Sampling-Reconstruction Procedure of Markov Chains with Continuous Time

V. Kazakov and Y. Goritskiy

***Abstract*—**At the first time the statistical description of the Sampling-Reconstruction Procedure of Markov Chains with continuous time and with an arbitrary number of states is given. The analytic expression for the conditional probability density of the jump time moment is obtained. On the basis of this probability density function the expression for the jump moment estimation is found. The methodology of the sampling interval choice is suggested. One illustrative example is considered.

***Keywords*—**Markov chain with continuous time, sampling – reconstruction, reconstruction error.

I. Introduction

THE mathematical models of Markov chains with continuous time are intensively used in the description of some real stochastic processes with jumps, for instance, impulse noise [1, 2]. This is the reason that it is necessary to know: how to sample, how to reconstruct and how to calculate the reconstruction errors of such processes. The Markov chains with continuous time and with limited number of states are processes with jumps. (Jumps can be occurred in continuous time moments.) So, the usual method of the Sampling- Reconstruction Procedure (SRP) investigation of continuous stochastic processes (i.e. the method of the conditional mathematical expectation rule) can not be applied directly.

The quantity of references dedicated to the problem of the statistical description of SRP of stochastic processes is rather huge. Nevertheless, the SRP of Markov Chains with continuous time is not practically investigated in the literature. Here we notice paper [3] concerned with binary processes and the certificate of invention [4]. The present paper is the generalization of the recent publication [5] devoted to the SRP of a particular case of the Markov chains with continuous time and with *two* states. Here the number of states is an arbitrary.

The statistical SRP description of Markov Chains with continuous time has some specific features. We need to take into account that the sampled realization of Markov chains with continuous time keeps its sampled value on the right and on the left of the sampling time during some random intervals. It means that we have to estimate the jump moment between two known different samples. In other words it is necessary to estimate *one* random variable during a sampling interval. (At the same time in the reconstruction of a continuous process we need to estimate random variables *permanently* at any instant moment between two neighbor samples.) Therefore the reconstruction error of a jump process will be characterized by the variance of the jump time estimation. This is the *first* specific feature of the SRP of the discussed processes. The *second* feature is connected with the absence of the problem of a reconstruction function because the shape of the reconstruction function is known: this is a straight line on the right and on the left of any known sample.

II. Problem Formulations

We consider a Markov chain $\xi(t)$ with continuous time and with the states *0, 1, 2,…,N*. This chain is completely described by the rates (or the intensities) of N +1 Poisson's flows λ_0, λ_1, λ_2, …, λ_N and by the matrix of the transfer probabilities $P_{ij}(P_{ii}=0)$ at the jumps moments. Let us designate $t_0, t_1, ..., t_n, t_{n+1}$ as sampling moments. Let us $\xi(t_n)=i$. It is necessary to find the time interval T_i determining the next sampling moment $t_{n+1}=t_n+T_i$ if the variance $\left(V\hat{\tau}_{ij}\right)$ of the estimation $\hat{\tau}_{ij}$ of the jump moment τ_{ij} from the state *i* into the state *j* ($j\neq i$) is not more than a given value σ^2 (the same for all *i* and *j*):

$$V\hat{\tau}_{ij} \le \sigma^2 \qquad (1)$$

Because we deal with Markov processes the reconstruction characteristics inside of sampling intervals are dependent on two neighbor samples only.

Below we shall find the estimation $\hat{\tau}_{ij}$, the variance $V\hat{\tau}_{ij}$ and the value of the interval T_i. In order to do this it is necessary to determine the conditional probability density function (pdf) of a jump moment between two different arbitrary states.

III. SRP in the Case of Two States

For the sake of simplicity we find the conditional pdf of a jump moment between two known different samples in the

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binary Markov process. Let us assume that the process $\xi(t)$ has two states 0 and 1. We designate the rates of two control Poisson flows as λ_0 and λ_1 correspondingly. Let us fix two samples $\xi(0)=0$ and $\xi(T)=1$. Then the conditional process inside of the sampling interval T does not depend on other possible samples on the left and on the right of these chosen samples owing to the Markov property. The sampling interval T must be restricted by the condition

$$T<\tau_c=\left(\lambda_0+\lambda_1\right)^{-1} \tag{2}$$

where τ_c is the covariance time of the sampled process.

In fact, if the value of T is rather big $\left(T>\tau_c\right)$ then during the sampling interval more than one transitions from one state to another can be occurred. It means that some states can not be detected inside of the interval *T*. In other words the condition (2) provides a possibility to neglect a probability of such events. Therefore we restrict our consideration by the analysis of another error. Let us designate τ as a random moment of the jump from the state 0 into the state 1, i.e. $0<\tau<T$. Then $p_\tau\left(t\middle|0,1\right)$ is a desirable conditional pdf in the interval (0,*T*) when one jump can be occurred. We write this pdf in the form

$$\begin{aligned}
&p_\tau(t|0,1)=p_\tau(t|\xi(0)=0,\xi(T)=1)=\\
&=\lim_{\Delta t\to 0}\frac{1}{\Delta t}P\left\{\tau\in(t-\Delta t,t)\middle|\xi(0)=0,\xi(T)=1\right\}=\\
&=C\lim_{\Delta t\to 0}\frac{1}{\Delta t}P\left\{\xi(0)=0,\tau\in(t-\Delta t,t),\xi(T)=1\right\}
\end{aligned} \tag{3}$$

where $C^{-1}=P\left\{\xi(0)=0,\xi(T)=1\right\}$ is the normalizing constant which can be determined from the equation:

$$\int_0^\infty p_\tau(t|0,1)dt=1. \tag{4}$$

Let us designate η_i as the random duration of process staying in the state $i\left(i=0,1\right)$. As is known pdf of this variable is

$$p_{\eta i}(t)=\lambda_i\cdot\exp(-\lambda_i t),\qquad t>0 \tag{5}$$

The event $\left\{\xi(0)=0,\tau\in(t-\Delta t,t),\xi(T)=1\right\}$ is equivalent to another event $\left\{\eta_0\in(t-\Delta t,t),\ \eta_1>T-t\right\}$. Then instead of (3) we shall calculate the probability:

$$\begin{aligned}
&P\left\{\eta_0\in\left(t-\Delta t,t\right),\eta_1>T-t\right\}=\\
&P\left\{\eta_0\in\left(t-\Delta t,t\right)\right\}P\left\{\eta_1>T-t\right\}
\end{aligned} \tag{6}$$

where the independence of two events $\left\{\eta_0\in\left(t-\Delta t,t\right)\right\}$ and $\left\{\eta_1>T-t\right\}$ is taken into account.

On the basis of (5) we calculate both probabilities in the right part of (6). The first probability will be

$$\begin{aligned}
P\left\{\eta_0\in\left(t-\Delta t,t\right)\right\}&=p_{\eta 0}\left(t\right)\Delta t+0(\Delta t)=\\
&=\lambda_0\exp\left(-\lambda_0 t\right)\Delta t+0(\Delta t)
\end{aligned} \tag{7}$$

The second probability is the integral:

$$P\left\{\eta_1>T-t\right\}=\int_{T-t}^\infty p_{\eta 1}\left(t'\right)dt'=\exp\left[-\lambda_1\left(T-t\right)\right] \tag{8}$$

Substituting (6) – (8) into (3) and designating $\mu=\lambda_0-\lambda_1$ we have

$$p_\tau(t|0,1)=\begin{cases}\dfrac{\mu\cdot\exp(-\mu t)}{1-\exp(-\mu T)}, & if\quad \mu\neq 0,\\ 1/T, & if\quad \mu=0,\end{cases}\quad (0<t<T) \tag{9}$$

The expression (9) is called by the cut exponential distribution. On the basis of (9) one can find the conditional mathematical expectation or the estimation of the jump moment τ:

$$\hat{\tau}=\begin{cases}\dfrac{1}{\mu}\left(1-\dfrac{\mu T}{e^{\mu T}-1}\right), & \mu\neq 0\\ T/2, & \mu=0\end{cases} \tag{10}$$

and the conditional variance or the estimation quality:

$$V\hat{\tau}=\begin{cases}\dfrac{1}{\mu^2}\left[1-\dfrac{\left(\mu T\right)^2}{\left(1-e^{-\mu T}\right)\left(e^{\mu T}-1\right)}\right], & \mu\neq 0\\ T^2/12, & \mu=0\end{cases} \tag{11}$$

The expressions (10) and (11) are final because they describe the reconstruction point (10) and the error of this reconstruction (11). Actually, when the precision of the estimation σ^2 is known ($\sigma<1/\mu$), from (11) one can determine the sampling interval *T*, after this following (10) we find the point of jump.

IV. SRP IN THE CASE OF AN ARBITRARY NUMBER OF STATES

Using the same methodology and generalizing the problem for an arbitrary number of states one can find the following analytical result:

$$p_{\tau ij}(t|i,j)=\begin{cases}\dfrac{\mu_{ij}\cdot\exp(-\mu_{ij}t)}{1-\exp(-\mu_{ij}T)}, & if \quad \mu_{ij}\neq 0,\\ 1/T, & if \quad \mu_{ij}=0,\end{cases}, \quad (0<t<T) \qquad (12)$$

where $\mu_{ij}=\lambda_i-\lambda_j$.

Once again, this is the cut exponential distribution. On the basis of (12) we find the conditional mean and the conditional variance. The estimation $\hat{\tau}_{ij}$ for the jump moment τ_{ij} is the conditional mean:

$$\hat{\tau}_{ij}=E\{\tau_{ij}|i,j\}==\begin{cases}\dfrac{1}{\mu_{ij}}\left(1-\dfrac{\mu_{ij}T}{e^{\mu_{ij}T}-1}\right), & \mu_{ij}\neq 0\\ T/2, & \mu_{ij}=0\end{cases} \qquad (13)$$

The variance of the estimation is the reconstruction error:

$$V\hat{\tau}_{ij}==\begin{cases}\dfrac{1}{\mu_{ij}^2}\left[1-\dfrac{(\mu_{ij}T)^2}{\left(1-e^{-\mu_{ij}T}\right)\left(e^{\mu_{ij}T}-1\right)}\right], & \mu_{ij}\neq 0\\ T^2/12, & \mu_{ij}=0\end{cases}. \qquad (14)$$

We notice that the conditional pdf (12) and the principal characteristics of the reconstruction procedure: the estimation (13) and the variance (14) do not depend on the transfer probabilities $P_{ij}(P_{ii}=0)$ because the events (marked by indexes *i* and *j*) have been already realized.

Now we can realize the consecutive procedure of sampling in order to satisfy the above mentioned condition (1). Because the average time of state staying can be very different the sampling procedure is non-uniform.

Because $\xi(0)=i$ and the value $\xi(T_i)$ is unknown we have to choose the sampling interval T_i from the maximum of the variance:

$$\max_{j,i\neq j,P_{ij}\neq 0} V\hat{\tau}_{ij}=\sigma^2 \qquad (15)$$

In order to find the maximum in (15) we need to take into account the states characterized by non zero transfer probabilities $P_{ij}\neq 0$. Other states can be ignored. Because the variance (14) is the even function with regards of μ_{ij} and it decreases with the rise of $|\mu_{ij}|$ one can see that the maximum in (15) can be reached with the minimum of $|\mu_{ij}|=|\lambda_i-\lambda_j|$ (when *j* is changed).

Let us designate

$$\mu_i^*=\min_{j,i\neq j,P_{ij}\neq 0}|\lambda_i-\lambda_j| \qquad (16)$$

Then taking into account (14) the condition (15) will have the view:

$$\frac{1}{\mu_i^{*2}}\left[1-\frac{(\mu_i^*T_i)^2}{\left(1-e^{-\mu_i^*T_i}\right)\left(e^{\mu_i^*T_i}-1\right)}\right]=\sigma^2 \qquad (17)$$

Let us introduce two values *x* and *z* normalized with regards of time $1/\mu_i^*$:

$$x\equiv\frac{T_i}{1/\mu_i^*}=T_i\mu_i^*, \qquad z\equiv\frac{\sigma}{1/\mu_i^*}=\sigma\mu_i^*$$

The equation (17) with respect of *x* is described in the form:

$$\sqrt{1-\frac{x^2}{(1-e^{-x})(e^x-1)}}=z \qquad (18)$$

This expression determines the function $z=f(x)$ or $\sigma\mu_i^*=f(T_i\mu_i^*)$.

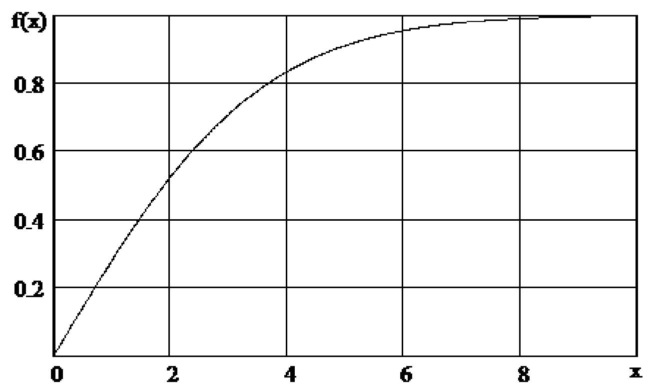

Fig. 1. The graph of the function *f(x)*.

Because the values σ and μ_i^* must be known, therefore the sampling interval T_i is determined from the inverse function

$$T_i=f^{-1}(\sigma\mu_i^*)/\mu_i^* \qquad (19)$$

The graph in Fig. 1 shows that the direct function $f(x)$ is one-valued, so there is not any difficulty to obtain the inverse function (19).

Putting (19) into (13) and (14), we get the required estimation of the jump moment (i.e. the reconstruction line) and the variance of the estimation (i.e. the reconstruction error).

Finally, the Sampling-Reconstruction Procedure of Markov chain with continuous time is described by the following steps: there is a sample in the state i, i.e. $\xi(t_n)=i$, (here t_n is the sample moment); we calculate the value μ^*_i following

(16); from the inverse function with respect to (18) we determine the interval T_i and therefore the next sampling moment $t_{n+1} = t_n + T_i$. If the next sample has another value (i.e. $\xi\left(t_{n+1}\right) = j$), one can evaluate the reconstruction quality following (14), putting $T = T_i$.

It is clear that the discussed sampling algorithm can be realized by some simple computing device with memory blocks and with a calculator of the operation (16) and of the inverse function (19).

V. Example

Let us consider the Markov chain with continuous time and with three states 0, 1 and 2. The rates of Poisson's flows are λ_0=1, λ_1=0.1 and λ_2=0.05. This means that this process is characterized by the following average times of staying in the states: 1, 10 and 20. Let us choose the following matrix of the transfer probabilities:

$$\boldsymbol{P} = \begin{bmatrix} 0 & P_{01} & 0 \\ P_{10} & 0 & P_{12} \\ 0 & P_{21} & 0 \end{bmatrix}$$

The matrix elements indicate four possible transfers among the states. Let us choose σ=1.

Following (17) we find the values μ_i^*:

$$\mu_0^* = 0.9, \quad \mu_1^* = 0.05, \quad \mu_2^* = 0.05.$$

Using the condition σ=1 from (19) we determine the required sampling intervals:

$$T_0 = 5.36,\ T_1 = 3.47,\ T_2 = 3.47.$$

As one can see, generally the sampling intervals have different values. In this concrete example the sampling interval for the state 0 has the length 5.36, and the intervals from the states 1 and 2 have the same length 3.47.

VI. Conclusion

At the first time the statistical description of the SRP of Markov chains with continuous time and with an arbitrary number of states is carried out. The new methodology of the choice of the sampling intervals is suggested.

References

[1] M. S Yarlykov and M. A. Mironov, *The Markov theory of Estimating Random processes*. New York Begell House, 1996.

[2] V. A. Kazakov, "One-dimensional kinetic equations for non-Markovian processes and statistical analysis problems of systems driven by the asymmetrical binary Markovian process, *Signal processing*, vol. 79, No. 1, pp. 87-95, 1999.

[3] S. J. Redman and D. G. Lampard, "Stochastic sampling of a binary random process," *IEEE Trans. on Circuit Theory*, vol. No. 10, pp. 3-24, March 1963.

[4] V. A. Kazakov, "Device for an interpolation of non continuous processes," USSR Patent 1023350, 1981.

[5] V. A. Kazakov and Y. Goritskiy. "Sampling-Reconstruction of the Binary Markov process". "Radioelectronics and Communication Systems.", Vol. 49, No 11, pp. 6-10, 2006. ISSN 0021-3470. Allerton Press Inc. N.Y

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