Does the higher order mean the better internal delay rational approximation?

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***Abstract*—**The aim of this contribution is to test by simulations whether the higher order rational approximation for exponential elements in linear time-invariant time-delay systems (LTI-TDS) automatically means the better (i.e. more accurate) finite dimensional approximating model. The presented approximations are utilized to the Laplace transfer function model in the form of fractions of so-called quasipolynomials and the methods are chosen so that they are easy to handle with. Namely, Padé approximation, shift operator approximations – Laguerre and Kautz shift - and Fourier analysis based method are introduced and benchmarked. The work is motivated i.a. by the fact that direct controller design for LTI-TDS based on such models is mostly rather intricate and there are no theoretical results for internal delays. Moreover, the authors intend to use the results for rationalization of so-called anisochronic controllers when their discretization. The quality of approximation is measured by the well known H_2 and H_∞ norms instead of exact analytic calculations since it is sufficient for practical engineering problems. Some simulation examples for anisochronic controllers by means of a developed program testing interface in Matlab-Simulink environment are presented as well.

***Keywords*—** Anisochronic controllers, Fourier analysis, Hardy space, LTI-TDS, rational approximation.

I. Introduction

IT is a well known fact that a large number of both hypothetic and real-life processes and systems in a wide spectrum of human activities (e.g. in biology, chemistry, economics, mechanics, information technologies, etc.) are affected by delay as their generic part. Delay within the meaning of a lag or latency has been usually assumed to take effect in input-output relations only, and moreover, in a one time instant. The Laplace transform then results in a transfer function expressed by a serial combination of a delayless term and a delay.

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However, this conception is somewhat restrictive in effort to fit and model the real plant dynamics since in many cases delay appears in process inner feedback loops (or/and it can be of a distributed or nature).

Anisochronic (or hereditary) TDS models [1] – [3], on the other hand, offer a more universal dynamics description applying both integrators and delay elements either in lumped or distributed form so that delays appear on the left side of a differential equation which is no longer ordinary (ODE) but rather functional (FDE) - this brings the concept of internal (or state) delays. Aftereffect phenomenon is included in many processes, see e.g. models in [4] – [8]. Capabilities and advantages of this class of models and controllers for modeling and process control were broadly discussed in [9]. LTI-TDS models can be used not only for description of those systems embodying internal delays but they are successfully capable to fit the dynamics of high-order systems and processes even without apparent delays [10].

LTI-TDS input-output models in the simple-input simple output (SISO) case can be expressed by the transfer function in the form of a fraction of so-called quasipolynomials which can be viewed as polynomials in s over the ring of exponentials polynomials (or over a ring of linear combinations of real numbers and exponential functions in s). Alternatively, the concept of pseudopolynomials [11], meromorphic functions [12], [13] or a special ring [14], [15] instead of quasipolynomials can be used; however, from the description point of view all these models are equivalent – their significance emerges while controller design.

Nevertheless, many controller design approaches based on quasipolynomial transfer function models are a rather complex. Because of this, one may approximate an infinite-dimensional model by a finite-dimensional one to obtain a rational transfer function which can be handled by a control approach for undelayed plants. Similarly, one way how to implement anisochronic controllers (i.e. those with internal delays) in PLC or PC is to find a delayless approximating model followed by a discretization.

In recent decades a huge number of papers and works have been focused on model reduction or rational approximation of LTI-TDS, see e.g. [16] – [18]. A fair overview of some methods and approaches has been published in [19]. An overwhelming majority of these methods, however, deals with input-output delays only ignoring internal or state delays on the left-hand side of differential equations, i.e. those transfer functions with exponential terms in the denominator [4], [20].

Contrariwise, the presented contribution focuses transfer function rational approximations for systems with internal (state) delays since there are no theoretical analytic results about approximations convergence and accuracy for such models, and one can thus expect interesting numerical comparative results. The appropriate methods are chosen so that they are easy to deal with and anyone can use them effortlessly. The common principle of all approaches consists in substitution of exponential terms in the transfer function of LTI-TDS with a rational fraction in the Laplace complex variable *s*.

There are no attempts to deal with analytic mathematical proofs in this paper. The efficiency and accuracy of the selected approaches is measured via norms in the Hardy space, namely, H_∞ and H_2 norms [21], the definitions of which and some calculation tricks are presented here as well.

A Matlab-Simulink user-interface application has been programmed to make a benchmark of approximations easier which enables to enter an approximated (nominal) plant transfer function, to select methods to be compared and the choice of norms by which the accuracy of approximation is calculated. It is also possible to specify an order of the approximation for each selected approximation approach. As the outputs, a table of calculated norms and the gain Bode plot (that is relevant to the norms) are displayed.

As mentioned above, the authors' intention is to utilize the obtained results while computer digital implementation of so called anisochronic controller received, specifically, by algebraic controller design in the ring of retarded-quasipolynomial meromophic functions ($\mathbf{R}_{MS}$) [12], [21] – [24]. Several study cases on approximations of anisochronic controllers' derived by this algebraic method are introduced and compared in this contribution.

II. Transfer Functions of LTI-TDS

Since the authors' interest lies in single-input single-output (SISO) LTI-TDS and their input-output models, the state-space description and general multivariable case are omitted, the reader is referred e.g. to [4]. Namely, transfer functions in the form of quasipolynomial fractions are taken as initial models to be approximated.

Even if there are distributed delays in the system, this formulation can be governed by the transfer function obtained directly from the Laplace transform of a state space model (considering zero initial conditions) as follows

$$G(s)=\frac{b(s)}{a(s)} \tag{1}$$

where $b(s)$, $a(s)$ are quasipolynomials of the general form

$$q(s)=s^n+\sum_{i=0}^{n}\sum_{j=1}^{h_i} m_{ij}s^i\exp\left(-s\vartheta_{ij}\right), \vartheta_{ij}\geq 0 \tag{2}$$

where *n* is the order of a quasipolynomial, x_{ij} are real numbers and $\vartheta_{ij}\geq 0$ represent delays. If $\sum_{j=1}^{h_n} m_{nj}\exp\left(-\vartheta_{nj}s\right)$ do not equals a real constant, the system is called neutral; otherwise, the system is retarded.

III. Selected Transfer Function Rational Approximations

Selected easy-handling transfer function rational approximation methods are described in this section. The emphasis was put on the engineering usability of the methods. They are all based on the substitution of exponential elements by a rational function.

A. Padé approximation

In the second half of the 19th century, a French mathematician Henry Padé devised a simple and, nowadays, one of the most used and favorite rational approximations which is based on the comparison of derivatives of the approximating and approximated functions in zero. More precisely, let $F(s)$ with $F(0)\neq 0$ be analytic in the neighborhood of zero. Then, the *n*-*n* Padé approximation is the function $\Phi(s)=N_n(s)/D_n(s)$ where $N_n(s)$, $D_n(s)$ are polynomials of the *n*th order with $D_n(0)=1$ and it holds that $F^{(i)}(0)=\Phi^{(i)}(0), i=1,\ldots,2n$.

For this contribution, Padé approximation of $F(s)=\exp(-sT)$, $T\geq 0$ is given by the following relation [19]

$$\begin{aligned}\exp(-sT)&\approx\frac{P(-s)}{P(s)}\\ P(s)&=\sum_{k=0}^{n}\binom{n}{k}\frac{(2n-k)!}{(2n)!}(sT)^k\end{aligned} \tag{3}$$

where *n* is the order of the approximation.

Sometimes a method called *diagonal* Padé approximation is distinguished, see [4], [18], which can be expressed as

$$\begin{aligned}\exp(-sT)&\approx\frac{P(s)}{P(-s)}\\ P(s)&=\sum_{k=0}^{n}\frac{(2n-k)!}{k!(n-k)!}(-sT)^k\end{aligned} \tag{4}$$

However, it is easy to verify, that (3) and (4) represents the same approximations. In fact,

$$\binom{n}{k}\frac{(2n-k)!}{(2n)!}=\frac{n!}{(2n)!}\frac{(2n-k)!}{k!(n-k)!} \tag{5}$$

and $n!/(2n)!$ is the common factor of both, the numerator and denominator, hence, the fraction is the same.

B. Shift operator approximations

These methods are based on the fact that a delay term

$\exp(-sT)$ can be perceived as a shift operator and it can be subjected to Maclaurin series expansion. Moreover, the variable *s* can be vied as a derivative operator.

Indeed

$$sf(t) := f'(t)$$
$$\exp(-sT)f(t) := f(t-T) = f(t) - Tf'(t) + \frac{T^2}{2}f''(t) - \frac{T^3}{6}f'''(t) + \ldots$$
$$\exp(-sT)f(t) := \exp\left(-\frac{\mathrm{d}}{\mathrm{d}t}T\right)f(t)$$
$$= \left(1 - T\frac{\mathrm{d}}{\mathrm{d}t} + \frac{T^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{T^3}{6}\frac{\mathrm{d}^3}{\mathrm{d}t^3} + \ldots\right)f(t) \tag{6}$$

A brief description of three the most important shift operator approaches follows.

1) Laguerre shift

The eventual relation for this shift operator approximant is given by the formula

$$\exp(-sT) = \lim_{n\to\infty}\left(\frac{1-\frac{sT}{2n}}{1+\frac{sT}{2n}}\right)^n \tag{7}$$

see [16] for details, where is also stated that the Laguerre shift can be successfully used in robust control.

2) Kautz shift

As presented in [16], the *n*th order Kautz shift reads

$$\exp(-sT) \approx \left(\frac{1-\frac{sT}{2n}+\frac{1}{2}\left(\frac{sT}{2n}\right)^2}{1+\frac{sT}{2n}+\frac{1}{2}\left(\frac{sT}{2n}\right)^2}\right)^n \tag{8}$$

It has been analytically proved for input-output delays in the source referenced above that the Kautz shift approximation is asymptotically twice more accurate than the Laguerre one.

3) Padé shift

This type of shift approximation is based on the second order Padé approximation [17], [18] and it is also called Padé-2. It can be formulated e.g. as follows

$$\exp(-sT) \approx \left(\frac{1-\frac{sT}{2n}+\frac{1}{3}\left(\frac{sT}{2n}\right)^2}{1+\frac{sT}{2n}+\frac{1}{3}\left(\frac{sT}{2n}\right)^2}\right)^n \tag{9}$$

C. Fourier analysis based approximation method approach

This type of approximation was derived from the analysis of the delay-in-feedback step response or, generally, the response to the input of the form $u(t) = t^k, t, k \geq 0$. It has been found that the feedback system response is a superposition of a periodic and an aperiodic signal. The Fourier series expansion of the periodic part of the response gives the resulting approximation. To demonstrate the initial idea more precisely, consider a delay term in the negative feedback. Then the whole feedback transfer function $W(s)$ reads

$$W(s) = \frac{Y(s)}{U(s)} = \frac{\exp(-sT)}{1+\exp(-sT)} \tag{10}$$

Hence

$$y(t) = y(t-T) + u(t-T) \tag{11}$$

and the output $y(t)$ is assumed to be a superposition of a polynomial $q(t)$ and a periodic part $p(t)$

$$y(t) = q(t) + p(t) \tag{12}$$

Inserting (12) into (11) yields

$$y(t-T) - y(t) = \text{constant} \tag{13}$$

If we, furthermore, subject $p(t)$ to the Fourier series expansion and make some other nontrivial calculations, it can be obtained the final relation

$$W(s) = \frac{1}{2} - \frac{1}{4}Ts + \frac{2}{\pi^2}T^3s^3 \sum_{k=0}^{n}\frac{1}{(2n+1)^2}\frac{1}{s^2T^2+(2n+1)\pi} \tag{14}$$

The exponential term can be then approximated by the inverse of (10), i.e.

$$\exp(-sT) = \frac{W(s)}{1-W(s)} \tag{15}$$

For more details, the reader is referred to [18]

IV. Approximation Quality Measurement

Rigorous analytic proofs are usually used for evaluation of the accuracy (quality) of rational approximation methods. However, these results have been obtained for delays in input-output relation only, ignoring the effect of internal delays, which is the main gap here.

Transfer function norms in the Hardy space known from robust control for unstructured uncertainty measurement instead of any complex analytic method for the comparison of approximation accuracy are used in this paper since this conception is sufficient for particular cases and for engineering practice.

First of all, define the objective to be minimized in various meanings. The difference between the nominal G and approximated $\hat{G}$ transfer functions is taken as a measured expression G_M, i.e.

$$G_M = G - \hat{G} \tag{16}$$

We chose two the most used norms in the Hardy space, namely, H_2 and H_∞ norms, which are capable to express the "proximity" of transfer functions in the frequency domain.

A. H_2 norm

The H_2 norm (also called quadratic norm) of a stable strictly proper transfer function is defined as

$$\|G\|_2 = \frac{1}{\sqrt{2\pi}} \sqrt{\int_{-\infty}^{\infty} |G(\mathrm{j}\omega)|^2 \,\mathrm{d}\omega} \tag{17}$$

The norm is finite for strictly proper stable systems having no pole on the imaginary axis, and the meaning of H_2 is energy of G. Note that for LTI-TDS with distributed delays, there can exist a denominator root of G which is not the system pole.

In computer (discrete data) practice, the improper integral in (17) is calculated as a sum within a finite range of nonnegative frequencies, $\omega \in [0, \omega_{\max}]$. The value of $\omega_{\max}$ can be chosen so that the frequency gain is small enough. Residual expansion can be used when analytic (and continuous) calculation of $\|G\|_2$ [25].

B. H_∞ norm

This norm is defined as

$$\|G\|_\infty = \sup_{\omega} |G(\mathrm{j}\omega)| \tag{18}$$

i.e. it expresses the supreme of the amplitude (gain) frequency characteristics of G. If the system is asymptotically stable and provides a finite H_∞ norm, it is said that it is H_∞ stable and lies in the space $H_\infty(\mathbb{C}^+)$ of functions analytic and bounded in the right-half complex plane.

The norm is also called L_2 gain, That is, the H_∞ stable system has finite $L_2(0,\infty)$ to $L_2(0,\infty)$ gain where $L_2(0,\infty)$ norm of an input or output signal $h(t)$ is defined as

$$\|h(t)\|_2 := \sqrt{\int_0^{\infty} |h(t)|^2 \,\mathrm{d}t} \tag{19}$$

The frequency characteristics supreme can be easily found by standard analytic means, or, by mapping the values of $G(\mathrm{j}\omega)$ when using digital computers.

Note, for instance, that a transfer function having no pole on the imaginary axis but a sequence of poles with real part converging to zero has an infinite H_∞ norm due to an unbounded gain [26].

V. User-Interface Testing Application

A user-friendly interface in Matlab-Simulink environment has been developed by the authors in order to provide testing and comparison of approximation approaches introduced above.

The application allows entering a nominal plant transfer function, selecting approximation methods to be compared and the choice of norms by which the accuracy of approximation is calculated. It is also possible to specify an order of the approximation for each selected method. The programme returns a table of calculated norms, a graphical comparison of all chosen amplitude frequency responses, and, in the text form, we can obtain approximated transfer functions, see Fig.1.

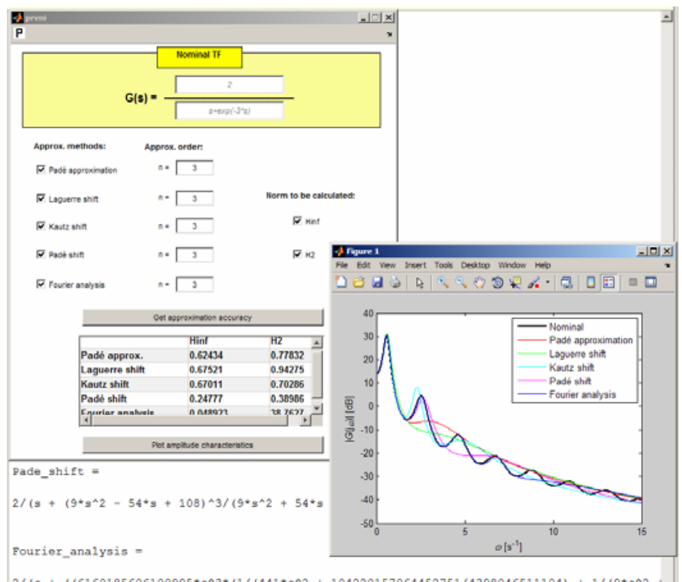

Fig. 1 User program interface in Matlab – Simulink

VI. Examples

The following section some simulation examples are presented to verify whether the introduced methods provides a sufficient rational approximation, and if the higher order of an approximation gives the more accurate estimation.

As mentioned above, results of the benchmark ought to be used by the authors for the rational approximation of controllers with internal delays (also called anisochronic) as a first step of controllers' discretization via the z-transform, for the computer implementation.

Now we present three examples. The first two give results for anisochronic controllers design for control of a simple LTI-TDS with stepwise and linearwise reference, respectively. The third one presents the approximation of an anisochronic controller for an attractive unstable system of a skater on a swaying bow.

A. Stable controlled LTI-TDS

Consider the well-known simple feedback control loop (Fig. 2) and a stable first order plant with internal and input-output delays governed by the transfer function

$$G(s)=\frac{b\exp(-\tau s)}{s+a\exp(-\vartheta s)};\ b,\tau,\vartheta\geq 0 \qquad (20)$$

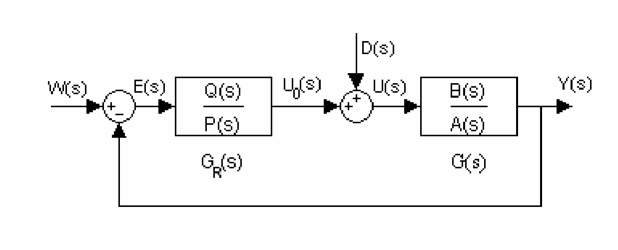

Fig. 2 Simple feedback control loop

Model (20) is able to fit the dynamics of a high order undelayed system; for instance, a tenth order system

$$G_S(s)=\frac{1}{(2s+1)^{10}} \qquad (21)$$

can be estimated by (20) with $a=b=6.5\cdot 10^{-2}, \tau=15.3,\ \vartheta=6.7$, see [27], [28] for details.

1) Stepwise reference

If step reference tracking is required, one can derive a controller give by

$$G_R(s)=\frac{m_0}{b}\frac{s+a\exp(-\vartheta s)}{s+m_0(1-\exp(-\tau s))} \qquad (22)$$

see details in e.g. [21], where $m_0>0$ is a selectable tuning parameter and let $m_0=0.05$.

Let $n\in\{1,2,3,4,5\}$, where n is the order of the approximation method, and test the accuracies for all methods introduced above. The best results for each of them measured by H_∞ and H_2 norms with the corresponding order are displayed in Table 1.

TABLE 1. COMPARISON OF RATIONAL APPROXIMATIONS OF (22) MEASURED BY H_∞ AND H_2 NORMS

Method	H_∞	*n*	H_2	*n*
Padé approx.	0.243	1	0.237	1
Laguerre shift	0.243	1	0.237	1
Kautz shift	0.408	5	0.299	3
Padé shift	0.277	1	0.29	1
Fourier analysis	0.38	1	0.354	5

Obviously, the best result for H_∞ is given alongside by the Padé approximation and Laguerre shift of the first order, whereas, amazingly, higher orders make results worse. The Fourier analysis based methods yields almost the same score for all studied orders.

The benchmark results for the H_2 norm with $\omega_{\max}=15$ displayed in Table 1 are almost identical with those for H_∞, i.e. the Padé approximation and Laguerre shift of the first order are the best and the Fourier analysis based methods gives almost the same results for all orders.

The corresponding gain frequency responses for the approximations of orders as in the last column in Table 1 are displayed in Fig. 2, and the approximating transfer function by Padé approximation and Laguerre shift of n = 1 is given by (23), which is a conventional PID controller.

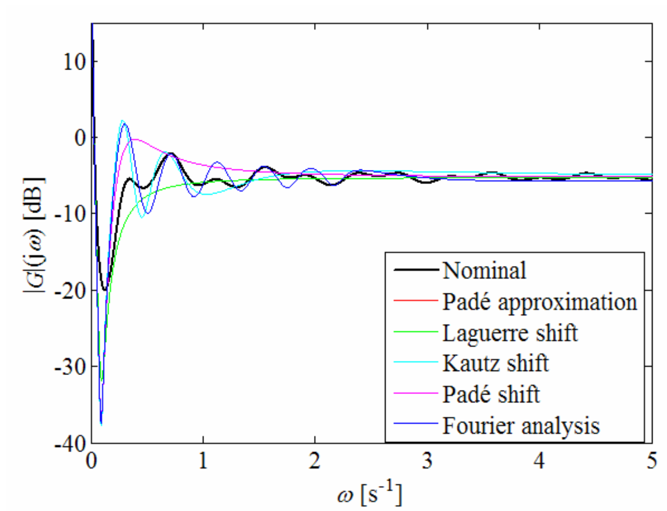

Fig. 2 Bode magnitude plots of the nominal and approximating systems for (22)

$$\hat{G}_R(s)=\frac{0.7692s^2+0.506s+0.0065}{s^2+0.2307s} \qquad (23)$$

2) Linearwise reference

The requirement of linearwise reference tracking and a stepwise load disturbance yields a rather more complex controller [23], [24] with the transfer function

$$G_R(s)= =\frac{(s+a\exp(-\vartheta s))((1-\beta_1)s+(m_1-\beta_0))}{s^2+(m_0+m_1)s+b(\beta_1-1)s\exp(-\tau s)+m_0m_1+b(m_1-\beta_0)} \qquad (24)$$

where

$$\beta_0 = b^{-1} m_1 (b - m_0) = 0.0115$$
$$\beta = b^{-1}(b - m_0 - m_1(1 + \tau m_0)) = -1.1269 \quad (25)$$

and $m_0 = m_1 = 0.05$.

Table 2 displays the best results for the approximations measured using H_∞ and H_2 norms.

TABLE 2. COMPARISON OF RATIONAL APPROXIMATIONS OF (24) MEASURED BY H_∞ AND H_2 NORMS

Method	H_∞	***n***	H_2	***n***
Padé approx.	3.629	3	1.2421	5
Laguerre shift	3.676	4	1.306	1
Kautz shift	3.4791	2	1.2463	5
Padé shift	3.629	3	1.0952	4
Fourier analysis	3.6291	5	1.968	5

The expected finding that the higher order approximation yields the better rational estimation is evident here only in some cases. The Kautz shift gives the best result for H_∞ for the second order approximation. Padé shift, on the other hand, provides the best result for the H_2 norm with the fourth order. Note that values for the Fourier analysis based method converges very slowly with the order, and, what is interesting, Padé approximation gives a very good result also for n = 1 (H_2 = 1.306).

Bode magnitude plots for the best orders for H_2 (the last column in Table 2) are figured in Fig. 3.

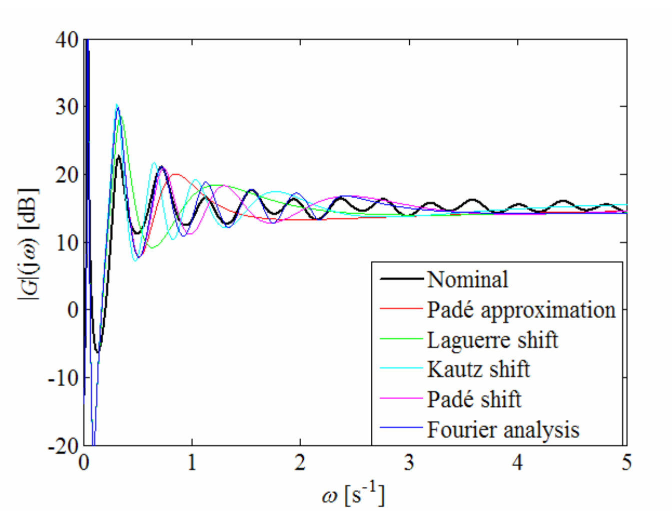

Fig. 3 Bode magnitude plots of the nominal and approximating systems for (24)

The approximating controller's model using the 4th order Padé shift is represented by the transfer function

$$\hat{G}_R(s) = \frac{\begin{array}{l} 2.127s^{10} + 13.522s^9 + 37.755s^8 + 68.848s^7 \\ + 69.937s^6 + 59.853s^5 + 23.378s^4 \\ + 9.446s^3 + 0.648s^2 + 0.071s + 0.001 \end{array}}{\begin{array}{l} s^{10} + 6.236s^9 + 19.544s^8 + 30.219s^7 + 41.756s^6 \\ + 24.168s^5 + 18.344s^4 + 3.124s^3 + 1.338s^2 \end{array}} \quad (26)$$

It is clear that the higher order approximation brings the more complex model. Because of this, it is better to use a lower order yet with a satisfactory norm values.

B. Unstable controlled LTI-TDS

Consider an unstable system describing roller skater on a swaying bow [29] given by the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b\exp(-(\tau + \vartheta)s)}{s^2(s^2 - a\exp(-\vartheta s))} \quad (27)$$

see Fig. 4, where $y(t)$ is the skater's deviation from the desired position, $u(t)$ expresses the slope angle of a bow caused by force P, delays τ, ϑ means the skater's and servo latencies and b, a are real parameters. Skater controls the servo driving by remote signals into servo electronics.

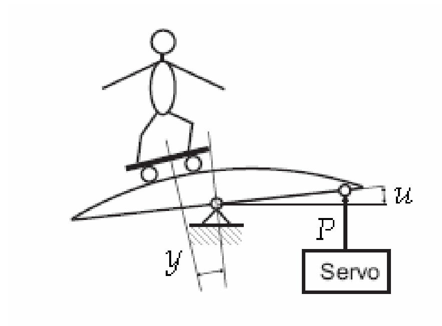

Fig. 4. The roller skater on a swaying bow

Let $b = 0.2$, $a = 1$, $\tau = 0.3$ s, $\vartheta = 0.1$ s, as in the literature, and design the controller structure according to the algebraic approach described e.g. in [21-24]. Consider the reference and load disturbance in the form of a step-wise function. The final controller has the following transfer function

$$G_R(s) = \frac{b(q_3 s^3 + q_2 s^2 + q_1 s + q_0)(s + m_0)^4 + p_0 m_0^4 s^2 (s^2 - a\exp(-\vartheta s))}{b[(s^3 + p_2 s^2 + p_1 s + p_0)(s + m_0)^4 - p_0 m_0^4 \exp(-(\tau + \vartheta)s)]} \quad (28)$$

where p_2, p_1, p_0, q_3, q_2, q_1, $q_0 \in \mathbb{R}$ are free parameters, see details in [30]. Using a quasi-optimal tuning algorithm, the parameters were set as

$$q_3 = 4.7587,\ q_2 = 2.1164,\ q_1 = 2.6252, q_0 = 0.4482$$
$$p_2 = 0.4636, p_1 = 0.529, p_0 = 4.6164 \quad (29)$$

and e.g. $m_0 = 5$.

The comparison of the best controller rational approximations can be found in Table 3.

TABLE 3. COMPARISON OF RATIONAL APPROXIMATIONS OF (28) MEASURED BY H_∞ AND H_2 NORMS

Method	H_∞	n	H_2	n
Padé approx.	5.6674	1	8.577	1
Laguerre shift	5.6674	1	8.577	1
Kautz shift	6.0847	5	9.7446	5
Padé shift	6.0668	1	9.674	1
Fourier analysis	5.6674	1	7.8114	5

Again, the method based on the Fourier series expansion is very slowly approaching the limit value of the H_2 norm (≈ 7.811) with the increasing *n*. The only method evincing the better asymptotical results with the higher order approximation is the Kautz shift. Again, the Padé and Laguerre approximations of the first order give very good results with the approximating controller transfer function

$$\hat{G}_R(s) = \frac{\begin{array}{l}4.7587s^8 + 121.084s^7 + 1245.21s^6 + 20969.8s^5 \\ + 90315.2s^4 + 39304.5s^3 + 56753.5s^2 + 9604.38s \\ + 1400.63\end{array}}{\begin{array}{l}s^8 + 25.4636s^7 + 262.119s^6 + 1383.74s^5 + 3952.16s^4 \\ + 6389.1s^3 + 8872.37s^2 + 18964.6s\end{array}} \tag{30}$$

Fig. 4 displays Bode magnitude plots for the best orders for H_2 (the last column in Table 3), which verifies a very good performance of all the approaches.

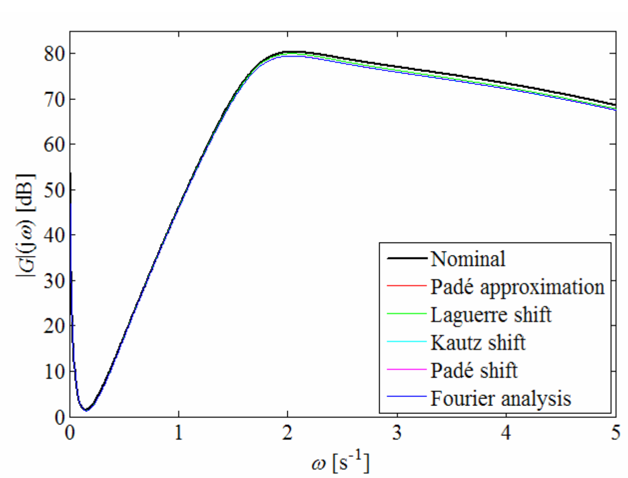

Fig. 4 Bode magnitude plots of the nominal and approximating systems for (28)

To conclude study cases above, it is startling that the best approximations measured by H_2 and H_∞ norms are mostly given by the well known and widely used Padé approximation of the first order which is identical with the first order Laguerre shift. By simulations, the higher order of an approximation does not generally yields the more accuracy finite dimensional model, which is in the contradiction with a general expectation and analytic results for rational approximations for input-outputs delays.

VII. CONCLUSION

A simulation analysis of transfer function rational approximations for systems with internal (state) delays has been the aim of this paper. Since there are no rigorous analytic results about the approximation efficiency and accuracy in the literature, the authors have decided to perform benchmark numerical tests in Matlab-Simulink on selected anisochronic controllers which are planned to be discretized after rationalization. Two norms in the Hardy space, namely, H_2 and H_∞ norms have been used as a measure for the accuracy and efficiency of the selected easy-handling approaches. Approximation methods have been selected so that they are serviceable also for people with basic mathematical background knowledge.

Numerical results have been very interesting and startling because the habitual Padé approximation has been the best one and, moreover, with the first or second order of the approximation. This means that the higher order approximation does not automatically mean the better result for the systems with internal delays.

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