An analysis of the beam bending problem with random beam height

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***Abstract*—**The standard beam bending problem has been obtained where the beam height is assumed to have spatial uncertainty. The formulation to determine the response variability of the beam due to randomness of the beam height is given. The concept of variablity response function is extended to beam bending problem where the beam height is considered to be one-dimensional, homogenous stochastic field. The randomness of the beam height has than influence not only on the flexural rigidity of the beam, but also on self-weight load of the beam. Through the proposed formulation it becomes possible for the weighted integral stochastic finite element analysis to consider complete influence of uncertain geometrical property on response variability. The coefficeint variation and variance of the response deflection was calculated as the function of the coefficeint of variation and variance of the beam height as the input quantity. Numerical example shows good agreement of the proposed weighted integral method with solution calculated by Monte Carlo simulation.

***Keywords*—**Beam bending, Response variability, uncertain geometrical property, weighted integral method.

I. Introduction

The assumption that structures have deterministic material properties, is implicitly involved in the most calculation of standard finite element structural analysis of the structures. The material and geometrical properties of real structures have uncertainties, which have to be considered in structural analysis. The uncertainties of the structures are than considered through the increase of the safety factors using deterministic analysis.

The concept of variability response function was introduced in [1] and used in [2,3,4]. The weighted integral method was introduced in [2] and generally applied in [3,4]. In those works the variability was involved through the variability of elastic modulus.

This study is concentrated on the randomness in geometrical parameters (the beam height) and its influence on the both side of equation, on the stiffness matrix and on the load vector. The first and second moment of the beam height are used to describe randomness of the input quantities. The new autocorrelation function is written for the flexural rigidity of the beam considering the randomness of the beam height. The variability response function is used to find spectral-distribution-free upper bounds of the response variability. The structural response variability is represented as second moment, variance and coefficient of variation, of the response deflection.

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II. Variability of Input Quantities

We consider a beam of length *L* with a spatially varying height *H(x)*. The beam height is assumed to constitute a homogenous one-dimensional random field in the following form:

$$H(x) = H\,\left[1 + h(x)\right] \; , \tag{1}$$

where $H_0=H_0(x)=const.$ is expectation of the beam height taken equal for any point at the beam and *h(x)* is homogenous one-dimensional random field with expectation equal to zero. This random field is represented with its variance s_{hh}^2 and autocorrelation function

$$R_{hh}(\chi) = \mathrm{E}\left[h(x+\chi)h(x)\right] \; , \tag{2}$$

what leads to variance and coefficient of variation of beam height *H(x)*

$$\mathrm{Var}\left[H(x)\right] = H_0^{\;2}\,\sigma_{\mathrm{hh}}^{\;2} \; , \; \mathrm{COV}\left[H(x)\right] = \sigma_{hh} \; . \tag{3}$$

The flexural rigidity of the beam, *EI(x)*, is now also random field of the form

$$EI(x) = EI_0\left[1 + d(x)\right] \; , \tag{4}$$

where *d(x)* is homogenous one-dimensional random field with expectation zero defined as

$$d(x) = \left[1 + h(x)\right]^3 - 1 = 3h(x) + 3h^2(x) + h^3(x) \; . \tag{5}$$

Autocorrelation function of this random field is then according to [5]

$$R_{dd}(\chi) = 9\sigma_{\mathrm{hh}}^{\;4} + \left(9 + 18\sigma_{\mathrm{hh}}^{\;2} + 9\sigma_{\mathrm{hh}}^{\;4}\right)R_{hh}(\chi) + 18R_{hh}^{\;2}(\chi) + 6R_{hh}^{\;3}(\chi) \; . \tag{6}$$

The variance and the coefficient of variation of random field *d(x)* are

$$\sigma_{dd}^{\;2} = 9\sigma_{hh}^{\;2} + 45\sigma_{hh}^{\;4} + 15\sigma_{hh}^{\;6} \tag{7}$$

$$\mathrm{COV}[d(x)] = 3\sigma_{hh}\sqrt{1 + 5\sigma_{hh}^{\;2} + \tfrac{5}{3}\sigma_{hh}^{\;4}} \; . \tag{8}$$

This was the calculation of the influence of uncertain beam height on the stiffness of the beam. But, the variability of the beam height, obviously, leads to the variability of loading. The self-weight of the beam directly depends on the beam height. We define the load as some prescribed load, $\bar{q} = \bar{q}(x) = const.$, added to self-weight calculated with constant weight density of the beam, $\gamma = \gamma(x) = const.$, what leads on load expressed as linear combination of deterministic and stochastic part,

$$q(x) = \gamma H(x) + \bar{q} = \gamma H_0 \left[1 + h(x)\right] + \bar{\mathrm{q}} = q_0\left[1 + \Delta q(x)\right], \tag{9}$$

where deterministic part is expressed as

$$q_0 = \gamma H_0 + \bar{\mathrm{q}} \ , \tag{10}$$

and stochastic part, with introduced supstitution $G = \frac{\gamma H_0}{\gamma H_0 + \bar{q}}$, is expressed as

$$\Delta q(x) = \frac{\gamma H_0}{\gamma H_0 + \bar{q}} h(x) = G h(x) \ . \tag{11}$$

Autocorrelation function of loading as random field is then

$$R_{qq}(\chi) = G^2 R_{hh}(\chi), \tag{12}$$

and the variance is

$$\sigma_{qq}^{\ 2} = G^2 \sigma_{hh}^{\ 2} \quad . \tag{13}$$

III. Finite Element Formulation

Standard deterministic finite element formulation of the beam bending problem is

$$\mathbf{K}_0 \mathbf{w} = \mathbf{q}_0 \ . \tag{14}$$

Involving the randomness of the beam properties, the finite element formulation for the stochastic analysis is according to [2]

$$\left(\mathbf{K}_0 + \Delta\mathbf{K}\right)\mathbf{w} = \mathbf{q}_0 + \Delta\mathbf{q} \ , \tag{15}$$

where $\mathbf{K}_0$ and $\mathbf{q}_0$ are deterministic stiffness matrix and load vector respectively and $\mathrm{D}\mathbf{K}$ and $\mathrm{D}\mathbf{q}$ are stochastic parts of stiffness matrix and load vector respectively. The displacement vector, with assumption that variance is sufficiently small, is approximated with

$$\mathbf{w} = \mathbf{K}^{-1}\mathbf{q} = \left[\mathbf{K}_0\left(\mathbf{I} + \mathbf{K}_0^{\ -1}\Delta\mathbf{K}\right)\right]\left(\mathbf{q}_0 + \Delta\mathbf{q}\right) \approx \mathbf{w}_0 - \mathbf{K}_0^{\ -1}\Delta\mathbf{K}\mathbf{w}_0 + \mathbf{K}_0^{\ -1}\Delta\mathbf{q}. \tag{16}$$

The stochastic part of displacement vector is now given according uncertain stiffnes matrix and uncertain load vector as

$$\Delta\mathbf{w} = -\mathbf{K}_0^{\ -1}\Delta\mathbf{K}\mathbf{w}_0 + \mathbf{K}_0^{\ -1}\Delta\mathbf{q} \, . \tag{17}$$

The expression for expectation of displacement vector is

$$\mathrm{E}\left[\mathbf{w}\right] = \mathbf{w}_0 \tag{18}$$

where $\mathbf{w}_0$ is the solution of standard deterministic finite element problem and the covariance matrix of the response deflection $\mathbf{w}$ is given as

$$\begin{aligned} \mathrm{Cov}\left[\mathbf{w},\mathbf{w}\right] = \mathrm{E}&\left[(\mathbf{w} - \mathbf{w}_0)\,(\mathbf{w} - \mathbf{w}_0)^{\mathrm{T}}\right] \\ &+ \mathrm{E}\left[\mathbf{K}_0^{\ -1}\Delta\mathbf{q}\Delta\mathbf{q}^{\mathrm{T}}\mathbf{K}_0^{\ -1}\right] \\ &- \mathrm{E}\left[\mathbf{K}_0^{\ -1}\Delta\mathbf{K}\mathbf{w}_0\Delta\mathbf{q}^{\mathrm{T}}\mathbf{K}_0^{\ -1}\right] \\ &- \mathrm{E}\left[\mathbf{K}_0^{\ -1}\Delta\mathbf{q}\mathbf{w}_0^{\ \mathrm{T}}\Delta\mathbf{K}\mathbf{K}_0^{\ -1}\right], \end{aligned} \tag{19}$$

where $\mathbf{W}_0 = \mathbf{w}_0\ \mathbf{w}_0^{\mathrm{T}}$. The first part, the first row, of (19) is same as in former analysis [2] with influence of uncertain beam properties only on stiffness matrix. The second part, the second row, includes the randomness of the loading. Last two parts, the third and the fourth row, exist only when stochastic field of stiffness and stochastic field of load is correlated. If we represent self-weight load as function of the beam height, those fields are strictly correlated.

IV. Weighted Integral Method

Weighted integral method was primary introduced in [3,4]. Stochastic part of element stiffness matrix is represented as linear combination of *NWI* random variables $X_i^{(e)}$ called weighted integrals,

$$\Delta\mathbf{K}^{(e)} = \sum_{k=0}^{NWI-1} X_k^{\ (e)} \Delta\mathbf{K}_k^{\ (e)} \, . \tag{20}$$

The number of weighted integrals, *NWI*, depends on the choice of finite element. Using the standard cubic finite element follows, (*NWI*=3), 3 weighted integrals. The weighted integrals $X_i^{(e)}$ are defined as

$$X_i^{\ (e)} = \int_0^{\mathrm{L}_x^{(e)}} \xi^{\mathrm{i}}\ d(\xi)\,\mathrm{d}\xi \quad . \tag{21}$$

The element matrices, the coefficients of linear combination $\mathrm{D}\mathbf{K}_i^{(e)}$, are all deterministic.

The stochastic part of element load vector is defined as

$$\Delta\mathbf{q}^{(e)} = \sum_{k=0}^{NWIQ-1} Y_k^{(e)} \Delta\mathbf{q}_k^{\ (e)} \, . \tag{22}$$

where *NWIQ* is the number of weighted integrals $Y_i^{(e)}$. Using the cubic finite element follows, (*NWIQ*=4), 4 weighted integrals. The weighted integrals $Y_i^{(e)}$ are defined as

$$Y_i^{(e)} = \int_0^{\mathrm{L}_x^{(e)}} \xi^{\mathrm{i}}\ h(\xi)\,\mathrm{d}\xi \quad . \tag{23}$$

where all vectors $\mathrm{D}\mathbf{q}_i^{(e)}$, the coefficients of linear combination, are deterministic.

V. Response Variability

The response vector, the vector of unknown displacements $\mathbf{w}$, could be approximated with linear part of Taylor series around the expectation of weighted integrals $X_k^{(e)}$ and $Y_p^{(e)}$,

$$\mathbf{w} = \mathbf{w}_0 + \sum_{(e)=1}^{N^{(e)}} \sum_k X_k^{(e)} \left[\frac{\partial \mathbf{w}}{\partial X_k^{(e)}} \right]_E + \sum_{(e)=1}^{N^{(e)}} \sum_p Y_p^{(e)} \left[\frac{\partial \mathbf{w}}{\partial Y_p^{(e)}} \right]_E$$

$$= \mathbf{w}_0 - \sum_{(e)=1}^{N^{(e)}} \sum_k \mathbf{K}_0^{-1} \Delta\mathbf{K}^{(e)} \mathbf{w}_0 X_k^{(e)} + \sum_{(e)=1}^{N^{(e)}} \sum_p \mathbf{K}_0^{-1} \Delta\mathbf{q}_p^{(e)} Y_p^{(e)} . \quad (24)$$

Using the first-order approximation around the zero-mean value of weighted integrals, first-order approximation of covariance matrix of response vector follows as

$$\mathrm{Cov}[\mathbf{w},\mathbf{w}] = \mathrm{E}\left[(\mathbf{w}-\mathbf{w}_0)(\mathbf{w}-\mathbf{w}_0)^{\mathrm{T}}\right]$$

$$= \sum_{(e),(f)=1}^{N^{(e)}} \sum_{k,m=1}^{N^{(e)}} \mathbf{K}_0^{-1} \Delta\mathbf{K}_k^{(e)} \mathbf{W}_0 \left(\Delta\mathbf{K}_m^{(f)}\right)^{\mathrm{T}} \left(\mathbf{K}_0^{-1}\right)^{\mathrm{T}} \mathrm{E}\left[X_k^{(e)} X_m^{(f)}\right]$$

$$- \sum_{(e),(f)=1}^{N^{(e)}} \sum_{k,r=1}^{N^{(e)}} \mathbf{K}_0^{-1} \Delta\mathbf{K}_k^{(e)} \mathbf{w}_0 \left(\Delta\mathbf{q}_r^{(f)}\right)^{\mathrm{T}} \left(\mathbf{K}_0^{-1}\right)^{\mathrm{T}} \mathrm{E}\left[X_k^{(e)} Y_r^{(f)}\right]$$

$$- \sum_{(e),(f)=1}^{N^{(e)}} \sum_{p,m=1}^{N^{(e)}} \mathbf{K}_0^{-1} \Delta\mathbf{q}_p^{(e)} \mathbf{w}_0^{\mathrm{T}} \left(\Delta\mathbf{K}_m^{(f)}\right)^{\mathrm{T}} \left(\mathbf{K}_0^{-1}\right)^{\mathrm{T}} \mathrm{E}\left[Y_p^{(e)} X_m^{(f)}\right]$$

$$+ \sum_{(e),(f)=1}^{N^{(e)}} \sum_{p,r=1}^{N^{(e)}} \mathbf{K}_0^{-1} \Delta\mathbf{q}_p^{(e)} \left(\Delta\mathbf{q}_r^{(f)}\right)^{\mathrm{T}} \left(\mathbf{K}_0^{-1}\right)^{\mathrm{T}} \mathrm{E}\left[Y_p^{(e)} Y_r^{(f)}\right] . \quad (25)$$

where the only unknowns are the values for expectations of weighted integrals products $\mathrm{E}[X_k^{(e)} X_m^{(f)}]$, $\mathrm{E}[X_k^{(e)} Y_r^{(f)}]$, $\mathrm{E}[Y_p^{(e)} X_m^{(f)}]$ and $\mathrm{E}[Y_p^{(e)} Y_r^{(f)}]$. The first expectation is to find as in [4]. Considering that the beam height for all finite elements are characterised by the same stochastic field *h(x)*, what leads than to the same stochastic field *d(x)* for flexural rigidity for all elements. The unknown expectation in the first part of equation (25) can be expressed with

$$\mathrm{E}\left[X_k^{(e)} X_m^{(f)}\right] = \mathrm{E}\left[\left(\int_0^{L^{(e)}} \xi_e^{\,k} d^{(e)}(\xi_e) \mathrm{d}\xi_e\right)\left(\int_0^{L^{(f)}} \xi_f^{\,m} d^{(f)}(\xi_f) \mathrm{d}\xi_f\right)\right]$$

$$= \int_0^{L^{(e)}} \int_0^{L^{(f)}} \xi_e^{\,k} \xi_f^{\,m} \mathrm{E}\left[d^{(e)}(\xi_e) d^{(f)}(\xi_f)\right] \mathrm{d}\xi_e \mathrm{d}\xi_f , \quad (26)$$

with

$$\mathrm{E}\left[d(\xi_e)\, d(\xi_f)\right] = R_{dd}\left(\Delta x_{fe} + L^{(f)}\xi_f - L^{(e)}\xi_e\right), \quad (27)$$

where $R_{dd}(\chi)$ is autocorrelation function of stochastic field $r(\chi)$, and Δx_{fe} is given as the distance of the first knots of finite elements *(e)* and *(f)* expressed as $\Delta x_{fe} = x_i^{(f)} - x_i^{(e)}$. After similar algebra and same simplification about field characterisation follow the other unknown expectations with

$$\mathrm{E}\left[X_k^{(e)} Y_r^{(f)}\right] = \mathrm{E}\left[\left(\int_0^{L^{(e)}} \xi_e^{\,k} d^{(e)}(\xi_e) \mathrm{d}\xi_e\right)\left(\int_0^{L^{(f)}} \xi_f^{\,r} h^{(f)}(\xi_f) \mathrm{d}\xi_f\right)\right]$$

$$= \int_0^{L^{(e)}} \int_0^{L^{(f)}} \xi_e^{\,k} \xi_f^{\,r} \mathrm{E}\left[d^{(e)}(\xi_e) h^{(f)}(\xi_f)\right] \mathrm{d}\xi_e \mathrm{d}\xi_f , \quad (28)$$

$$\mathrm{E}\left[Y_p^{(e)} X_m^{(f)}\right] = \mathrm{E}\left[\left(\int_0^{L^{(e)}} \xi_e^{\,p} h^{(e)}(\xi_e) \mathrm{d}\xi_e\right)\left(\int_0^{L^{(f)}} \xi_f^{\,m} d^{(f)}(\xi_f) \mathrm{d}\xi_f\right)\right]$$

$$= \int_0^{L^{(e)}} \int_0^{L^{(f)}} \xi_e^{\,p} \xi_f^{\,m} \mathrm{E}\left[h^{(e)}(\xi_e) d^{(f)}(\xi_f)\right] \mathrm{d}\xi_e \mathrm{d}\xi_f . \quad (29)$$

$$\mathrm{E}\left[Y_p^{(e)} Y_r^{(f)}\right] = \mathrm{E}\left[\left(\int_0^{L^{(e)}} \xi_e^{\,p} h^{(e)}(\xi_e) \mathrm{d}\xi_e\right)\left(\int_0^{L^{(f)}} \xi_f^{\,r} h^{(f)}(\xi_f) \mathrm{d}\xi_f\right)\right]$$

$$= \int_0^{L^{(e)}} \int_0^{L^{(f)}} \xi_e^{\,p} \xi_f^{\,r} \mathrm{E}\left[h^{(e)}(\xi_e) h^{(f)}(\xi_f)\right] \mathrm{d}\xi_e \mathrm{d}\xi_f . \quad (30)$$

with

$$\mathrm{E}\left[d(\xi_e) h(\xi_f)\right] = R_{dh}\left(\Delta x_{fe} + L^{(f)}\xi_f - L^{(e)}\xi_e\right), \quad (31)$$

$$\mathrm{E}\left[h(\xi_e) d(\xi_f)\right] = R_{hd}\left(\Delta x_{fe} + L^{(f)}\xi_f - L^{(e)}\xi_e\right), \quad (32)$$

$$\mathrm{E}\left[h(\xi_e) h(\xi_f)\right] = R_{hh}\left(\Delta x_{fe} + L^{(f)}\xi_f - L^{(e)}\xi_e\right), \quad (33)$$

with introduced autocorrelation function and its variance

$$R_{dh}(\chi) = R_{hh}(\chi)\left(1 + \sigma_{hh}^{\,2}\right),$$
$$\sigma_{dh}^{\,2} = \sigma_{hh}^{\,2} + \sigma_{hh}^{\,4} . \quad (34)$$

The variance vector of response displacement vector **w** is than evaluated as

$$\mathrm{Var}[\mathbf{w}] = \int_{-\infty}^{\infty} S_{dd}(\kappa) \mathrm{VRF}_1(\kappa) \mathrm{d}\kappa$$
$$- \int_{-\infty}^{\infty} S_{dh}(\kappa)\left[\mathrm{VRF}_2(\kappa) + \mathrm{VRF}_3(\kappa)\right] \mathrm{d}\kappa \quad (35)$$
$$\int_{-\infty}^{\infty} S_{hh}(\kappa) \mathrm{VRF}_4(\kappa) \mathrm{d}\kappa,$$

where $S_{dd}(\kappa)$, $S_{dh}(\kappa)$ and $S_{hh}(\kappa)$ are spectral density function and the vectors $\mathrm{VRF}_i(\kappa)$ are the first-order approximations of the variability response function parts respectively

$$\mathrm{VRF}_1(\kappa) = \sum_{(e),(f)} \sum_{k,m} \mathrm{diag}\left(\mathbf{K}_0^{-1} \Delta\mathbf{K}_k^{(e)} \mathbf{w}_0\right) \mathbf{K}_0^{-1} \Delta\mathbf{K}_m^{(f)} \mathbf{w}_0$$
$$\cdot \left[\left(\mathrm{CI}_k^{(e)} \mathrm{CI}_m^{(f)} + \mathrm{SI}_k^{(e)} \mathrm{SI}_m^{(f)}\right) \cos\left(\Delta x_{fe}\kappa\right) - \left(\mathrm{SI}_k^{(e)} \mathrm{CI}_m^{(f)} - \mathrm{CI}_k^{(e)} \mathrm{SI}_m^{(f)}\right) \sin\left(\Delta x_{fe}\kappa\right) \right] , \quad (36)$$

$$\mathrm{VRF}_2(\kappa) = \sum_{(e),(f)} \sum_{k,r} \mathrm{diag}\left(\mathbf{K}_0^{-1} \Delta\mathbf{K}_k^{(e)} \mathbf{w}_0\right) \mathbf{K}_0^{-1} \Delta\mathbf{q}_r^{(f)}$$
$$\cdot \left[\left(\mathrm{CI}_k^{(e)} \mathrm{CI}_r^{(f)} + \mathrm{SI}_k^{(e)} \mathrm{SI}_r^{(f)}\right) \cos\left(\Delta x_{fe}\kappa\right) - \left(\mathrm{SI}_k^{(e)} \mathrm{CI}_r^{(f)} - \mathrm{CI}_k^{(e)} \mathrm{SI}_r^{(f)}\right) \sin\left(\Delta x_{fe}\kappa\right) \right] , \quad (37)$$

$$\mathrm{VRF}_3(\kappa) = \sum_{(e),(f)}\sum_{p,m} \mathrm{diag}\left(\mathbf{K}_0^{-1}\Delta\mathbf{q}_p^{(e)}\right)\mathbf{K}_0^{-1}\Delta\mathbf{K}_m^{(f)}\mathbf{w}_0 \cdot \Big[\left(\mathrm{CI}_p^{(e)}\mathrm{CI}_m^{(f)} + \mathrm{SI}_p^{(e)}\mathrm{SI}_m^{(f)}\right)\cos\left(\Delta x_{fe}\kappa\right) - \left(\mathrm{SI}_p^{(e)}\mathrm{CI}_m^{(f)} - \mathrm{CI}_p^{(e)}\mathrm{SI}_m^{(f)}\right)\sin\left(\Delta x_{fe}\kappa\right)\Big] \ , \tag{38}$$

$$\mathrm{VRF}_4(\kappa) = \sum_{(e),(f)}\sum_{p,r} \mathrm{diag}\left(\mathbf{K}_0^{-1}\Delta\mathbf{q}_p^{(e)}\right)\mathbf{K}_0^{-1}\Delta\mathbf{q}_r^{(f)} \cdot \Big[\left(\mathrm{CI}_p^{(e)}\mathrm{CI}_r^{(f)} + \mathrm{SI}_p^{(e)}\mathrm{SI}_r^{(f)}\right)\cos\left(\Delta x_{fe}\kappa\right) - \left(\mathrm{SI}_p^{(e)}\mathrm{CI}_r^{(f)} - \mathrm{CI}_p^{(e)}\mathrm{SI}_r^{(f)}\right)\sin\left(\Delta x_{fe}\kappa\right)\Big] , \tag{39}$$

and for *(g)= (e),(f)* follow the expressions for CI_k and SI_k respectively

$$\mathrm{CI}_k^{(g)} = \int_0^{L^{(g)}} \xi_g^{\,k}\cos\left(\kappa\xi_g\right)d\xi_g \tag{40}$$

$$\mathrm{SI}_k^{(g)} = \int_0^{L^{(g)}} \xi_g^{\,k}\sin\left(\kappa\xi_g\right)d\xi_g \tag{41}$$

Consider a specific degree of freedom w_i and corresponding component of according response variabilities $\mathrm{VRF}_j^i(k)$, j=1…4, the coefficient of variation is bounded as

$$\mathrm{COV}[\,w_i\,] \le \sigma_{hh}\frac{\sqrt{V^{*,i}}}{\left\|\mathrm{E}\left[w_i\right]\right\|} \ , \tag{42}$$

where

$$V^{*,i} = \left[9 + 45\sigma_{hh}^{\ 2} + 15\sigma_{hh}^{\ 4}\right]\mathrm{VRF}_1^{\,i}(\kappa^*) + \mathrm{VRF}_4^{\,i}(\kappa^*) - \left(1+\sigma_{hh}^{\ 2}\right)\left[\mathrm{VRF}_2^{\,i}(\kappa^*) + \mathrm{VRF}_3^{\,i}(\kappa^*)\right], \tag{43}$$

and k^* is the point at which the function under square root takes its maximum value.

VI. Numerical Example

The simply-supported beam of unit length under uniform load is considered in numerical example. The variation of the beam height is taken s_{hh} =0.1 what leads to variation of the flexural rigidity of the beam as $\sigma_{dd} \approx 0.3074$. The variability response function is calculated for the deflection in the middle of the beam, *w*(0.5), by weighted integral method with 4 finite elements. The results of the proposed weighted integral method are compared arecompared with results of the classical Monte Carlo simulation (MCS).

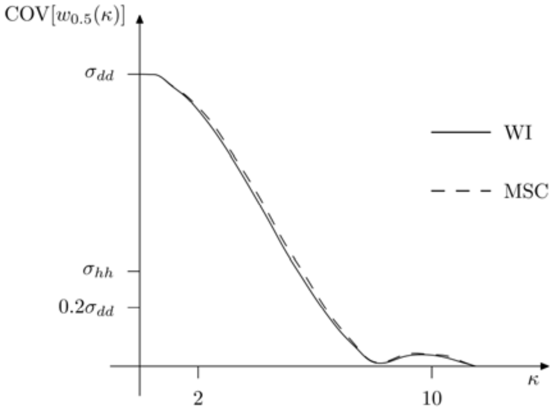

Fig. 1 Coefficient of variation of the response deflection *w*(0.5)

The results show that the randomness of the beam height has the influence on the response deflection of the beam. It is also obvious that results obtained with finite elements by using weighted integral method are in good agreement with results obtained by MCS.

VII. Conclusion

The concept of the variability response function based on weighted integral and local average method was extended to the beam bending problem with random beam height. It has been shown that randomness of the beam height has influence not only on the randomness of the flexural rigidity what is expressed as the randomness of the stiffness matrix than also on the randomness of self-weight load what is expressed as the randomness of the load vector. The influence on the variability of the response deflection was calculated according the concept of the variability response function. It has been shown very good agreement of the results calculated with weighted integral method with results obtained by MCS.

With proposed weighted integral formulation, that includes randomness of the beam height on the randomnes of the flexural rigidity and loading, it becomes possible for the weighted integral stochastic finite element analysis to consider complete influence of uncertain geometrical property on the structural response variability.

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