2 Two algorithms for decomposition of numerical semigroups as an intersection of irreducibles

Let S be a numerical semigroup. We say that an element $x \in \mathbb{Z}$ is a **pseudo-Frobenius number** of S if $x \notin S$ but $x + s \in S$ for all $s \in S \setminus 0$. We denote by $\mathrm{Pg(S)}$ the set of pseudo-Frobenius numbers of S.

We define in S the following partial order: $a \leq_S b$ if $b - a \in S$.

In [3, Proposition 7] is proved the following result showing the connection between the pseudo-Frobenius number and the Apéry set of s_1 in S.

Lemma 1 *If S is a numerical semigroup, $s_1 \in S \setminus \{0\}$ and $\{w_{i_1}, \ldots, w_{i_t}\} = \mathrm{maximals}_{\leq_S} \mathrm{Ap}(S, s_1)$, then* $\mathrm{Pg(S)} = \{w_{i_1} - s_1, \ldots, w_{i_t} - s_1\}$.

Given a numerical semigroup S, denote by

$$\mathrm{H}(S) = \mathbb{N} \setminus S$$

$$\mathrm{EH}(S) = \{x \in \mathrm{H}(S) \,:\, 2x \in S, x + s \in S \text{ for all } s \in S \setminus \{0\}\}.$$

The set $\mathrm{EH}(S)$ is a subset of $\mathrm{Pg(S)}$ and thus $\#\mathrm{EH}(S) \leq \#PG(S) \leq m(S)-1$. From definition of $\mathrm{EH}(S)$, it easy to prove the next result, which describes those elements that added to a numerical semigroup yield a numerical semigroup.

Proposition 2 *Let S be a numerical semigroup and $x \notin S$. Then $x \in \mathrm{EH}(S)$ if only if $S \cup \{x\}$ is a numerical semigroup.*

Every numerical semigroup containing properly the numerical semigroup S must contain an element of $\mathrm{EH}(S)$. Let $\mathrm{EH}(S) = \{x_1, \ldots, x_k\}$, from an numerical semigroup S adding only an element of $\mathrm{EH}(S)$ we get new numerical semigroups $S \cup \{x_1\}, \ldots, S \cup \{x_k\}$. Thus we can compute a finite family of numerical semigroups that contain S, denote it by $\mathcal{V}(S)$.

In [5] it is presented that S is irreducible if and only if S is maximal in the set of numerical semigroups not containing $\mathrm{g}(S)$. Then we get the following result:

Corollary 3 *A numerical semigroup S is irreducible if and only if $\#\mathrm{EH}(S) = 1$.*

The idea is the following one: from S we compute the set $\mathrm{EH}(S) = \{x_1, x_2, \cdots, x_k\}$, and thus we obtain the numerical semigroups $S \cup \{x_i\}$; for each $S \cup \{x_i\}$ we make the same. We stop when $\#\mathrm{EH}(S_i) = 1$, and thus we get all irreducible numerical semigroups that contain S.

Using the above results from S we can compute $\mathcal{V}(S)$ and thus $S = \bigcap_i S_i$, with $S_i \in \mathcal{V}(S)$ and S_i irreducibles. If we remove the irreducibles that not minimal, we have the following:

Proposition 4 *Let S be a numerical semigroup. Then*

$$S = S_1 \cap \ldots \cap S_n.$$

such that $S_1, \ldots, S_n$ are the minimal irreducible elements in $\mathcal{V}(S)$.

Our objective is to compare two different ways to obtain a semigroup S as intersection of irreducible semigroups. This algorithm is presented in [8] and it needs to construct the set $\mathrm{EH}(S)$. We start by describing two different algorithms to compute the set $\mathrm{EH}(S)$. Suppose that $S = \{0, s_1, s_2, \ldots, s_r, \longrightarrow\}$ is a semigroup represented as a set starting at 0 and has all elements of S until $s_r = \mathrm{g}(S) + 1$. From the definition we can easily see that the set $\mathrm{EH}(S)$ is finite.

Algorithm EH 1

INPUT: A semigroup $S = \{0, s_1, s_2, \ldots, s_r, \longrightarrow\}$

1. Compute the set $H(S) = \mathbb{N} \setminus S$.
2. Compute the set $D(S) = \{x \in H(S) : 2x \in S\}$
3. Compute the set $\mathrm{EH}(S)$ by checking if $x + s \in S$ for all $x \in D(S)$ and $s \in S$.

OUTPUT: The set $\mathrm{EH}(S)$.

Proposition 5 *The Algorithm EH 1, computes the set* $\mathrm{EH}(S)$.

Proof.
It is obvious from de definition of $\mathrm{EH}(S)$.

Algorithm EH 2

INPUT: A semigroup $S = \{0, s_1, s_2, \ldots, s_r, \longrightarrow\}$

1. Compute $\mathrm{Ap}(S, s_1) = \{Ap_1, \ldots, Ap_{s_1}\}$, Apéry set of S
2. Compute the set $E(S) = \max(\mathrm{Ap}(S))$ with respect to the partial order $<$ in $\mathrm{Ap}(S)$).
3. Compute $PGS(S)$, the pseudo Frobenius numbers of S.
4. Compute $\mathrm{EH}(S) = \{x \in PGS(S) : 2x \in S\}$

OUTPUT: The set $EH(S)$.

Proposition 6 *The Algorithm EH 2, computes the set* $EH(S)$.

Proof.
From de definition of $PGS(S)$ and $\mathrm{EH}(S)$.

The main algorithm is the following:

Algorithm Intersection 1/2

INPUT: A semigroup $S = \{0, s_1, s_2, \ldots, s_r, \longrightarrow\}$

1. Set $R = \{\}$ and $E = \{\}$.
2. Compute $\mathrm{EH}(S) = \{e_1, \ldots, e_p\}$, using algorithm EH 1/2.
3. If $p = 1$ then $RF(S) = \{S\}$ and goto step 10.
4. Set $R_i = S \cup \{e_i\}$ for $i = 1, \ldots, p$ and $R = R \cup \{R_i\}$
5. Set $j = 1$ and $t = p$

6. Compute $\mathrm{EH}(R_j) = \{e_{j,1}, \ldots, e_{j,p_j}\}$ and set $E = E \cup \{EH(R_j)\}$
7. Set $R_{\#R+1} = R_j \cup \{e_{j,k}\}$ for $k = 1, \ldots, p_j$ and $R = R \cup \{R_{\#R+1}\}$
8. If $t \neq \#R$ then set $t = t+1$, $j = j+1$ and goto step 6.
9. Set $RF(S) = \{R_{i_1}, \ldots R_{i_q}\}$ where q is minimal such that $S = \bigcap_{X \in RF(S)} X$.
10. Return $RF(S)$.

OUTPUT: A list $RF(S)$ of semigroups such that $S = \bigcap_{X \in RF(S)} X$.

Proposition 7 *Let S be a semigroup. Algorithm Intersection 1/2 computes a minimal set of semigroups which intersection is S.*

Proof.
From the proposition 4.

2.1 The Complexity

The complexity of these algorithms will be expressed as function of $\mathrm{g}(S)$, $s_1, \ldots, s_p$ the set of generators of S and the size of the tree of semigroups.

The semigroup S is given by its generators so we have complexity $O(g(S))$ to write S in the forme described above. To compute the Apéry set of S we have again complexity $O(g(S))$.

The complexity of Algorithm EH 1

First, we compute $D(S)$ with complexity $O(g(S) - \frac{s_1}{2})$. Now to compute $\mathrm{EH}(S)$ we must test if $x - s \in S$ for all $s \in S$ and $x \in D(S)$ so we achieve this with complexity $O(g(S)^2 - \frac{s_1}{2}g(S))$. We conclude that the complexity of Algorithm EH 1 is $O(g(S) - \frac{s_1}{2} + g(S)^2 - \frac{s_1}{2}g(S)) = O(g(S)^2 - \frac{s_1}{2}g(S))$

The complexity of Algorithm EH 2

First, we compute $Ap(S, s_1)$ with complexity $O(g(S) + s_1)$. We have that $\#Ap(S, s_1) = s_1$ and so the complexity of ordering this set, to compute $E(S)$, is $O(s_1(s_1 - 1)) = O(s_1^2)$. The set $PGS(S)$ is computed with $O(\#Ap(S, s_1)) = O(s_1)$ complexity. Finally the set $EH(S)$ is computed with complexity $O(s_1)$. Hence the total complexity of this algorithm is $O(g(S) + s_1 + s_1^2 + s_1) = O(g(S) + s_1^2)$

Remark 8 *Note that if* $S = < s_1, s_2 >$ *then* $\mathrm{g}(S) = s_1 s_2 - (s_1 + s_2)$ *or in the case where* $S = < s_1, \ldots, s_p >$ *is a* $MED - semigroup$ *then* $\mathrm{g}(S) = s_p - s_1$ *and thus we can use this result above. We can see that in these cases the complexity of Algorithm EH 1 is greater than the complexity of Algorithm EH 2.*

The complexity of Algorithm Intersection 1/2

We start by computing $EH(S)$ (using **Algorithm EH 1/2**) with complexity $O(EH)$ described above. Then after constructing the semigroups $R_i = S \cup \{e_i\}$ we compute $EH(R_i)$ and repeat this process until there are no new semigroups that appear. This is done with complexity $O(T)O(EH)$, where T is the total number of semigroups to intersect. Finally we eliminate those which are redundant. So the final complexity of these algorithms are $O(T(g(S)^2 - \frac{s_1}{2}g(S)))$ and $O(T(g(S) + s_1^2))$ respectively.

The value of T is not predictable. Meaning that we do not know any upper bound for it because it arises from a tree structure (see [8]). We will indicate in the experimental results the maximum value of T for each set of tested semigroups.

2.2 Experimental results

In order to test the efficiency of both algorithms we defined 200 random semigroups with 3 up to 10 generators bounded by 100, 200 and 300. We computed the maximum running time (MRT) of each algorithm and the overall average running time (ART). The results (given in seconds) are summarized in the following tables:

- For generators with values up to 100:

Generators ;	3	4	5	6	7	8	9	10
MRT for Alg 1	16.3580	14.5800	13.5790	5.2340	5.9220	3.1560	5.5320	5.1710
MRT for Alg 2	16.3140	14.3740	13.2650	5.1410	5.7670	3.0470	5.4200	5.0790
ART for Alg 1	1.2381	0.9071	0.7167	0.3954	0.5231	0.2343	0.2497	0.2343
ART for Alg 2	1.2228	0.8918	0.7126	0.3877	0.5135	0.2275	0.2457	0.2315
Max T	11	23	22	20	24	22	24	25
Average T	4.290	5.750	6.945	6.305	7.725	5.970	6.040	5.895

- For generators with values up to 200:

Generators	3	4	5	6	7	8	9	10
MRT for Alg 1	183.286	116.424	107.250	73.407	57.735	162.221	35.420	50.720
MRT for Alg 2	184.296	116.139	107.283	72.641	57.216	162.093	35.704	50.061
ART for Alg 1	15.808	13.261	9.892	7.004	4.923	6.405	2.772	2.460
ART for Alg 2	15.804	13.116	9.843	6.921	4.865	6.316	2.736	2.421
Max T	11	24	34	38	37	45	33	41
Average T	4.930	9.710	11.105	12.540	11.860	12.595	10.695	9.960

- For generators with values up to 300:

Generators	; 3	4	5	6	7	8	9	10
MRT for Alg 1	1139.221	561.259	456.063	346.933	798.463	321.449	100.201	275.079
MRT for Alg 2	1153.356	563.537	459.562	346.766	653.346	351.020	99.637	274.061
ART for Alg 1	65.575	43.807	35.525	27.618	31.366	17.984	10.404	12.120
ART for Alg 2	63.842	43.675	35.420	27.601	30.389	17.795	10.270	12.005
Max T	11	36	40	42	55	51	42	55
Average T	5.540	10.790	14.750	14.970	17.370	16.620	15.430	14.850

2.3 Conclusion

The experimental data show us two different things. The first one is that Algorithm 2 is in practice faster then Algorithm 1,(comparing the corresponding ART). The second is that, surprisingly, it is only slightly faster, indeed the difference between the corresponding ART is quite small (approximately around 0.5%). The worst case scenario complexity, of the two algorithms, are not comparable in general. This happens because there are no known relations between de frobenius number and the multiplicity of a semigroup. But for the particular semigroups, presented in section 2.1 remark 8, this relation is known and hence we are able to compare them.

3 The intersection of two numerical semigroups with embedding dimension two and the same multiplicity

In this section we study a special case of the intersection of irreducible numerical semigroups. Note that if S has embedding dimension two, then S is irreducible (i.e, S is symmetric or pseudo-symmetric)) [5]), hence we have the following o result:

Lemma 9 *Let $S = \langle s_1, s_2\rangle$ be a numerical semigroup. Then* $\mathrm{Ap}(S,s_1) = \{0, s_2, 2s_2, \ldots, (s_1-1)s_2\}$.

With the next result, we can obtain a algorithm for computing the intersection of two numerical semigroups with embedding dimension two and the same multiplicity.

Proposition 10 *Let $S_1 = \langle s_1, s_2\rangle$ and $S_2 = \langle s_1, s_3\rangle$ then $S_1 \cap S_2 = \langle s_1, a_1, \ldots a_{s_1-1}\rangle$, with $a_i = \max\{w_1(i), w_2(i)\}$ and $w_1(i) \in \mathrm{Ap}(S_1,s_1)$, $w_2(i) \in \mathrm{Ap}(S_2,s_1)$.*

Proof.

It is sufficiency to prove that

$\mathrm{Ap}(S_1 \cap S_2, s_1) = \{0, \max\{w_1(1), w_2(1)\}, \cdots \max\{w_1(s_1-1), w_2(s_1-1)\}$,

because $\langle \mathrm{Ap}(S_1 \cap S_2, s_1) \cup \{s_1\}\rangle = S_1 \cap S_2$.

We assume that $w_{i_1}(i) \equiv w_{i_2}(i) \equiv i \ mod\ s_1$ with $w_{i_1} \in S_1$ and $w_{i_2} \in S_2$, then $\max\{w_1(i), w_2(i)\} \in S_1 \cap S_2$ and $\max\{w_1(i), w_2(i)\} - s_1 \notin S_1 \cap S_2$. Because if $w_2 \in S_2$ is the maximum, as $w_1 = s_1 k + i \in S_1$ with $k \in \mathbb{N}$, then $w_2 \in S_1$. On other hand if $w_2 - s_1 \notin S_2$ then $w_2 - s_1 \notin S_1 \cap S_2$. Therefore $\max\{w_1(i), w_2(i)\} \in \mathrm{Ap}(S_1 \cap S_2, s_1)$.

We illustrate this procedure with an example:

Example: Suppose that $S_1 = \langle 5, 7\rangle$ and $S_2 = \langle 5, 9\rangle$ we compute the $\mathrm{Ap}(S_1, 5)$ and $\mathrm{Ap}(S_2, 5)$ and

(i)	1	2	3	4
$\mathrm{Ap}(S_1,5)$	21	7	28	14
$\mathrm{Ap}(S_2,5)$	36	27	18	9

Therefore, we have that

$$S_1 \cap S_2 = \langle \mathrm{Ap}(S_1 \cap S_2, s_1) \cup \{5\}\rangle = \langle 5, 14, 28, 27, 36\rangle =$$

$$\{0, 5, 10, 14, 15, 19, 20, 24, 25, 27, 28, 29, 30, 32, 33, 34, 35, 36, \longrightarrow\}.$$

Now, we compare the classical intersection algorithm of numerical semigroups with embedding dimension two and the same multiplicity and new algorithm presented above:

The classical intersection algorithm (CIA):

INPUT: A pair of semigroups $S_1 = \langle n_0, n_1\rangle$ and $S_2 = \langle n_0, n_2\rangle$

1. Compute the set $S_1' = \{0, s_1, s_2, \ldots, s_p, \longrightarrow\}$.
2. Compute the set $S_2' = \{0, r_1, r_2, \ldots, r_q, \longrightarrow\}$.
3. Return $S' = S_1' \cap S_2'$.

OUTPUT: A list S' of the elements of the intersection semigroup.

The improved intersection algorithm (IIA):

INPUT: A pair of semigroups $S_1 = \langle n_0, n_1\rangle$ and $S_2 = \langle n_0, n_2\rangle$

1. Compute the set $Ap(S_1, n_0) = \{0, n_1, 2n_1, \ldots, (n_0-1)n_1\}$.
2. Compute the set $Ap(S_2, n_0) = \{0, n_2, 2n_2, \ldots, (n_0-1)n_2\}$.
3. Reorder the sets $Ap(S_1, n_0)$ and $Ap(S_2, n_0)$ modulo n_0 and set $Ap(S_1, n_0) = \{0, s_1, s_2, \ldots, s_{n_0-1}\}$ and $Ap(S_2, n_0) = \{0, r_1, r_2, \ldots, r_{n_0-1}\}$.
4. Compute $R = \{max(s_i, r_i), i = 1, \ldots, n_0 - 1\} \cup \{n_0\}$.
5. Set $S' = \langle R\rangle$
6. Return $S' = \{0, s_1', s_2', \ldots, s_t', \longrightarrow\}$.

OUTPUT: A list S' of the elements of the intersection semigroup.

Remark: Proposition 10 assure us that in the end of this algorithm we obtain the desired result.

3.1 The Complexity

Given a semigroup $S = \langle s_1, s_2\rangle$, the complexity of generating the set of elements in S which are smaller than $g(S)$ is $O(g(S))$. The complexities of these algorithms are linear functions of $\max\{g(S_1), g(S_2)\}$. In the case of the algorithm CIA we have complexity $O(2\max\{g(S_1), g(S_2)\})$. The case of the algorithm IIA we need a little more work to compute this. So the complexity of computing the sets $Ap(S_i, s_1)$ is $O(s_1)$. To reorder each of these sets we have complexity $O(s_1)$. Finally we have to write the elements in the intersection semigroup. So we have that the complexity of the algorithm IIA is $O(4s_1 + \max\{g(S_1), g(S_2)\})$. Remark 8 give us an explicit formula for $g(S_i)$, $i = 1, 2$.

3.2 Experimental results

To compare, from the practical point of view, the two algorithms we preformed in the following way: We generated 100 random pairs (S_1, S_2) of semigroups of the form $S_1 = \langle s_1, s_2 \rangle$ and $S_2 = \langle s_1, s_3 \rangle$. We start by choosing a random s_1 smaller than $20, 50$ and 100 and than we choose randomly s_2 and s_3 bigger than s_1 and smaller than $150, 200$ and 300. The results obtained are contained in the following table:

Range of Generators	MRT for CIA	MRT for IIA	ART for CIA	ART for IIA
$(20, 150)$	2.297	1.187	0.36911	0.19963
$(20, 200)$	1.937	1.047	0.48024	0.25993
$(20, 300)$	6.594	3.515	1.07048	0.57141
$(50, 150)$	11.515	6.344	2.44154	1.36268
$(50, 200)$	25.375	12.437	3.78908	2.06749
$(50, 300)$	48.422	25.953	7.22870	3.91006
$(100, 150)$	49.640	27.783	13.65713	7.60992
$(100, 200)$	95.720	54.705	22.65495	12.56754
$(100, 300)$	246.391	134.329	33.29890	18.17655

We can check that as the theoretical worst case complexity it is actually an average behaviour. The algorithm CIA takes an average of 1.83 more time than IIA.

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