

An Operatorial Approach to Sturm-Liouville Theory with Application to the Problem of a Spherical Conductor Embedded in a Uniform Field

J. Quartieri^(°)(*), L. Sirignano(*), A. Troisi(*), C. Guarnaccia^(°)(*), S. D'Ambrosio(*)

Abstract— The algebra of operator and its application to a certain set of differential equations is presented. In particular it is considered the possibility to transform a Sturm-Liouville problem in an equivalent one which exploits the property of symmetric operators and is capable to furnish “by eye” problem solutions. Such a method is then applied to a typical problem of electrostatics, i.e. a spherical conductor embedded in a uniform external field. As an example, the Laplace equation is considered and solved with this powerful technique in a very elegant way.

Keywords— Electrostatic Problem , Operator Algebra, PDE.

I. INTRODUCTION AND OPERATOR ALGEBRA

THIS work has a twofold aim: to present an operatorial technique for solving partial differential equations and, at the same time, to make detailed calculations on the electrostatic field generated by a charged sphere placed in a uniform external field. As for the first problem we recall the followings: Laplace equation often is the end point of transformations starting from different equations (Poisson, Helmholtz, etc.). This equation can be solved by separation of variables method, if a suitable coordinates transformation is performed, e.g. writing the problem in one of the eleven system of coordinates that can be built using second degree surfaces and requiring that they are locally orthogonal. This is usefully accomplished only if the physical problem at hands has the same well definite symmetry. For example the problem we are going to present, i.e. a sphere in a uniform field, has a well definite symmetry which allows us to replace the Laplace equation with two Sturm-Liouville's equations. In an old paper, one of us [1] showed an operatorial way of

transforming a second rank differential form, with variable coefficients of whatsoever degree, into a Sturm-Liouville's operator. This technique allows one to automatically find out the weight function and to infer the functional space that render that operator a Hermitian operator.

For easy reference some main ideas are here reported. Consider the following expression (where coefficients are real functions):

$$\begin{aligned} P(x)\frac{d^2}{dx^2} + Q(x)\frac{d}{dx} + R(x) &\equiv \\ &\equiv PD^2 + QD + R \end{aligned} \quad (1)$$

and, remembering that the commutator of a derivative and a function is just the derivative of the function: $[D, F(x)] = F'(x)$, one can put the expression in its quasi-symmetric form, by the following manoeuvres:

$$\begin{aligned} PD^2 + QD + R &= (PD + Q)D + R = \\ &= (DP + Q - P')D + R = \\ &= \left(D + \frac{Q - P'}{P} \right) PD + R = \\ &\equiv (D + G)PD + R = W^{-1}DWPD + R \end{aligned}$$

where we used the following operatorial equality:

$$\begin{aligned} D + \frac{Q - P'}{P} &= e^{-\int \frac{Q - P'}{P} dx} D e^{\int \frac{Q - P'}{P} dx} \equiv \\ &\equiv W^{-1}DW \end{aligned} \quad (2)$$

In view of finding the self-adjoint (and Hermitian) properties of the expression (2), and omitting for a while function W^{-1} , we name function $WP \equiv Z$ and $WR \equiv Y$, obtaining

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[°] Department of Physics “E. Caianiello”, University of Salerno, Via S. Allende, I – 84081 Baronissi (SA), Italy.

* Faculty of Engineering, University of Salerno, Via Ponte don Melillo, I - 84084 Fisciano (SA), Italy. (corresponding authors QUARTIERI J.: 0039 089 969356; e-mail: quartieri@unisa.it; GUARNACCIA C.: 0039 089 969356; e-mail: guarnaccia@sa.infn.it).

$$W^{-1}DWPD + R \rightarrow DZD + Y \tag{3}$$

In this form the quasi-symmetry of the expression emerges. "Quasi", because we need to free the expression from W^{-1} before obtaining its full symmetric form. This is always possible multiplying both sides by W . In order to show the symmetry of (3), let us remember transposition rules for (both limited and unlimited) operators: $(AB)^T = B^T A^T$, $(D)^T = -D$ and $(f(x))^T = +f(x)$, where A and B are generic limited operators, D and $f(x)$ represent respectively the unlimited operators pertaining to d/dx and $f(x)$.

So one has: $(DZD + Y)^T = (-D)Z(-D) + Y = DZD + Y$ which proves that the unlimited operator, $DZD + Y$, we are studying, is self-adjoint.

More over if we choose to work in a suitable Hilbert functional space the unlimited self-adjoint operator becomes also Hermitian [2], [3]. For this to happen is sufficient to eventually integrate between two (proper or improper) points in which Z vanishes [1], [2]. Besides, W results to be just the weight function of that functional space.

Resuming, we showed that the (Sturm-Liouville) operator

$$PD^2 + QD + R, \tag{4}$$

can be put in the symmetrical form

$$DZD + Y, \tag{5}$$

after having freed it from a multiplicative function. In what said till now, we had no need to limit the algebraic degree of the coefficients. Now, if we put some more conditions on the degree of coefficients, namely P is up to a second degree polynomial, Q is up to a first degree polynomial, while R is a constant, then, we have the solutions of the Sturm-Liouville problem

$$(PD^2 + QD + R)\Psi_n = f_n \Psi_n \tag{6}$$

"at first sight" in operatorial form. They are:

$$\Psi_n = D^n Z^n \Psi_0 \quad (-\infty < n < \infty), \tag{7}$$

where Ψ_0 could also be the identity function.

Obviously also $\Phi_n = K g(n) \Psi_n$, where K is a constant and $g(n)$ an arbitrary function, is still a solution of (6). As it is well known this degree of freedom can be used to normalize Ψ_n . As a matter of fact, by means of the procedure previously described, one can "at sight" obtain almost all the orthogonal polynomials of the classical "special" functions

and, at the same time, one can link Hermitian properties of O.D.E.s to their symmetry properties.

Let us very briefly discuss why the Sturm-Liouville operator is Hermitian. Actually, a symmetric 2-nd order operator, like $DZD + Y$, is Hermitian in the manifold of its own eigenvectors if they are definite and continuous (up to the second derivatives) in $[a, b]$, a and b being two zeros of the central function Z . In particular, we have the statement: if $Z(a) = Z(b) = 0$ then $\langle X_1 | DZD + Y | X_2 \rangle = \langle X_2 | DZD + Y | X_1 \rangle$. In fact,

$$\int_a^b X_1 D(ZX_2') = X_1 ZX_2' \Big|_a^b - \int_a^b X_1' ZX_2' = \int_a^b X_1' ZX_2'$$

which, being symmetric with respect to the two indices, reveals the Hermitian nature of the operator¹.

Now, as previously said, we are going to apply these ideas to the solution of a simple physical problem like that arising in the case of a charged conductor sphere embedded in a uniform electrostatic field. In such a case, in order to investigate the physics of the problem, one has to solve a Laplace equation which describes the behaviour of the electrostatic potential defined in the space.

Let us remember that the electrostatic problem is characterized by means of the Poisson equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0},$$

where V is the electrostatic potential defined

by $\vec{E} = -\vec{\nabla} V$. In the domain where electrostatic charges are lacking, this equation becomes its well known homogeneous associate expression

$$\nabla^2 V = 0, \tag{8}$$

i.e. a Laplace equation. Obviously, in order to find the solution of such an equation, one has to add proper boundary conditions. At this aim one has to remember that:

Field lines have to be perpendicular to the tangent plane defined at each point of the conductor surface. Such a geometrical property is equivalent, from the physical point of view, to state that the conductor has to be characterized by the same electrostatic potential at each point of its surface (*equipotential surface*).

Lines of force have to be parallel to the external field when measurements are performed very far from the conductor system. Equivalently, the same phenomenon is in order when system dimensions are negligible with respect to the geometrical size of the problem.

¹Notice that this result only holds if we work in a real space.

So, let us consider a charged spherical conductor with radius R embedded in an external uniform electrostatic field \vec{E} (see Fig.1). According to the physical properties of a conductor system we can assign an electric potential V_0 to the whole system. Actually, the flow lines of a localized electrostatic field, in presence of a conductor, will be affected by

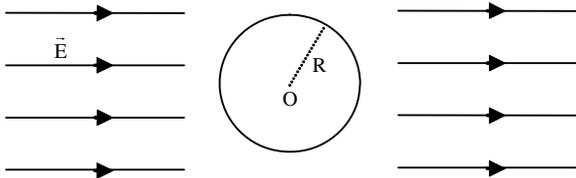


Fig. 1: A spherical conductor with R radius embedded in an external electrostatic field \vec{E} . The electrostatic potential is V_0 .

the particular geometry of the system. In particular, remembering that each line of force has to be perpendicular to the surface tangent when it matches the conductor, this will characterize the structure of the field in the neighbourhood of the system. Since we are dealing with a spherical system, the problem admits a symmetry when considering a straight line passing for the centre O of the sphere which is parallel to the external field \vec{E} . As we said in the introduction, the problem of solving what is the shape of the electrostatic field around a whatever conductor system is related to the so called “electrostatic problem” which requires solving the Poisson equation with well defined boundary conditions. Since we want to obtain the configuration of the electrostatic outside the conductor² one has to take into account the Laplace equation (8) which does not consider charges.

Let us consider a polar coordinate system $\{r, \vartheta, \phi\}$ centred in the conductor centre O which embodies the spherical symmetry in an intrinsic way (Fig.2).

The electrostatic potential at the position P distant r from the source origin, because of the symmetry of the problem, depends only on the variables $\{r, \vartheta\}$ i.e. $V = V(r, \vartheta)$. In fact, being embedded in an external field the conductor sphere turns out to be polarized and an asymmetry with respect to one of the angles arises. As a matter of fact the whole system depends on both $\{r, \vartheta\}$. Eq.(8) can be explicitly written in spherical coordinates as follows:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial V}{\partial \vartheta} \right) = 0 \quad (9)$$

² It has to be remarked that the electrostatic field inside a conductor system is vanishing. This very well known property is related to the solenoidality of the electrostatic field in vacuum.

Multiplying both sides of Eq.(9) by r^2 and rewriting the resulting expression in operatorial form (which means that each operator acts on all other operators at its right if no parenthesis is present), we have:

$$(\partial_r r^2 \partial_r + \partial_{\cos \vartheta} \sin \vartheta \partial_{\vartheta}) V(r, \vartheta) \hat{=} 0, \quad (10)$$

where $\hat{=}$ reminds us the operatorial nature of the equation. Realizing that

$$\sin \vartheta \partial_{\vartheta} = \sin^2 \vartheta \frac{\partial}{\partial \cos \vartheta} = \sin^2 \vartheta \partial_{\cos \vartheta}, \quad (11)$$

we can write:

$$(\partial_r r^2 \partial_r + \partial_{\cos \vartheta} (1 - \cos^2 \vartheta) \partial_{\cos \vartheta}) V(r, \cos \vartheta) = 0. \quad (12)$$

Putting $\cos \vartheta = x$ we finally find eq.(10) in its symmetrical and with separated variables form:

$$(\partial_r r^2 \partial_r + \partial_x (1 - x^2) \partial_x) V(r, x) = 0. \quad (13)$$

Now, thanks to the symmetry matching between the physical problem and the coordinate system chosen, we can write:

$$V(r, x) = h(r)g(x),$$

and write (13) as the equivalent system

$$\begin{cases} D_r r^2 D_r h(r) = -k h(r) \\ D_x (1 - x^2) D_x g(x) = k g(x) \end{cases} \quad (14)$$

which represent two distinct Sturm-Liouville problems. It is evident that both the operators $D_r r^2 D_r$ and $D_x (1 - x^2) D_x$ (Legendre operator) are already in the foreseen form that allows one to apply the “solution at sight” formula, thus we have:

$$h_n(r) \propto D_r^n r^{2n} 1, \quad (15)$$

$$g_n(r) \propto D_x^n (1 - x^2)^n 1, \quad (16)$$

where as initial function we choose $h_0 = g_0 = 1$ and “ n ” generally varies into the range $]-\infty, \infty[$.

Let us remark that, in our approach, negative values of n , when considered with respect to the derivative operator, imply an integration. With these premises in mind, if we perform calculations and search for orthonormal eigenfunctions it is

possible to obtain a complete set of such solutions. Eq. (15) provides the complete solution for $h(r)$, as :

$$h_n(r) \propto \frac{r^n}{n!}, \quad n \geq 0$$

$$h_n(r) \propto \frac{(-1)^n (n+1)!}{(2n+1)! r^{n+1}},$$
(17)

On the other side Eq.(16) furnishes the solution for $g_n(x)$. In particular, in the case of positive n one has

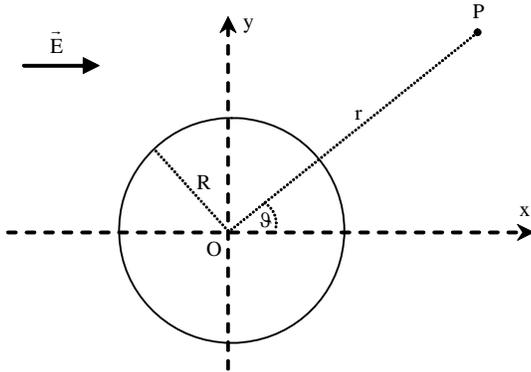


Fig. 2: A polar coordinate system embodying spherical symmetry centred in the spherical conductor under consideration.

$$g_n(x) \propto \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} (1-x^2)^n, \quad (18)$$

with $x = \cos \vartheta$ which can be easily recognized as the Legendre polynomials given by the Rodriguez formula [4]. While the second set of solutions fulfilling field equations

$$g_n(x) \propto \underbrace{\int \int \dots \int}_{n \text{ times}} \frac{1}{(1-x^2)^n} 1, \quad (19)$$

which are obtained from (16) considering negative n , as well as the Legendre polynomials of second kind, turn out to be physically ill defined since they explode on the boundaries of the integration interval which determines these polynomials. As a matter of fact this second set of solution must be discarded.

Actually, solutions (17) and (18) are defined for less than a multiplying constant which can be determined imposing boundary conditions. Finally, the complete solution of (13) can be written as

$$V(r, \vartheta) = \sum_{n=0}^{+\infty} \left[A_n r^n + \frac{B_n}{r^{n+1}} \right] P_n(\cos \vartheta), \quad (20)$$

introducing generic constants condensing also the normalizing factors. Such expression coincides with the well known textbooks solution [4] and it has been obtained without recurring to very lengthy calculations as typically done in

differential equation resolution.

At this point, in order to completely determinate the solution one needs to fix constants considering the (20) in relation with boundary conditions. Let us summarize the physical conditions the electrostatic potential has to satisfy :

The electrostatic potential has to be constant on each point of the spherical conductor surface and inside the conductor itself. In term, of mathematical relations: $V(r, \vartheta) = V_0, \quad \forall \vartheta \in [0, \pi]$.

The electrostatic potential at infinity is determined by the uniform electrostatic field outside the conductor system. Since $\vec{E} = -\vec{\nabla} V$, one obtains

$$V = -\int E dx = -E \int d(r \cos \vartheta) = -Er \cos \vartheta$$

for less than an additive constant term. Thus :

$$\lim_{r \rightarrow \infty} V(r, \vartheta) = -Er \cos \vartheta \quad \forall \vartheta \in [0, \pi].$$

In the external regions with respect to the conductor system the (20) reads

$$V(r, \vartheta) = \left[A_0 + \frac{B_0}{r} \right] P_0(\cos \vartheta) +$$

$$+ \left[A_1 r + \frac{B_1}{r^2} \right] P_1(\cos \vartheta)$$

$$+ \sum_{n=2}^{+\infty} \left[A_n r^n + \frac{B_n}{r^{n+1}} \right] P_n(\cos \vartheta),$$

so that, if we introduce the explicit form of Legendre polynomials $P_0(x)$; $P_1(x)$ and omitting the additive constant A_0 , we obtain

$$V(r, \vartheta) = \frac{B_0}{r} + \left[A_1 r + \frac{B_1}{r^2} \right] \cos \vartheta +$$

$$+ \sum_{n=2}^{+\infty} \left[A_n r^n + \frac{B_n}{r^{n+1}} \right] P_n(\cos \vartheta)$$

In order to fulfil the boundary condition (ii) it has to be : $A_n = 0$, with $n = 2, 3, 4, 5, \dots$ so that the potential $V(r, \vartheta)$ at infinity behaves as well as $V(r, \vartheta) = A_1 r \cos \vartheta$. Therefore, the condition (ii) implies $A_1 = -E$ and one gets

$$V(r, \vartheta) = \frac{B_0}{r} + \left[ER + \frac{B_1}{r^2} \right] \cos \vartheta + \sum_{n=2}^{+\infty} \frac{B_n}{r^{n+1}} P_n(\cos \vartheta). \tag{21}$$

The condition (i), on the other side, suggests that :

$$V(r, \vartheta) = \frac{B_0}{R} + \left[ER + \frac{B_1}{R^2} \right] \cos \vartheta + \sum_{n=2}^{+\infty} \frac{B_n}{R^{n+1}} P_n(\cos \vartheta).$$

which is true $\forall \vartheta \in [0, \pi]$ with

$$\begin{cases} V_0 = B_0 / R \\ -ER + B_1 / R^2 = 0 \\ B_n = 0 \quad n = 2,3,4,5,\dots \end{cases} \tag{22}$$

thus

$$\begin{cases} B_0 = V_0 R, \quad b_1 = ER^3 \\ B_n = 0 \quad n = 2,3,4,5,\dots \end{cases}$$

Finally the (21) becomes:

$$V(r, \vartheta) = \frac{V_0 R}{r} - E \left[1 - \frac{R^3}{r^3} \right] r \cos \vartheta. \tag{23}$$

II. SURFACE CHARGE DISTRIBUTION

Let us notice that the since we are dealing with a charged system the total amount of charge present on the conductor is

$$Q = 4\pi\epsilon_0 V_0 R. \tag{24}$$

Such a charge, also in the presence of an external electrostatic field, turns out to be constant in time (Principle of charge conservation). The radial component of the electrostatic field is:

$$E_r = -\frac{\partial V}{\partial r} = \frac{V_0 R}{r^2} + E \left(1 + 2 \frac{R^3}{r^3} \right) \cos \vartheta$$

which evaluated in $r=R$ provides:

$$(E_r)_{r=R} = \frac{V_0}{R} + 3E \cos \vartheta$$

Actually, Coulomb theorem states that the charge density on the conductor's surface is: $\sigma = \epsilon_0 (E_r)_{r=R}$, that is:

$$\sigma = \frac{\epsilon_0 V_0}{R} + 3\epsilon_0 E \cos \vartheta. \tag{25}$$

Let us investigate the sign of σ with respect to variations in term of the potential V_0 . We assume that $V_0 \geq 0$ (the opposite case $V_0 \leq 0$ can be straightforwardly deduced from the first one). In such a case, the charge density σ on the surface is positive if :

$$\cos \vartheta > -\frac{V_0}{3ER}. \tag{26}$$

- $V_0 > 3ER$

In such a case Eq.(26) provides : $0 \leq \vartheta \leq \pi$. In each point of the spherical conductor surface the charge density is positive (see Fig.3.) The maximum

value of the charge by effect of the external field is obtained at the point P_1 ($\vartheta = 0$); while the minimum is obtained at the point P_2 ($\vartheta = \pi$) :

$$\begin{cases} \sigma_{P_1} = \epsilon_0 (V_0 / R + 3E) \\ \sigma_{P_2} = \epsilon_0 (V_0 / R - 3E) \end{cases}$$

One can now calculate the amount of positive charge on the conductor surface. Let dS be the infinitesimal surface spotted by an infinitesimal angle $d\vartheta$, this quantity, in term of spherical coordinates, can be written down as $dS = 2\pi R^2 \sin \vartheta d\vartheta$. Thus, according with Eq.(25) the positive charge is:

$$q^{(+)} = \int \sigma dS = 2\pi\epsilon_0 R V_0 \int_0^\pi \sin \vartheta d\vartheta + 6\pi\epsilon_0 E R^2 \int_0^\pi \sin \vartheta \cos \vartheta d\vartheta = 4\pi\epsilon_0 V_0 R$$

which is coherent with (24).

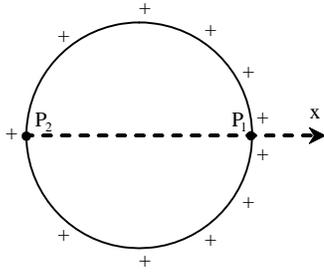


Fig. 3: The spherical conductor in presence of a positive charge.

- $V_0 = 3ER$

From Eq.(26) turns out that $0 \leq \vartheta < \pi$. Each point of the conductor surface experiences a positive distribution of charge except for the point P_2 ($\vartheta = \pi$) where $\sigma_{P_2} = 0$. The maximum of the charge distribution is obtained at the point P_1 ($\vartheta = 0$);

$$\begin{cases} \sigma_{P_1} = 6\varepsilon_0 V_0 \\ \sigma_{P_2} = 0 \end{cases}$$

- $0 < V_0 < 3ER$

In such a case, from Eq.(26) descends: $0 \leq \vartheta < \alpha$ with $\alpha = \cos^{-1}(-V_0/3ER)$. Points of conductor surface such that $\vartheta = \alpha$ imply that surface density charge is vanishing. On the other side, in the regions where $0 \leq \vartheta < \pi$ the density of surface charge is positive. Finally, regions where $\alpha < \vartheta \leq \pi$ (Fig.4) experience a negative charge. The maximum of positive density of charge distribution is obtained at the point P_1 ($\vartheta = 0$) while the maximum of negative charge distribution is obtained at the point P_2 ($\vartheta = \pi$):

$$\begin{cases} \sigma_{P_1} = \varepsilon_0(V_0/R + 3E) \\ \sigma_{P_2} = \varepsilon_0(V_0/R - 3E) \end{cases}$$

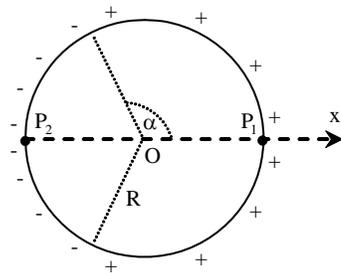


Fig. 4: The spherical conductor in presence of both a

positive charge and a negative charge.

The amount of positive charge is

$$\begin{aligned} q^{(+)} &= 2\pi\varepsilon_0 R V_0 \int_0^\alpha \sin \vartheta d\vartheta + \\ &+ 6\pi\varepsilon_0 E R^2 \int_0^\alpha \sin \vartheta \cos \vartheta d\vartheta = \\ &= 3\pi\varepsilon_0 E R^2 \left(1 + \frac{V_0}{3ER}\right)^2 \end{aligned}$$

while the amount of negative contribute is:

$$\begin{aligned} q^{(-)} &= 2\pi\varepsilon_0 R V_0 \int_\alpha^\pi \sin \vartheta d\vartheta + \\ &+ 6\pi\varepsilon_0 E R^2 \int_\alpha^\pi \sin \vartheta \cos \vartheta d\vartheta =, \\ &= -3\pi\varepsilon_0 E R^2 \left(1 - \frac{V_0}{3ER}\right)^2 \end{aligned}$$

the total charge is of course :

$$q = q^{(+)} + q^{(-)} = 4\pi\varepsilon_0 V_0 R$$

independent by the external electric field as it should be in relation to Eq.(24).

$$V_0 = 0$$

This case coincides with a conductor initially uncharged. From Eq.(26) it is obtained that $0 < \vartheta \leq \pi/2$. The conductor surface by effect of the external field turns out to be half-charged with a positive contribute ($\alpha \leq \vartheta < \pi$) while a negative charge is distributed on the other half side of the conductor ($\pi/2 < \vartheta \leq \pi$). When $\vartheta = \pi/2$ the density distribution of surface charge is vanishing. The maximum of positive charge is obtained when P_1 ($\vartheta = 0$), whereas the maximum of the negative charge is obtained at P_2 ($\vartheta = \pi$):

$$\begin{cases} \sigma_{P_1} = 3\varepsilon_0 V_0, \\ \sigma_{P_2} = -3\varepsilon_0 V_0. \end{cases}$$

The total charge will be :

$$\begin{cases} q^{(+)} = Q/2 = 2\pi\varepsilon_0 V_0 R, \\ q^{(-)} = -Q/2 = -2\pi\varepsilon_0 V_0 R. \end{cases}$$

III. LINES OF FORCE OF THE ELECTROSTATIC FIELD

Let us recall here the potential expression (23)

$$V(r, \vartheta) = \frac{V_0 R}{r} - E \left[1 - \frac{R^3}{r^3} \right] r \cos \vartheta,$$

from this relation on obtains that the components of the electrostatic field along the directions labelled by r and ϑ coordinate read :

$$E_r = -\frac{\partial V}{\partial r} = \frac{V_0 R}{r^2} + E \left[1 + 2 \frac{R^3}{r^3} \right] \cos \vartheta,$$

$$E_\vartheta = -\frac{1}{r} \frac{\partial V}{\partial \vartheta} = -E \left[1 - \frac{R^3}{r^3} \right] \sin \vartheta.$$

Moving from polar coordinate to Cartesian ones we actually have

$$\begin{cases} E_{rx} = E_r \cos \vartheta = \frac{V_0 R}{r^2} \cos \vartheta + E \left[1 + 2 \frac{R^3}{r^3} \right] \cos^2 \vartheta, \\ E_{ry} = E_r \sin \vartheta = \frac{V_0 R}{r^2} \sin \vartheta + E \left[1 + 2 \frac{R^3}{r^3} \right] \sin \vartheta \cos \vartheta, \\ E_{\vartheta x} = E_\vartheta \cos \frac{\pi}{2} + \vartheta = E \left[1 - \frac{R^3}{r^3} \right] \sin^2 \vartheta, \\ E_{\vartheta y} = E_\vartheta \cos \frac{\pi}{2} + \vartheta = -E \left[1 - \frac{R^3}{r^3} \right] \sin \vartheta \cos \vartheta. \end{cases}$$

At the end of the day summing up the two components respectively with respect to x and y , one obtains the whole Cartesian contribute of the electrostatic field

$$\begin{cases} E_x = E_{rx} + E_{\vartheta x} = \\ = \frac{V_0 R}{r^2} \cos \vartheta + E \left[1 + 2 \frac{R^3}{r^3} - 3 \frac{R^3}{r^3} \sin^2 \vartheta \right], \\ E_y = E_{ry} + E_{\vartheta y} = \\ = \frac{V_0 R}{r^2} \sin \vartheta + 3E \frac{R^3}{r^3} \sin \vartheta \cos \vartheta. \end{cases}$$

By definition, the lines of force of a whatever field theory are characterized in such a way that considered a generic point on these lines the tangent shows the same direction of the field. Thus, if $y = y(x)$ is the equation of the line of force, its derivative has to follow the field direction, that is:

$$\frac{dy}{dx} = \frac{E_y}{E_x}.$$

Being

$$\frac{dy}{dx} = \frac{d(r \sin \vartheta)}{d(r \cos \vartheta)} = \frac{r' \sin \vartheta + r \cos \vartheta}{r' \cos \vartheta - r \sin \vartheta}, \tag{29}$$

Eq.(28) together the relations (27) for E_x and E_y provides:

$$r' = \frac{r(r^3 + 2R^3) \cos \vartheta + \frac{V_0 R}{E} r^2}{(R^3 - r^3) \sin \vartheta}. \tag{30}$$

Considering $r = \xi R$ the last expression can be made dimensionless as:

$$\xi' = \frac{\xi(\xi^3 + 2R^3) \cos \vartheta + \frac{V_0 R}{E} \xi^2}{(R^3 - \xi^3) \sin \vartheta}, \tag{31}$$

where ξ is depending on ϑ . If $V_0 = 0$, the (31) reads :

$$\xi' = \frac{\xi(\xi^3 + 2)}{(1 - \xi^3)} \cot \vartheta,$$

which can be solved quite immediately by separating variables

$$\frac{(1 - \xi^3)}{\xi(\xi^3 + 2)} d\xi = \cot \vartheta d\vartheta,$$

$$-\frac{1}{2} \ln \frac{(\xi^3 + 2)}{\xi} = \ln \sin \vartheta + c,$$

$$\frac{\xi^3 + 2}{\xi} \sin^2 \vartheta = k.$$

Such equation provides a family of curves $\xi = \xi(\vartheta, k)$ with $\xi \geq 1$ and $k \geq 0$. Now, in order to construct the cartesian form of such family of curves one has to remember that :

$$\xi = \frac{r}{R} = \frac{\sqrt{x^2 + y^2}}{R},$$

$$\sin \vartheta = \frac{y}{R} = \frac{y}{\sqrt{x^2 + y^2}}.$$

As a matter of fact one obtains:

$$\left[\left(\frac{y}{R} \right)^2 - k \right],$$

$$\left[\left(\frac{x}{R} \right)^2 + \left(\frac{y}{R} \right)^2 \right]^{3/2} + 2 \left(\frac{y}{R} \right)^2 = 0, \quad (32)$$

with

$$\left(\frac{x}{R} \right)^2 + \left(\frac{y}{R} \right)^2 \geq 1 \text{ and } k \geq 1.$$

In Fig.(5), we show the lines of force (32) when it is considered a spherical conductor with a unitary radius (we have settled in this case the potential with the value $V_0 = 0$), and with k :

- $k = 0$ (x-axis), $k = 0.5$, $k = 1.0$,
- $k = 1.5$, $k = 2.0$, $k = 2.5$,
- $k = 3.0$, $k = 3.2$, $k = 3.5$, $k = 4.0$,

Observing Fig.5 one can notice that the lines of force approach a line parallel to the x -axis when k increases. In other words, the field lines approach the direction of the external electrostatic field when k increases according with the boundary conditions (ii).

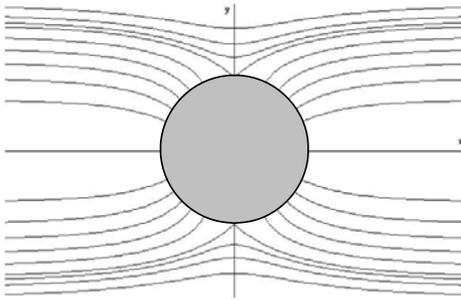


Fig. 5: The lines of force of the electrostatic field obtained setting $V_0 = 0$.

If one considers $V \neq 0$, which means to consider a charged spherical system, Eq.(31) cannot be solved in a straightforward way anymore. In particular one has to resort to numerical calculations in order to obtain the solution of the field lines problem. To this purpose we employed the Runge-Kutta algorithm and the software DERIVE.

Fig.6, shows the lines of force of the electrostatic field around a charged spherical conductor when this is equipped with a unitary radius. In particular we display the case

$$\frac{V_0}{ER} = 4.$$

In such case, since $V > 3ER$, the conductor is characterized in each point of its surface with a positive charge. This means that the lines of force will be leaving the conductor surface a moving towards infinity.

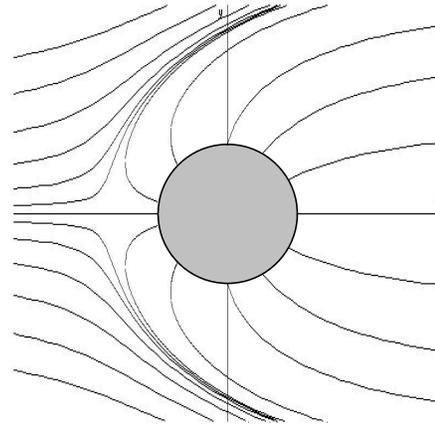


Fig. 6: The lines of force of the electrostatic field when $V_0 = 4ER$.

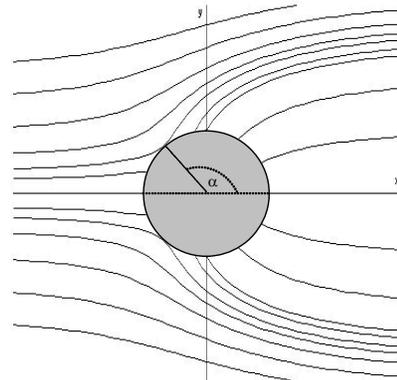


Fig.7: Lines of force of the electrostatic field when $V_0 = 2ER$.

Fig.(7) shows the case $\frac{V_0}{ER} = 2$. Now, since $0 < V_0 < 3ER$, the conductor surface is partially positively charged and negative elsewhere. The angulus α which divides the two regions holds:

$$\alpha = \cos^{-1}(V_0 / 3ER) \approx 131.48. \quad (33)$$

As matter of fact, they will be lines moving from regions of the conductor where σ is positive and others approaching the conductor surface in the regions where σ is negative. Let us remark that positions where the system experiences a vanishing charge are characterized by cusps.

REFERENCES:

- [1] J. Quartieri, Can the Application of Heavyside's Method Shed New Light into a Class of Sturm-Liouville's Equations Relevant in the Mathematical-Physics Field?, *Il Nuovo Cim. B* 8, 809 (1994).
- [2] R. Shankar, "*Principles of Quantum Mechanics*", Plenum Press – New York and London.
- [3] E. Kreyszig, "*Introduction functional analysis with application*" John Wiley and Sons, Inc..
- [4] J.D. Jackson, "*Classical Electrodynamics*", John Wiley and Sons, Inc..
- [5] Quartieri J., Steri S., A New Method for Solving 2D Nonlinear PDEs Relevant in Physics, *Int. J. Nonlin. Sci. Num. Sim.*, 4, n.4 (2003), 423-427.
- [6] Guida M., Quartieri J., Steri S., Groebner's integration of a special class of nonlinear Cauchy problems, *Int. J. Nonlin. Sci. Num. Sim.*, 4 (2003), 99-102.
- [7] S. D'Ambrosio, C. Guarnaccia, M. Guida, J. Quartieri, S. Steri; A biomathematical study of a controlled birth and death process describing malignancy; *International Journal of Biology and Biomedical Engineering*, Issue 1, Vol.1, pp 17-24 (2007) ISSN: 1998-4520.
- [8] Steri S., Quartieri J., Volzone G., Guarnaccia C., Evolutionary Processes Solved with Lie Series and by Picard Iteration Approach, *International Journal of Mathematical Models and Methods in Applied Sciences*, Issue 2, Vol. 2, pp 262-268 (2008), ISSN: 1998-0140.
- [9] S. D'Ambrosio, C. Guarnaccia, D. Guida, T.L.L. Lenza, J. Quartieri, Lie Series Application to the Identification of a Multibody Mechanical System , Proceeding of the 9th WSEAS Int. Conf. ICAI '08, Bucharest (Romania), 24-26 June 2008, pp 480-483.
- [10] Quartieri J., Steri S., Volzone G., Guarnaccia C., A Model on Controlled Evolution of Malignant Cells and on Drug Balance between Blood and Tumor, *International Journal of Biology and Biomedical Engineering*, Issue 3, Vol.1, pp 62-67 (2007) ISSN: 1998-4520.