Travelling wave solutions for a generalized Benjamin-Bona-Mahony-Burgers equation

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***Abstract*— In this paper, we consider a generalized Benjamin-Bona-Mahony-Burgers equation. Classical symmetries of this equation are considered. The functional forms, for which the BBMB equation can be reduced to ordinary differential equations by classical Lie symmetries, are obtained. A catalogue of symmetry reductions and a catalogue of exact solutions are given. A set of new solitons, kinks, antikinks, compactons and Wadati solitons are derived.

I. Introduction

In this paper we solve a group classification problem for equation

$$\Delta \equiv u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + (g(u))_x = 0, \qquad (1)$$

where $u(x,t)$ represents the fluid velocity in the horizontal direction x, α is a positive constant, $\beta \in \mathbb{R}$ and $g(u)$ is a C^2-smooth nonlinear function [2]. We study the functional forms $g(u)$ for which equation (1) admits the classical symmetry group.

When $g(u) = uu_x$ with $\alpha = 0$ and $\beta = 1$ equation (1) is the alternative regularized long-wave equation proposed by Peregrine [15] and Benjamin [3]. Equation (1) feature a balance between nonlinear and dispersive effects, but takes no account of dissipation. In the physical sense, equation (1) with the dissipative term αu_{xx} is proposed if the good predictive power is desired, such problem arises in the phenomena for both the bore propagation and the water waves.

In [2], Khaled-Momani-Alawneh implemented the Adomian's decomposition method for obtaining explicit and numerical solutions of the BBMB equation (1).

By applying the classical Lie method of infinitesimals Bruzón and Gandarias [4] obtained, for a generalization of a family of BBM equations, many exact solutions expressed by various single and combined nondegenerative Jacobi elliptic functions.

Tari and Ganji, [16], have applied two methods for solving nonlinear differential equations known as "variational iteration" and "homotopy perturbation" methods in order to derive approximate explicit solutions for (1) with $g(u) = \frac{u^2}{2}$.

El–Wakil–Abdou–Hendi [9] used the "exp-function" method with the aid of symbolic computational system to obtain the generalized solitary solutions and periodic solutions for (1) with $g(u) = \frac{u^2}{2}$. In [10] Fakhari *et al.* solved the resulting nonlinear differential equation by homotopy analysis method to evaluate the nonlinear equation (1) with $g(u) = \frac{u^2}{2}$, $\alpha = 0$ and $\beta = 1$.

The classical theory of Lie point symmetries for differential equations describes the groups of infinitesimal transformations in a space of dependent and independent variables that leave the manifold associated with the equation unchanged [11], [13], [14]. The fundamental basis of this method is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. For partial differential equations (PDEs) with two independent variables a single group reduction transforms the PDE into a ordinary differential equations (ODEs), which are generally easier to solve. Since the relevant calculations are usually rather laborious, they can be conveniently carried out by means of symbolic computations. In our work, we used the MACSYMA program symmgrp.max [8]. Most of the required theory and description of the method can be found in [13], [14].

Wadati [17], [18], [19] developed solitons for the Korteweg-de Vries (KdV) and the modified KdV (MKdV) equations. He proved that if

$$u(x,t) = 2\partial_x \arctan\left(\frac{c\sin(Nx+\delta t)}{N\cosh(cx+\gamma t)}\right),$$

$\gamma = c(3N^2 - c^2)$ and $\delta = N(N^2 - 3c^2)$, then u is solution of the MKdV

$$u_t + u_{xxx} + 6u^2 u_x = 0.$$

In [5], [7] the authors obtained solutions in terms of Wadati solitons for some models.

The structure of the work is as follows: In Sec. II we find conditions on $g(u)$ such that it allows symmetries. We use the classical Lie method to find these symmetries, in particular those beyond the translational symmetries of the independent variables. We obtain the symmetry reductions, similarity variables and the reduced ODEs. In Sec. III we derive, for some functions $g(u)$, exact solutions which describe solitons, kinks, anti-kinks, compactons and Wadati solitons. Finally, in Sec. IV some conclusions are presented.

II. Classical Symmetries

To apply the Lie classical method to equation (1) we consider the one-parameter Lie group of infinitesimal transformations in (x,t,u) given by

$$x^* = x + \epsilon\xi(x,t,u) + O(\epsilon^2), \qquad (2)$$
$$t^* = t + \epsilon\tau(x,t,u) + O(\epsilon^2), \qquad (3)$$
$$u^* = u + \epsilon\eta(x,t,u) + O(\epsilon^2), \qquad (4)$$

where ϵ is the group parameter. We require that this transformation leaves invariant the set of solutions of equation (1). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \quad (5)$$

The functions $u = u(x,t)$, which are invariant under the infinitesimal transformations $\mathbf{v}$, are, in essence, solutions to an equation arising as the "invariant surface condition":

$$\eta(x,t,u) - \xi(x,t,u)\frac{\partial u}{\partial x} - \tau(x,t,u)\frac{\partial u}{\partial t} = 0. \quad (6)$$

The symmetry variables are found by solving the invariant surface condition. The reduction transforms the PDE into ODEs.

We consider the classical Lie group symmetry analysis of equation (1). The set of solutions of equation (1) is invariant under the transformation (2)-(4) provided that

$$\mathrm{pr}^{(3)}\mathbf{v}(\Delta) = 0 \quad \text{when} \quad \Delta = 0,$$

where $\mathrm{pr}^{(3)}\mathbf{v}$ is the third prolongation of the vector field (5). Hence we obtain the following ten determining equations for the infinitesimals:

$$\begin{aligned}
\tau_u &= 0,\\
\tau_x &= 0,\\
\xi_u &= 0,\\
\xi_t &= 0,\\
\eta_{uu} &= 0,\\
\alpha\tau_t + \eta_{tu} &= 0,\\
2\eta_{ux} - \xi_{xx} &= 0,\\
\eta_{uxx} - 2\xi_x &= 0,\\
\eta_x g_u - \alpha\eta_{xx} + \beta\eta_x - \eta_{txx} + \eta_t &= 0,\\
-\alpha\xi_{xx} - g_u\xi_x - \beta\xi_x - g_u\tau_t - \beta\tau_t - \eta g_{uu} &\\
+2\alpha\eta_{ux} + 2\eta_{tux} &= 0.
\end{aligned} \quad (7)$$

From system (7) $\xi = \xi(x)$, $\tau = \tau(t)$ and $\eta = \gamma(x,t)u + \delta(x,t)$ where α, β, ξ, τ, γ, δ and g satisfy

$$\begin{aligned}
\gamma_t + \alpha\,\tau_t &= 0,\\
2\,\gamma_x - \xi_{xx} &= 0,\\
\gamma_{xx} - 2\,\xi_x &= 0,\\
2\,\alpha\,\gamma_x + 2\,\gamma_{tx} - g_{uu}\,u\,\gamma - \alpha\xi_{xx} - g_u\,\xi_x - \beta &\\
\xi_x - g_u\,\tau_t - \beta\,\tau_t - \delta\,g_{uu} &= 0,\\
-\alpha u\,\gamma_{xx} + g_u\,u\,\gamma_x + \beta\,u\,\gamma_x - u\,\gamma_{txx} + u &\\
\gamma_t + \delta_x\,g_u - \alpha\,\delta_{xx} + \beta\delta_x - \delta_{txx} + \delta_t &= 0.
\end{aligned} \quad (8)$$

From (8) we obtain

$$\gamma = \frac{e^{-2x}}{8} \quad \left((k_4 + 2\,k_3)\,e^{4\,x} + (4k_1 - 8\alpha\tau)\,e^{2x} - k_4 + 2k_3\right),$$

$$\xi = \frac{(k_4 + 2k_3)\;e^{2x}}{8} + \frac{(k_4 - 2k_3)\;e^{-2x}}{8} - \frac{k_4 - 4k_2}{4},$$

and α, β, τ, δ and g are related by the following conditions:

$$\begin{aligned}
&((g_u + \beta - 2\alpha)\,k_4 + (2g_u + 2\,\beta - 4\alpha)\,k_3)\,ue^{4x}\\
&+ (-4\alpha\tau_t u + \delta_x\,(4g_u + 4\beta) - 4\alpha\delta_{xx} - 4\delta_{txx}\\
&+4\,\delta_t)\;e^{2\,x} + ((g_u + \beta + 2\,\alpha)\;k_4 + (-2\,g_u\\
&-2\,\beta - 4\,\alpha)\;k_3)\;u = 0,
\end{aligned} \quad (9)$$

$$\begin{aligned}
&((g_{uu}\,k_4 + 2\,g_{uu}\,k_3)\;u + (2\,g_u + 2\,\beta)\;k_4\\
&+ (4g_u + 4\beta)\,k_3)\,e^{4\,x} + ((4g_{uu}k_1 - 8\alpha g_{uu}\tau)\,u\\
&+8g_u\tau_t + 8\,\beta\,\tau_t + 8\,\delta g_{uu})\;e^{2\,x}\\
&+ (2\,g_{uu}k_3 - g_{uu}k_4)\;u + (-2g_u - 2\,\beta)\;k_4\\
&+ (4g_u + 4\,\beta)\;k_3 = 0.
\end{aligned} \quad (10)$$

Solving system (9)-(10) we obtain that if g is an arbitrary function the only symmetries admitted by (1) are

$$\xi = k_1, \quad \tau = k_2, \quad \eta = 0, \quad (11)$$

which are defined by the group of space and time translations,

$$\mathbf{v}_1 = \lambda\frac{\partial}{\partial x}, \qquad \mathbf{v}_2 = \mu\frac{\partial}{\partial t}.$$

Substituting (11) in the invariant surface condition (6) we obtain the similarity variable and the similarity solution

$$\begin{aligned}
z &= \mu x - \lambda t,\\
u(x,t) &= h(z).
\end{aligned} \quad (12)$$

Substituting (12) into (1) we obtain

$$\lambda\mu^2 h''' - \alpha\mu h'' + (\beta\mu - \lambda)h' + \mu h' g_h(h) = 0. \quad (13)$$

Integrating (13) once we get

$$\lambda\mu^2 h'' - \alpha\mu h' + (\beta\mu - \lambda)h + \mu g(h) + k = 0. \quad (14)$$

In the following cases equation (1) have extra symmetries:

(i) If $\alpha = 0$, $g(u) = -\beta u + \frac{k}{a(n+1)}(au+b)^{n+1}$, $a \neq 0$,

$$\xi = k_1, \quad \tau = k_2 t + k_3, \quad \eta = -\frac{k_2}{an}(au+b).$$

Besides $\mathbf{v}_1$ and $\mathbf{v}_2$, we obtain the infinitesimal generator

$$\mathbf{v}_3 = t\partial_t - \frac{au+b}{an}\partial_u.$$

(ii) If $\alpha \neq 0$, $\beta \neq 0$ and $g(u) = au + b$,

$$\xi = k_1, \quad \tau = k_2, \quad \eta = \delta(x,t),$$

where δ satisfy

$$\alpha\delta_{xx} - g_u\delta_x - \beta\delta_x + \delta_{txx} - \delta_t = 0.$$

We do not considerer case (ii) because in this case equation (1) is a linear PDE.

In order to determine solutions of PDE (1) that are not equivalent by the action of the group, we must calculate the one-dimensional optimal system [13]. The generators of the nontrivial one-dimensional optimal system are the set

$$\mu\mathbf{v}_1 + \lambda\mathbf{v}_2, \quad \mathbf{v}_3, \quad \mathbf{v}_1 + \mathbf{v}_3.$$

Since equation (1) has additional symmetries and the reductions that correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ have already been derived, we must determine the similarity variables and similarity solutions corresponding to the generators $\mathbf{v}_3$ and $\mathbf{v}_1 + \mathbf{v}_3$.

• $\mathbf{v}_3$: We obtain the reduction

$$z = x, \qquad u = t^{-\frac{1}{n}} h(x) - \frac{b}{a},$$

where $h(t)$ satisfies

$$h'' + kna^n h^n h' - h = 0. \tag{15}$$

Equation (15) does not admit Lie symmetries. By making the change of variables

$$y(s) = h'(z), \quad s = h(z)$$

equation (15) becomes

$$y'y + kna^n s^n y - s = 0.$$

•$\mathbf{v}_1 + \mathbf{v}_3$: The reduction is

$$z = x - \ln|t|, \qquad u = t^{-\frac{1}{n}} h(z) - \frac{b}{a}. \tag{16}$$

The reduced ODE is

$$nh''' + h'' - nh' + nka^n h^n h' - h = 0. \tag{17}$$

Equation (17) does not admit Lie symmetries.
We can observe that, for the reduction (16), we have that

$$u(x,t) = t^{-\frac{1}{n}} h(x - \ln|t|) - \frac{b}{a}.$$

This solution describes a travelling wave with decaying velocity $v = \frac{1}{t}$ and decaying amplitude $t^{-\frac{1}{n}}$ if $n > 0$.

III. Travelling wave solutions

If g is an **arbitrary function** the similarity variables are given by $z = \mu x - \lambda t$, $u = h$, so that $u(x,t) = h(z) = h(\mu x - \lambda t)$. Consequently the corresponding solutions of (14) are travelling-wave solutions.

As the derivative of trigonometric, hyperbolic and exponential functions can be expressed in terms of themselves, we can choose g as an algebraic function of h, so that the equation (14) admits the trigonometric functions ($p\sin^q z$, $p\cos^q z$, $p\tan^q z$, $p\sinh^q z$, $p\cosh^q z$, $p\tanh^q z$), hyperbolic functions ($p\,\mathrm{sn}^q(z|m)$, $p\,\mathrm{cn}^q(z|m)$, $p\,\mathrm{dn}^q(z|m)$) and exponential function ($\exp(z)$), as solutions. In the following we present same exact solutions of equation (14) for $k = 0$.

•

$$h(z) = p\,\sin^q(z)$$

is solution of equation (14) for

$$g(h) = -\frac{\mu p^{\frac{2}{q}} q^2 \lambda}{h^{\frac{2}{q}-1}} + h\,\mu\, q^2 \lambda + \frac{\mu p^{\frac{2}{q}} q \lambda}{h^{\frac{2}{q}-1}} + \frac{h\lambda}{\mu} + \frac{\alpha\sqrt{p^{\frac{2}{q}} - h^{\frac{2}{q}}}\,q}{h^{\frac{1}{q}-1}} - \beta h. \tag{18}$$

Consequently, an exact solution of equation (1), where $g(u)$ is obtained substituting h by u in (18), is

$$u(x,t) = p\sin^q(\mu x - \lambda t). \tag{19}$$

For $\mu = \lambda = \frac{k}{2}$, $k = \sqrt{\frac{5}{12}}$, $p = 1$, $q = 2$, the solution

$$u(x,t) = \begin{cases} \sin^2(\mu x - \lambda t) & |x - t| \le \frac{2\pi}{k}, \\ 0 & |x - t| > \frac{2\pi}{k} \end{cases}$$

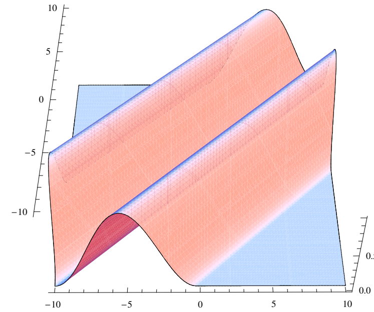

Fig. 1. Solution (19) for $\mu = \lambda = \frac{k}{2}$, $k = \sqrt{\frac{5}{12}}$, $p = 1$ and $q = 2$.

is a sine-type double compacton (that is solution which has two peaks, see Fig. 1)

• For

$$g(h) = -\frac{\mu p^{\frac{2}{q}} q^2 \lambda}{h^{\frac{2}{q}-1}} + h\,\mu\, q^2 \lambda + \frac{\mu p^{\frac{2}{q}} q \lambda}{h^{\frac{2}{q}-1}} + \frac{h\lambda}{\mu} - \frac{\alpha\sqrt{p^{\frac{2}{q}} - h^{\frac{2}{q}}}\,q}{h^{\frac{1}{q}-1}} - \beta h \tag{20}$$

a solution of (14) is

$$h(z) = p\,\cos^q(z).$$

So an exact solution of equation (1) is

$$u(x,t) = p\cos^q(\mu x - \lambda t), \tag{21}$$

where $g(u)$ is obtained substituting h by u in (20). For $\mu = \lambda = \frac{k}{2}$, $k = \sqrt{\frac{5}{12}}$, $p = 1$ and $q = 2$, the solution

$$u(x,t) = \begin{cases} \cos^2(\mu x - \lambda t) & |x - t| \le \frac{\pi}{k}, \\ 0 & |x - t| > \frac{\pi}{k} \end{cases}$$

is a compacton solution with a single peak, (see Fig.2).

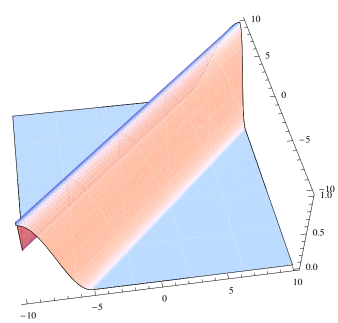

Fig. 2. Solution (21) for $\mu = \lambda = \frac{k}{2}$, $k = \sqrt{\frac{5}{12}}$, $p = 1$ and $q = 2$.

• For

$$g(h) = \frac{\mu p^{\frac{2}{q}} q\lambda(1-q)}{h^{\frac{2}{q}-1}} - \frac{h^{\frac{2}{q}+1}\mu q\lambda(q+1)}{p^{\frac{2}{q}}} - 2h\,\mu\, q^2\lambda + \frac{h\lambda}{\mu} + \frac{\alpha p^{\frac{1}{q}} q}{h^{\frac{1}{q}-1}} + \frac{\alpha h^{\frac{1}{q}+1} q}{p^{\frac{1}{q}}} - \beta h \tag{22}$$

a solution of (14) is

$$h(z) = p\,\tan^q(z).$$

So an exact solution of equation (1), where $g(u)$ is obtained substituting h by u in (22), is

$$u(x,t) = p\tan^q(\mu x - \lambda t). \qquad (23)$$

- For

$$g(h) = \frac{\mu p^{\frac{2}{q}} q\lambda(1-q)}{h^{\frac{2}{q}-1}} - h\mu q^2\lambda + \frac{h\lambda}{\mu} - \beta h + \frac{\alpha\sqrt{p^{\frac{2}{q}} + h^{\frac{2}{q}} q}}{h^{\frac{1}{q}-1}}, \qquad (24)$$

a solution of equation (14) is

$$h(z) = p\sinh^q(z)$$

So an exact solution of equation (1), where $g(u)$ is obtained substituting h by u in (24), is

$$u(x,t) = p\sinh^q(\mu x - \lambda t). \qquad (25)$$

- For

$$g(h) = \frac{\mu p^{\frac{2}{q}} q\lambda(q-1)}{h^{\frac{2}{q}-1}} - h\mu q^2\lambda + \frac{h\lambda}{\mu} - \beta h + \frac{\alpha\sqrt{p^{\frac{2}{q}} + h^{\frac{2}{q}} q}}{h^{\frac{1}{q}-1}} \qquad (26)$$

a solution of equation (14) is

$$h(z) = p\cosh^q(z).$$

Consequently, an exact solution of equation (1), where $g(u)$ is obtained substituting h by u in (26), is

$$u(x,t) = p\cosh^q(\mu x - \lambda t). \qquad (27)$$

For $\lambda = \mu = 1$, $p = 1$ and $q = -2$ the solution

$$u(x,t) = \text{sech}^2(x - t)$$

describes a soliton moving along a line with constant velocity (see Fig.3).

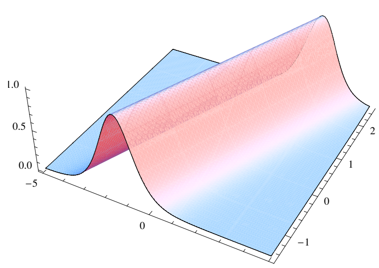

Fig. 3. Solution (27) for $\lambda = \mu = 1$, $p = 1$ and $q = -2$.

- For

$$g(h) = \frac{\mu p^{\frac{2}{q}} q\lambda(1-q)}{h^{\frac{2}{q}-1}} + \frac{2\mu p^{\frac{1}{q}} q\lambda(q-1)}{h^{\frac{1}{q}-1}} - h\mu q^2\lambda - \frac{2h^{\frac{1}{q}+1}\mu q\lambda}{p^{\frac{1}{q}}} + 3h\mu q\lambda + \frac{h\lambda}{\mu} + \frac{\alpha p^{\frac{1}{q}} q}{h^{\frac{1}{q}-1}} - \alpha h q - \beta h, \qquad (28)$$

a solution of equation (14) is

$$h(z) = p\tanh^q(z).$$

Consequently, an exact solution of equation (1), where $g(u)$ is obtained substituting h by u in (28), is

$$u(x,t) = p\tanh^q(\mu x - \lambda t). \qquad (29)$$

For $\mu = 1$, $\lambda = \frac{1}{2}$, $p = \frac{1}{4}$ and $q = 1$ the solution

$$u(x,t) = \frac{1}{4}\tanh\left(x - \frac{t}{2}\right)$$

describes a kink solution (see Fig.4).

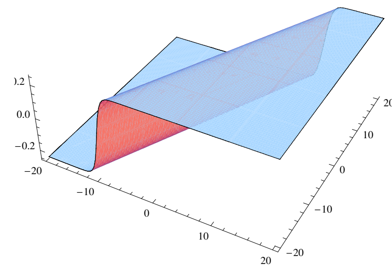

Fig. 4. Solution (29) for $\mu = 1$, $\lambda = \frac{1}{2}$, $p = \frac{1}{4}$ and $q = 1$.

For $\mu = 1$, $\lambda = \frac{1}{2}$, $p = 1$ and $q = 3$ the solution

$$u(x,t) = \tanh^3\left(x - \frac{t}{2}\right)$$

describes an anti-kink solution (see Fig.5).

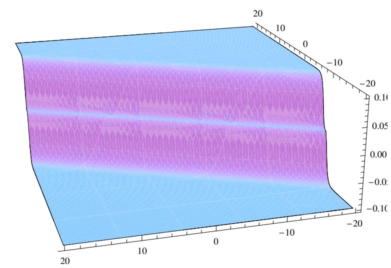

Fig. 5. Solution (29) for $\mu = 1$, $\lambda = \frac{1}{2}$, $p = 1$ and $q = 3$.

- For

$$g(h) = -\frac{h\left(\mu^2 q^2\lambda - \lambda - \alpha\mu q + \beta\mu\right)}{\mu}, \qquad (30)$$

a solution of (14) is

$$h(z) = p\exp(qz).$$

Consequently, an exact solution of equation (1), where $g(u)$ is obtained substituting h by u in (30), is

$$u(x,t) = p\exp[q(\mu x - \lambda t)]. \qquad (31)$$

- For

$$g(h) = \frac{\mu p^{\frac{2}{q}} q\lambda(1-q)}{h^{\frac{2}{q}-1}} - \frac{h^{\frac{2}{q}+1} m\mu q\lambda(m+1)(q+1)}{p^{\frac{2}{q}}} + hm^2\mu q^2\lambda + h\mu q^2\lambda - \beta h - hm^2\mu q\lambda + hm\mu q\lambda\frac{h\lambda}{\mu} + \frac{\alpha\sqrt{p^{\frac{2}{q}} - h^{\frac{2}{q}}}\sqrt{p^{\frac{2}{q}} - h^{\frac{2}{q}} m q}}{h^{\frac{1}{q}-1} p^{\frac{1}{q}}}, \qquad (32)$$

a solution of equation (14) is

$$h(z) = p\,\mathrm{sn}^q(z|m).$$

Consequently, an exact solution of equation (1), where $g(u)$ is obtained substituting h by u in (32), is

$$u(x,t) = p\,\mathrm{sn}^q(\mu x - \lambda t|m). \tag{33}$$

For $\mu = \lambda = p = q = 1$ and $m = 0.996$ the solution

$$u(x,t) = \mathrm{sn}(x - t|0.996)$$

shows a stable nonlinear nonharmonic oscillatory periodic wave (see Fig.6).

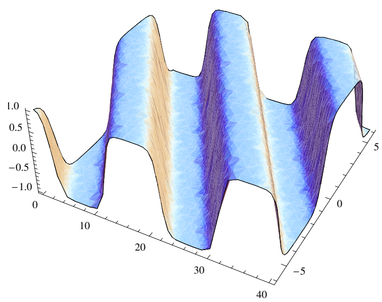

Fig. 6. Solution (33) for $\mu = \lambda = p = q = 1$ and $m = 0.996$.

• For

$$\begin{aligned} g(h) = \ & \frac{(m^2-1)\,\mu\, p^{\frac{2}{q}}\, q^2\,\lambda}{h^{\frac{2}{q}-1}} + \frac{h^{\frac{2}{q}+1}\, m\mu\, q\,\lambda(m+1)(q+1)}{p^{\frac{2}{q}}} \\ & + h\,\mu\, q^2\,\lambda - \frac{(1-m^2)\,\mu\, p^{\frac{2}{q}}\, q\,\lambda}{h^{\frac{2}{q}-1}} - \beta\, h \\ & + h\, m^2\,\mu\, q\,\lambda(1-2q) - h\, m\,\mu\, q\,\lambda + \frac{h\,\lambda}{\mu} \\ & - \frac{\alpha\sqrt{p^{\frac{2}{q}} - h^{\frac{2}{q}}}\,\sqrt{-m^2\, p^{\frac{2}{q}} + p^{\frac{2}{q}} + h^{\frac{2}{q}}\, m^2}\, q}{h^{\frac{1}{q}-1}\, p^{\frac{1}{q}}}, \end{aligned} \tag{34}$$

a solution of equation (14) is

$$h(z) = p\,\mathrm{cn}^q(z|m).$$

Consequently, an exact solution of equation (1), where $g(u)$ is obtained substituting h by u in (34), is

$$u(x,t) = p\,\mathrm{cn}^q(\mu x - \lambda t|m).$$

• For

$$\begin{aligned} g(h) = \ & \frac{\mu\, p^{\frac{2}{q}}\, q^2\,\lambda}{h^{\frac{2}{q}-1}\, m^2} - \frac{\mu\, p^{\frac{2}{q}}\, q^2\,\lambda}{h^{\frac{2}{q}-1}} + \frac{h^{\frac{2}{q}+1}\,\mu\, q^2\,\lambda}{m^2\, p^{\frac{2}{q}}} - \frac{2\, h\,\mu\, q^2\,\lambda}{m^2} \\ & + h\,\mu\, q^2\,\lambda - \frac{\mu\, p^{\frac{2}{q}}\, q\,\lambda}{h^{\frac{2}{q}-1}\, m^2} + \frac{\mu\, p^{\frac{2}{q}}\, q\,\lambda}{h^{\frac{2}{q}-1}} + \frac{2\, h^{\frac{2}{q}+1}\,\mu\, q\,\lambda}{m\, p^{\frac{2}{q}}} \\ & - \frac{h^{\frac{2}{q}+1}\,\mu\, q\,\lambda}{m^2\, p^{\frac{2}{q}}} + h\, m\,\mu\, q\,\lambda - \frac{2\, h\,\mu\, q\,\lambda}{m} + \frac{2\, h\,\mu\, q\,\lambda}{m^2} \\ & - \frac{\alpha\sqrt{p^{\frac{1}{q}} - h^{\frac{1}{q}}}\,\sqrt{p^{\frac{1}{q}} + h^{\frac{1}{q}}}\,\sqrt{m^2\, p^{\frac{2}{q}} - p^{\frac{2}{q}} + h^{\frac{2}{q}}}\, q}{h^{\frac{1}{q}-1}\, m\, p^{\frac{1}{q}}} \\ & - h\,\mu\, q\,\lambda + \frac{h\,\lambda}{\mu} - \beta\, h, \end{aligned} \tag{35}$$

a solution of equation (14) is

$$h(z) = p\,\mathrm{dn}^q(z|m).$$

Consequently, an exact solution of equation (1), where $g(u)$ is obtained substituting h by u in (35), is

$$u(x,t) = p\,\mathrm{dn}^q(\mu x - \lambda t|m).$$

• In order to obtain Wadati solitons for the BBMB equation we consider solutions of equation (14) in the form

$$h(z) = 2\partial_x \arctan\left(\frac{c\sin(nz)}{n\cosh(cz)}\right). \tag{36}$$

If we take $c = 2$ and $n = i$ in (36)

$$h(z) = \frac{4(\cosh(3z) - 3\cosh(z))}{3 - 4\cosh(2z) - \cosh(4z)}.$$

If we set $w = \cosh(z)$ then

$$h = \frac{12w - 8w^3}{-3 + 4w^4}. \tag{37}$$

We obtain

$$h' = \frac{4\sqrt{w^2-1}\left(8w^6 - 36w^4 + 18w^2 - 9\right)}{(4w^4-3)^2}$$

and

$$\begin{aligned} h'' \ = \ & \frac{4w}{(4w^4-3)^3}\left[135 - 2w^2(441 - 684w^2\right. \\ & \left. +552w^4 - 248w^6 + 16w^8)\right]. \end{aligned}$$

Substituting h, h' and h'' into equation (14) we get

$$\begin{aligned} g = \ & \frac{4w\left(32w^{10} - 496w^8 + 1104w^6 - 1368w^4 + 882w^2 - 135\right)\lambda\mu^2}{\mu(4w^4-3)^3} \\ & + \frac{4\sqrt{w^2-1}\left(8w^6 - 36w^4 + 18w^2 - 9\right)\alpha\mu}{\mu(4w^4-3)^2} - \frac{k}{\mu} \\ & + \frac{4w\left(2w^2-3\right)(\beta\mu - \lambda)}{\mu(4w^4-3)}. \end{aligned} \tag{38}$$

From (37) we obtain

$$w = \pm\frac{1}{2}\sqrt{\frac{-h^2+\theta^2+2}{h\theta} + \frac{1}{h^2}} \pm \frac{1}{2}\sqrt{\frac{\alpha_1+\alpha_2}{h^3}} - \frac{1}{2h}, \tag{39}$$

where $\theta = \left(3h + \sqrt{h^6 - 6h^4 + 21h^2 - 8}\right)^{1/3}$,

$$\alpha_1 = -h^2\theta + 2h + \frac{h^2(h^2-2)}{\theta}, \quad \alpha_2 = \frac{2 - 6h^2}{\sqrt{\frac{1}{h^2} + \frac{2-h^2+\theta^2}{h\theta}}}.$$

By substituting (39) into (38) we obtain $g(h)$ and from (12) a solution of the PDE (1) is

$$u(x,t) = \frac{4(\cosh(3(x-\lambda t)) - 3\cosh(x-\lambda t))}{3 - 4\cosh(2(x-\lambda t)) - \cosh(4(x-\lambda t))}. \tag{40}$$

In Fig. 7 we show solution (40) for $\lambda = -2$
If we take $c = 1, n = 7$ in (36) then

$$h(z) = \frac{28(\sin(7z)\sinh(z) - 7\cos(7z)\cosh(z))}{\cos(14z) - 49\cosh(2z) - 50}. \tag{41}$$

In Fig. 8 we show the solution (41) obtained
• Equation (14) for $\alpha = 0$, $k = A\lambda\mu^2$ and $\beta = c + \frac{\lambda}{\mu}$ becomes

$$h'' + ch + g(h) + A = 0. \tag{42}$$

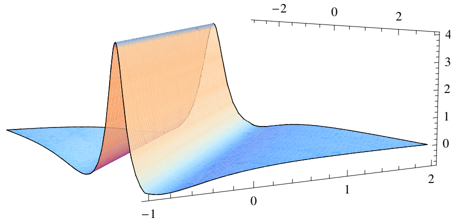

Fig. 7. Solution (40) for $\lambda = -2$.

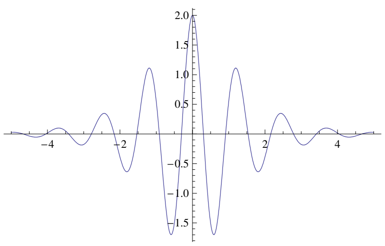

Fig. 8. Solution (41).

From [12] we see that either $g(h)$ is quadratic in h, then (42) may be solved in terms of the Weierstrass elliptic functions, or $g(h)$ is cubic in h and (42) may be solved in terms of Jacobi elliptic functions. For non-algebraic $g(h)$ we consider here the two special cases which are non-algebraic above, namely $g(h) = e^h$ and $g(h) = \ln h$.

- If $g(h) = e^h$ we differentiate (42) and make the transformation $h(z) = \ln w(z)$ so that (42) becomes rational

$$w^2 w''' - 3\, w w' w'' + 2\,(w')^3 + w^2\,(w + c)\, w' = 0. \quad (43)$$

- If $g(h) = \ln w$ we again differentiate (42), to get a rational equation

$$w w''' + w'(1 + c w') = 0. \quad (44)$$

Equation (42) falls into the classification of Painlevé and his colleagues who look for equations that are of Painlevé-type, for algebraic $g(h)$. Equations (43) and (44) are not of Painlevé-type [6].

IV. Conclusions

In this paper we have seen a classification of symmetry reductions of a generalized Benjamin–Bona–Mahony–Burgers equation, depending on the values of the constants α and β, and the function $g(u)$, by making use of the theory of symmetry reductions in differential equations. We have found the functions $g(u)$ for which we have obtained the Lie group of point transformations. We have constructed all the invariant solutions with regard to the one-dimensional system of subalgebras. Besides the travelling wave solutions, we have found new similarity reductions for this equation. We have constructed all the ODEs to which (1) is reduced. We have obtained for some functions many exact solutions which are solitons, kinks, anti-kinks, compactons and Wadati solitons.

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