Some algorithms for generating receipts in the cutting-covering problem

Paul Iacob, Daniela Marinescu, and Kinga Kiss-Iakab*

***Abstract*—** We consider a cutting-covering problem, defined by us in previous papers, the problem of covering a rectangular support with rectangular pieces cut from a roll. We first prove that our algorithm for the rectangular cutting-covering problem without losses is not optimal. Starting from a decomposition of a natural number in sums of naturals we developed an algorithm for a better solution for the rectangular cutting-covering problem. Now we continue with the modeling of the cutting-covering problem like an integer linear optimization problem. For solving this problem we can use a branch-and-bound algorithm. Because, this algorithm has a high time complexity, we construct other algorithm, which generates receipts for the cutting-covering problem with losses, but much faster.

***Keywords*—**algorithms, complexity, covering, optimization, rectangular cutting.

I. Introduction

Cutting and Covering problems belong to an old and very well known family. Such problems are also commonly referred to as Cutting and Packing problems, called CP in [1], [7]. This is a family of natural combinatorial optimization problems, admitted in numerous real world applications from computer science, industrial engineering, logistics, manufacturing, management, production process, etc. If we know the dimensions of the pieces then we are dealing with a classical cutting-stock problem, which can be modeled as a mixed 0-1 programming problem [6].

Cutting-stock problems have many applications in production processes in paper, glass, metal and timber cutting industries. Many different versions of cutting problems exist as there are one, two and three dimensional cuttings with or without several constraints. Such a constraint is the guillotine restriction where one can separate the pieces from the cutting model through a number of guillotine cuts (i.e. cuts through the 2 whole submodels).

The two-dimensional covering problem is related to the two-dimensional cutting-stock problem, which is also richly documented.

II. Problem formulation

We consider a cutting-covering problem, defined by us in [3] and [4], where we do not know the dimensions of the pieces for covering. From the material in a roll of fixed width and infinite length, we cut rectangular pieces by guillotine-cuts, so that we cover a rectangle of dimensions a, b, without losses or overlapping and with a minimum number of pieces.

In [3] and [4] we gave an algorithm due to which a rectangle of dimensions a=5, b=50 can be covered by a roll of the width x=6, with six pieces, as in Fig. 1.

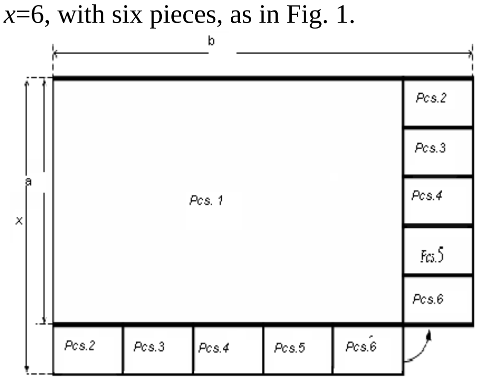

Fig. 1 Covering the rectangle with 6 pieces

We have considered the following problem: Can we cover the support with fewer pieces?

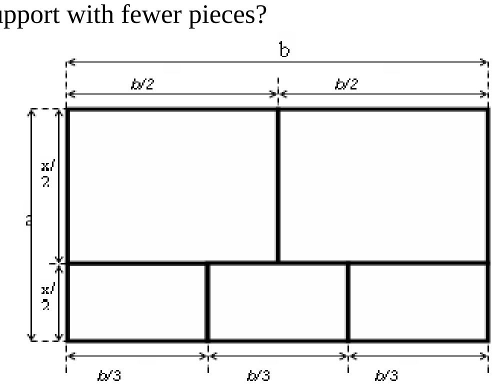

Fig. 2 Covering a rectangle of dimensions axb with five pieces

One solution is the one from Fig. 2.

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The solution is found by cutting the material from the roll of width x=6 as it shown in Fig. 3.

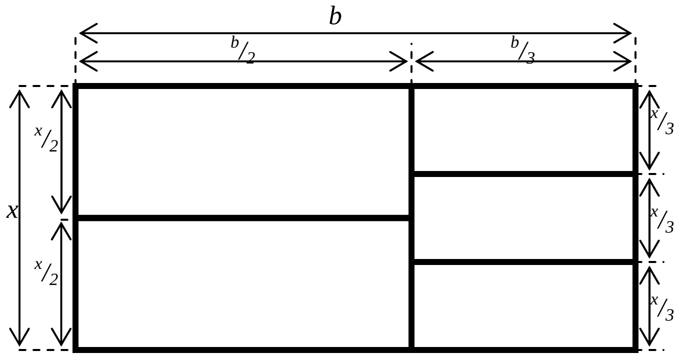

Fig. 3 Cutting of the five pieces for the covering of the rectangle of dimensions axb

We can see that we can effectuate the covering only with five pieces and the solution does not depend on the size of b, but only on the ratio $\frac{a}{x} = \frac{5}{6}$.

III. THE RELATION BETWEEN THE CUTTING-COVERING RECEIPT AND THE DECOMPOSITON OF INTEGER NUMBERS INTO INTEGERS

We will use a very simply decomposition of a natural number in sums of naturals. This decomposition was already presented in [5], but for the sake of completeness we give here a short overview. Each decomposition will represent a cutting-covering model as it follows.

For the cutting of the six pieces, we decompose the natural number 6 in the following eight sums:

$$\begin{array}{ll} 1+1+1+1+1+1 & (a) \\ 1+1+1+1+2 & (b) \\ 1+1+1+3 & (c) \\ 1+1+4 & (d) \\ 1+2+3 & (e) \\ 1+5 & (f) \\ 3+3 & (g) \\ 6 & (h) \end{array} \qquad (1)$$

Every decomposition gives us a cutting-covering model with six pieces as in the following:

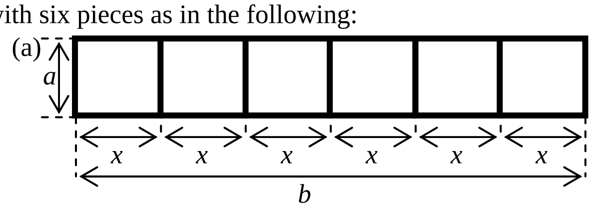

Fig. 4 Case $b = 6x$

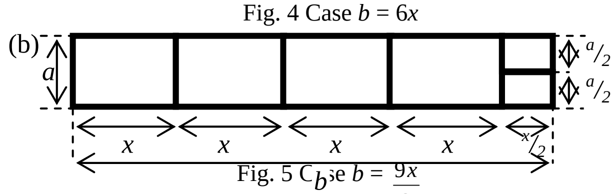

Fig. 5 Case $b = \frac{9x}{2}$

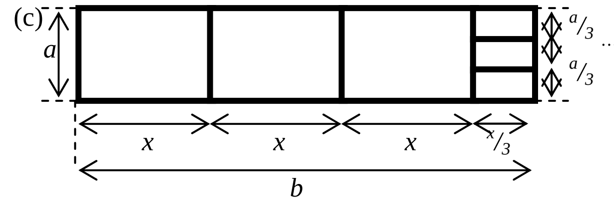

Fig. 6 Case $b = \frac{10x}{3}$

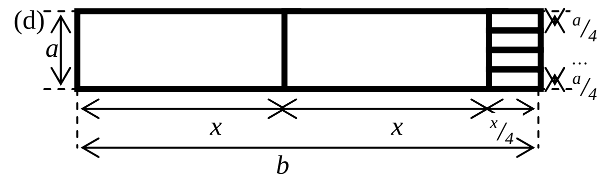

Fig. 7 Case $b = \frac{9x}{4}$

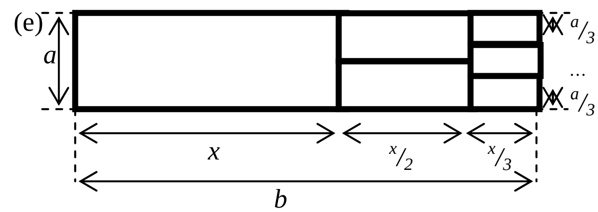

Fig. 8 Case $b = \frac{11x}{6}$

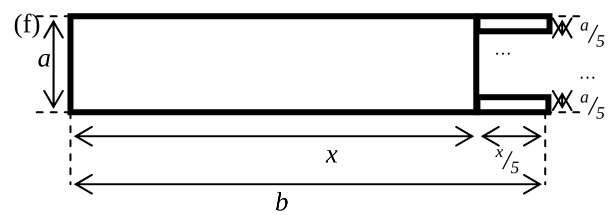

Fig. 9 Case $b = \frac{6x}{5}$

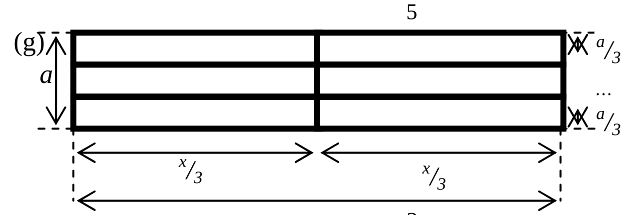

Fig. 10 Case $b = \frac{2x}{3}$

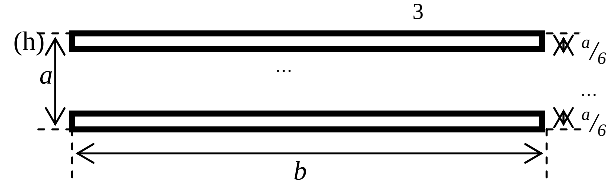

Fig. 11 Case $b = \frac{x}{6}$

Now we cut pieces of the width x first and what really counts is the fractional part.

The decomposition has the form:

$$(\underbrace{1,1,\ldots,1}_{s\ times}, \underbrace{p_1, p_2, \ldots, p_n}_{residual\ \ part}) \qquad (2)$$

We presume that $b = x\left(s + \sum_{i=1}^{n} p_i\right)$ and then the cutting-covering scheme will be the one from Fig. 12.

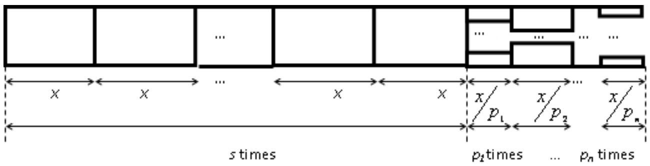

Fig. 12 The general covering scheme

We note with A the pieces of length x and with B_i the pieces of length x/p_i, for i = 1, 2, ..., n. The covering model from Fig. 12 is a model with guillotine restrictions and due to [4], it can be represented as a word of a pictural language, as it follows:

$$\underbrace{A|\ldots|A}_{s\ times}|\underbrace{(B_1 =\ldots= B_1)}_{p_1 times}|\underbrace{(B_2 =\ldots= B_2)}_{p_2 times}|\ldots|\underbrace{(B_n =\ldots= B_n)}_{p_n times}$$

In the formula, we note with |, the column concatenation and with = the row concatenation between two rectangles.

A. Algorithms for finding the model with minimum number of pieces and losses

We observe that an optimal model depends only on the fact that $\frac{a}{x}$ or $\frac{b}{x}$ can be written as $\left(s + \sum_{i=1}^{n} p_i\right)$ where p_i are ascending sorted integers, p_i appears at most $p_i - 1$ times, and the covering will contain $\left(s + \sum_{i=1}^{n} p_i\right)$ pieces. Because s is not important for the matching, it results that it is required to have one of the following equalities:

$$\mathcal{F}r\left(\frac{a}{x}\right) = \mathcal{F}r\left(\sum_{i=1}^{n} \frac{1}{p_i}\right) \quad (3)$$

$$\mathcal{F}r\left(\frac{b}{x}\right) = \mathcal{F}r\left(\sum_{i=1}^{m} \frac{1}{p_i}\right) \quad (4)$$

where we note with $\mathcal{F}r\left(\frac{a}{x}\right)$ the fractional part of a/x. We will use the following notations:

$S[i] = \left(p_1^i, \ldots, p_{n_i}^i\right)$ the row vectors for a fractional parts of a decomposition like (2),

$R[i] = \sum_{j=1}^{n_i} \frac{1}{p_j^i}$, a column vector,

$N_b[i] = \sum_{j=1}^{n_i} p_j^i$, a column vector for the number of the fractional pieces,

$\mathbb{S}$ the matrix of vectors $S[i]$, where each $S[i]$ is completed by zero components to the dimensions $\max\{n_i\}$.

We present first an algorithm for creating a database that will be used in the algorithm for finding the model with minimum number of pieces and without losses.

*B. Algorithm **Datab***

The ***Datab*** algorithm will create one table for each N, the number of the fractional pieces.

Input data: N_{max}, the maximum number of the fractional pieces.
Output data: The database.

*The **Datab** Algorithm*

for $N = 2$ to N_{max}
{for i=1, 2, . . .
{generate the sequence of fractional parts $S[i] = \left(p_1^i, \ldots, p_{n_i}^i\right)$, so that the residual part of the decomposition in sums to have at most N pieces
}
- generate the two vectors R and N_b and the matrix **S**
- generate the T matrix by adding the column vectors R and N_b to matrix **S**.
- sort the rows of the matrix T, on ascending mode by 2 keys:
  - the first key - the components of the vector R,
  - the second key- the components of the vector N_b.
-generate the reduced matrix T' by the following operations:
  - eliminate from T of the rows with $R[i]=0$,
  - eliminate from T of the repetitions in R, storing only the row corresponding to the component with minimum $N_b[i]$ for the same $R[i]$.
- memorize in a database of the matrix T' as T'_N, of the dimensions r_N and c_N.
}

Using this database created by the algorithm ***Datab*** we can define the algorithm for finding the rectangular cutting-covering model with the minimum number of pieces and without losses.

C. Algorithm MinMod

Input data:-The dimensions a and b of the supporting plate and the dimension x, the width of the material roll;
-The databases created by **Datab**

Output data:-N and the matrix T'_N, representing the cutting-covering model.

*The **MinMod** Algorithm*

N=2;

repeat
{k=0;
for i=1 to r_N do
{if ($R(i)=\Im r\left(\frac{a}{x}\right)$ or $R(i)=\Im r\left(\frac{b}{x}\right)$)
then k=1;
}
N=N+1;
}
until(k=1 or $N>N_{max}$);
if $N>N_{max}$ then write *„there is no cutting-covering model with at most N_{max} pieces"*
else write the model corresponding to (1)

An example can be found in [5].

IV. Integer linear optimization model

We consider the problem $P\left(\frac{r}{q}\right)$, where the fraction $\frac{r}{q}$ is an irreducible fraction.

$$\min\left(\sum_{i=1}^{t} y_i * p_i\right)$$

with the following constrains:

$$\frac{r}{q}=\left(\sum_{i=1}^{t}\frac{y_i}{p_i}\right)$$

where $y_i \in \{0,1,2,..., p_i-1\}$ and p_i are all divisiors of q, except 1.

If we solve the problem using branch-and-bound the time complexity will be $\prod_{i=1}^{t} p_i$.

The algorithm for the determination of one receipt for the cutting-covering problem will be:

Step 1. Write $\frac{a}{x}$ in the form of $\frac{a}{x}=s_a+\frac{r_a}{q_a}$. This form results from the integer division
$a=x*s_a+rest_a$
and by the rewriting the fraction $\frac{rest_a}{x}$ in the irreducible form of $\frac{r_a}{q_a}$.

Step 2. Solve the problem $P\left(\frac{r_a}{q_a}\right)$ which gives the solutions y_i^a with i from 1 to n_a.

Calculate $nr_a=s_a+\sum_{i=1}^{n_a} p_i \bullet y_i^a$.

Step 3. Write $\frac{b}{x}$ in the form of $\frac{b}{x}=s_b+\frac{r_b}{q_b}$.

This form results form the integer division
$b=x*s_b+rest_b$
and by the rewriting of the fraction $\frac{rest_b}{x}$ in the irreducible form of $\frac{r_b}{q_b}$.

Step 4. Solve the problem $P\left(\frac{r_b}{q_b}\right)$ which gives the solutions y_i^b with i from 1 to n_b.

Calculate $nr_b=s_b+\sum_{i=1}^{n_b} p_i \bullet y_i^b$.

Step 5. Chose the minimum between nr_a and nr_b and construct the corresponding receipt as we shown in section III.

Example 1.

Let a=13, b=107, x=10. Then a=1*10+3 and $\frac{a}{x}=1+\frac{3}{10}$.

$P\left(\frac{3}{10}\right)$ is the problem:

$$\min(2y_1+5y_2+10y_3)$$

with the constrain:

$$5y_1+2y_2+y_3=3$$

where $y_1\in\{0, 1\}$, $y_2\in\{0, 1, 2, 3, 4\}$, $y_3\in\{0, 1, 3, 7, 9\}$.

The feasible solutions are: (0, 0, 3) and (0, 1, 1) which give min(30, 15)=15. So nr_a=1+16=16.

For b=10*10+7, $\frac{b}{x}=10+\frac{7}{10}$.

$P\left(\frac{7}{10}\right)$ is

$$\min(2y_1+5y_2+10y_3)$$

with the constrain:

$$5y_1+2y_2+y_3=7$$

where $y_1\in\{0, 1\}$, $y_2\in\{0, 1, 2, 3, 4\}$, $y_3\in\{0, 1, 3, 7, 9\}$.

The feasible solutions are: (0, 0, 7), (0, 1, 5) (0, 2, 3), (0, 3, 1), (1, 0, 2) and (1, 1, 0), which give min(70, 55, 40, 25, 22, 7)=7. So nr_a=10+7=17.

The minimum between 16 and 17 is 16 so, using this method, the cutting-covering receipt without losses will have 16 pieces.

V. Cutting-covering algorithm with rest and better time complexity

The algorithm described in section IV has two inconveniences:

- exponential time complexity,
- returns receipts only without losses, but with losses we could obtain a receipt with significantly fewer pieces as we can see in the following example.

In the previous example let increase a, c=a+2=15.

Then c=1*10+5, $\frac{c}{x}=1+\frac{1}{2}$.

The problem $P\left(\frac{1}{2}\right)$ is

$$\min(2y_1)$$

with the constrains:

$$y_1 = 1$$

where $y_1 \in \{0, 1\}$.

The feasible solution is (y_1=1) which gives min(2)=2.
So nr_c=1+2=3 and the covering can be done by 3 pieces but with the loss

$$(0{,}5\text{-}0{,}3)b=0{,}2*107=21{,}4.$$

Because many receipts can be repeated, we conceived an algorithm which is founded on the construction of a database for receipts.

We limit us to receipts which have in the non-integer part at most N pieces.

We will generate all combinations without repetition, without number one (which would correspond to the whole pieces) and a number p_i repeated at most p_i-1 times, so we do not generate an already generated receipt.

The algorithm will have the following steps:

Step 1 Generate of the sequences S_i = (p_1, p_2, …, p_n) where $\sum_{i=1}^{m} p_i \le N$, the sequence is ascending

$$p_1 \le p_2 \le \ldots \le p_n$$

and one number p_i is repeated at most p_i-1 times.

Step 2 Generate the two vectors $N_{buc} = \sum_{i=1}^{m} p_i$ and $R = \left(\sum_{i=1}^{m} \frac{1}{p_i}\right)$ and a matrix S with the lines S_i

Step 3 Sort S in ascending mode by R and S_i and eliminate of repetitions in R correspondingly to N_{buc} and S, then storing the two vectors and the matrix S in one database.

Step 4 Rewriting $\frac{a}{x}$ in the form of $\frac{a}{x}=s_a+\frac{r_a}{q_a}$ and $\frac{b}{x}$ in the form of $\frac{b}{x}=s_b+\frac{r_b}{q_b}$ like in section III.

Step 5 Find by bisection of the equality

$$R(i_a) = \left(\frac{r_a}{q_a}\right) \text{ or } R(i_b) = \frac{r_b}{q_b}.$$

We will have the number of pieces $nr_a=s_a+N_{buc}(i_a)$ and $nr_b=s_b+N_{buc}(i_b)$.
If we have equality then we have the receipt as in section II for the number of pieces min(nr_a, nr_b).

Step 6 If we do not have equality, then

$$R(i_a\text{-}1) < \left(\frac{r_a}{q_a}\right) < R(i_a)$$

and chosing values $j > i_a$ for R, we find cutting receipts with loss of $b*(R(j)\text{-}\left(\frac{r_a}{q_a}\right))$, for $j \ge i_a$. We choose the receipt after we do the same test for $\frac{r_b}{q_b}$ with loss of $a*(R(j)\text{-}\frac{r_b}{q_b})$.

VI. Complexity

As we showed in [4], the complexity of the database construction is nonpolynomial, but the search by bisection, the operation which is repeated actually is polynomial for the number of receipts generated in the database.

For example, for the maximum 26 pieces in the fractionary part (N=26), continuing the previous example (a=13, b=107, x=10), we find in the table:

TABLE I

Nbuc	***R***	***S***
22	0.003968254	[2, 4, 7, 9, 0, 0, 0]
…	…	…
15	**0.3**	**[5, 10, 0, 0, 0, 0, 0]**
…	…	…
3	0.333333333	[3, 0, 0, 0, 0, 0, 0]
…	…	…
2	0.5	[2, 0, 0, 0, 0, 0, 0]
…	…	…

As we can see in Table I $R(i_a)$=0,3 without losses, with 15 pieces; R=0,333 with 3 pieces and with the loss of (0,333-0,3)107=3,51; and R=0,5 with 2 pieces and with the loss of (0,5-0,3)107= 21,4.

VII. Conclusion

The cutting-covering model introduced by us in [3], [4] is a different one to the classical models because at the beginning, we know nothing about the dimensions of the pieces from the model. We will find these dimensions and, in the same time, the arrangement of the pieces on the support, only after the application of an algorithm. We can find an optimal solution without losses in a polynomial time or a solution with losses, using a data bases, in a nonpolynomial time.

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