Stability Conditions for a Retarded Quasipolynomial and their Applications

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***Abstract*—** Non-delay real parameter stability and stabilization for a quasipolynomial of a retarded structure is studied in this contribution. In the sense of this paper, quasipolynomials are considered to be over real coefficients and in only one variable. Unlike some other methods and analyses, a non-delay real parameter is being to set in a quasipolynomial with two independent delay terms. Retarded quasipolynomial stability is given by the requirement that all its roots (of an infinite spectrum) are located in the open left-half complex plane. The proposed stabilization methodology is based on the argument principle, i.e. on the Mikhaylov stability criterion.

This problem has many applications especially in the control theory since such a quasipolynomial can characterize the dynamics of a closed-loop system with delays. In the presented paper, we introduce two problems connected with stabilization and control of time delay systems. The first one deals with a coprime factorization for algebraic controller design, the second one propose stabilization of an anisochronic model of a high order system. Stability features and application problems are accompanied with simulation examples in the Matlab-Simulink environment.

***Keywords*—**Stabilization, quasipolynomial, argument principle, time delay systems, simulations.

I. Introduction

THE existence of delays, latencies and distributed parameters in dynamics of continuous-time linear systems is known to engineers and scientists for decades. For example, heating and thermal processes are frequently introduced in the literature, see e.g in [1]-[3]. Time delay systems (TDS) and models are considered not only with input-output delays, but it is mainly supposed that internal (state) delays appear in their dynamics [4]. Considering continuous and linear (or linearized) models, one can naturally utilize the Laplace transform yielding in single-input single-output case the transfer function as a ratio of so-called quasipolynomials [5]. Unlike some other modeling and analytic approaches, which consider quasipolynomials in two independent variables s and z [6], we utilize only one complex variable s [7]-[8]. Roots of the transfer function denominator (i.e. poles) decide (except some cases of distributed delays [9]) about the asymptotic stability as in the case of polynomials; however, the system spectrum is infinite.

A deep interest in problems of stability and stabilization of delayed systems in the recent years can be observed, e.g. in [10-13] where the task was solved using various analytical and numerical tools. The decision about asymptotic stability of plants, feedback stabilization or control of systems with internal delays can be done via studying of the corresponding characteristic quasipolynomial. For these purposes, a powerful tool is the conventional argument principle (i.e. the Mikhaylov criterion) which holds for characteristic quasipolynomials of delayed systems of retarded type. Note that neutral delayed systems require rather modified Mikhaylov criterion [14].

In this contribution, we address the investigation the stability of the selected retarded quasipolynomial with two independent delays. The aim is to find lower and upper bounds for a real selectable non-delay parameter so that all its zeros are located in the open left-half complex plane, which implies the asymptotic stability of the quasipolynomial. In contrast to other authors who usually have studied the stability w.r.t. the delay, not w.r.t. a non-delay parameter, the presented paper deal with stabilization by a non-delay parameter and two independent delays. Presented derivations and calculations are based on the argument principle (i.e. the Mikhaylov criterion) and the desired shape of the Mikhaylov plot. The result can serve engineers in setting the unknown controller parameter in the characteristic quasipolynomial of a delayed system properly, or to decide about system or quasipolynomial stability.

The paper is organized as follows: The studied quasipolynomial stability and stabilization problem is introduced in Chapter II. Chapter III contains a brief overview of analytic tools for stability analysis. In Chapter IV, lemmas, propositions and theorems explaining the stability properties are derived. As a result, the acceptable stability interval for a non-delay parameter is obtained. The application of presented results on problems of a coprime factorization and the stabilization of an anisochronic model of a high order system is demonstrated in Chapter V. This chapter is supported by some simulation examples. Conclusions and references finalize the presented paper.

II. Retarded Quasipolynomial

It is a well known fact that single-input single-output linear time-invariant systems with time delays can be expresses in the input-output relation by a ration of two *quasipolynomials*. Input-output and internal point (lumped) or distributed delays in the Laplace transform yield exponentials which appear in the transfer function. Some authors comprehend a

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quasipolynomial as a polynomial in two independent variables s and $z = \exp(-s)$ over real numbers. However, there is no reason to introduce two variables (which the analyses more difficult) and thus one complex variable *s* in a quasipolynomial can be considered.

Hence, quasipolynomials have a general form

$$m(s) = s^n + \sum_{i=0}^{n} \sum_{j=1}^{h_i} m_{ij} s^i \exp(-s\vartheta_{ij}) \quad (1)$$

There are two basic types of quasipolynomials. A *neutral* type is obtained if $m_{nj} \neq 0$, a *retarded* one if $m_{nj} = 0$. It is said that a quasipolynomial is asymptotically stable if there is *no* root σ of (1) such that

$$\mathrm{Re}\{\sigma\} \geq 0 : m(\sigma) = 0 \quad (2)$$

Both quasipolynomial types have different spectral properties; namely, neutral quasipolynomials can have an infinite number on unstable poles which create "stripes" asymptotically tracing a vertical line in a complex plane. Moreover, a small delay perturbation can move even stable spectrum into the right-half plane which gives rise to the notion of *strong stability* [15], [16]. On the other side, retarded quasipolynomials are free from these two unpleasant properties.

The quasipolynomial denominator of a transfer function decides about system stability except some cases of distributed delays where the numerator and denominator have some identical unstable roots. In this case, there exists a stable state-space realization avoiding these unstable transfer function poles.

The main goal of this paper is to found upper and lower bounds for the parameter $q \neq 0 \in \nabla$ such that the quasipolynomial with two independent delays

$$m(s) = s + a\exp(-\vartheta s) + kq\exp(-\tau s) \quad (3)$$

is stable, where $a \neq 0 \in \nabla$; $k, \vartheta, \tau > 0 \in \nabla$. This quasipolynomial can represent, for example, the feedback stabilization problem when control a system with both input-output delay τ and state (internal) delay ϑ by a proportional controller q.

III. Argument-Increment Based Stability Criterion

During last decades a various techniques for time-delay system and quasipolynomial stability analysis have been investigated, see e.g. [7], [12]-[14], [17], [18], either for neutral and retarded cases. In this chapter a very important fact about TDS and quasipolynomial stability is recalled which is usable if a frequency-domain description is available. Since a general quasipolynomial (1) is a complex variable analytic function, the argument principle based on the evaluation of the increment of $m(s)$-argument resulting from a closed counter-clockwise (positive) Jordan curve running around all quasipolynomial zeros can be used. Namely, if it is required that all zeros are located in the open left-half plane, a curve identical with the imaginary axis can be taken. It can be shown that for a general *retarded* quasipolynomial the following statement holds.

Assumption 1. If retarded $m(s)$ has no zero on the imaginary axis, then $m(s)$ has no zero in the right-half *s*-plane if and only if

$$\Delta \arg_{s=\omega j, \omega\in[0,\infty)} m(s) = \frac{n\pi}{2} \quad (4)$$

see e.g. [8], [17].

This principle is the well-known Mikahylov stability criterion. The idea used in this contribution is a sort of "loop shaping" methodology. Thus, one can calculate the desired number of quadrants in the complex plane which the Mikhaylov plot has to pass according to (4), and then to restrict quasipolynomial parameters in order to obtain the desired shape of the curve.

IV. Main Results

Result (4) is now used when stability analysis and non-delay parameter stabilization of quasipolynomial (3). According to the criterion, the Mikhaylov curve of (3) for $\omega \in [0, \infty)$ must produce the overall argument change equal to $\pi/2$. Quasipolynomial stability investigation via lemmas, propositions and theorems follows; however, due to a rather high complexity of such type of quasipolynomials, some statements remain unproven.

Lemma 1. For $\omega = 0$, the imaginary part of the Mikhaylov curve of quasipolynomial (3) equals zero and it approaches infinity for $\omega \to \infty$.

Proof. Decompose $m(\mathrm{j}\omega)$ into real and imaginary parts as follows

$$\mathrm{Re}\{m(\mathrm{j}\omega)\} = a\cos(\vartheta\omega) + kq\cos(\tau\omega) \quad (4)$$

$$\mathrm{Im}\{m(\mathrm{j}\omega)\} = \omega - a\sin(\vartheta\omega) - kq\sin(\tau\omega) \quad (5)$$

Obviously

$$\mathrm{Im}\{m(\mathrm{j}\omega)\}\big|_{\omega=0} = 0$$

$$\lim_{\omega\to\infty} \mathrm{Im}\{m(\mathrm{j}\omega)\} = \infty \qquad \square$$

Lemma 2. If (3) is stable, the following inequality holds

$$q > \frac{-a}{k} \quad (6)$$

and thus the Mikhaylov curve starts on positive real axis.

Proof. If (3) is stable, the overall argument change equals to $\pi/2$ according to (2). Moreover, Lemma 1 states that the imaginary part goes to infinity. These two requirements imply that for stable quasipolynomial is

$$\operatorname{Re}\{m(\mathrm{j}\omega)\}\big|_{\omega=0} > 0 \tag{7}$$

By application of (7) onto (4) the condition (6) is obtained. □

Lemma 3. A point on the Mikhaylov curve of (3) lies in the first quadrant for an infinitesimally small $\omega = \Delta > 0$ if and only if

$$a\vartheta + kq\tau \le 1 \tag{8}$$

This point lies in the fourth quadrant if and only if

$$a\vartheta + kq\tau > 1 \tag{9}$$

Proof. (Necessity.) If the point goes to the first quadrant for an infinitesimally small $\omega = \Delta > 0$, then the change of function $\operatorname{Im}\{m(\mathrm{j}\omega)\}$ in $\omega = 0$ is positive or this function is increasing in $\omega = \Delta$. It is known fact that this is satisfied if either

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} > 0 \tag{10}$$

or there exists *even* $n \in \mathbb{N}$ such that

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}\omega}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = \ldots = \frac{\mathrm{d}^{n-1}}{\mathrm{d}\omega^{n-1}}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = 0,\\ \frac{\mathrm{d}^{n}}{\mathrm{d}\omega^{n}}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} > 0\end{gathered} \tag{11}$$

(i.e. there is a local minimum of $\operatorname{Im}\{m(\mathrm{j}\omega)\}$ in $\omega = 0$),
or there is *odd* $n \ge 3 \in \mathbb{N}$ such that

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}\omega}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = \ldots = \frac{\mathrm{d}^{n-1}}{\mathrm{d}\omega^{n-1}}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = 0,\\ \frac{\mathrm{d}^{n}}{\mathrm{d}\omega^{n}}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} \ne 0,\ \frac{\mathrm{d}}{\mathrm{d}\omega}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=\Delta} > 0\end{gathered} \tag{12}$$

(i.e. there is a point of inflexion of $\operatorname{Im}\{m(\mathrm{j}\omega)\}$ in $\omega = 0$; however, the function is increasing in $\omega = \Delta$).

Analyze now the previous three conditions. First, relation (10) w.r.t. (5) reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\omega}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} &= 1 - a\vartheta\cos(\vartheta\omega) - kq\tau\cos(\tau\omega)\big|_{\omega=0}\\ &= 1 - a\vartheta - kq\tau > 0\end{aligned} \tag{13}$$

which gives $a\vartheta + kq\tau < 1$.

Second, condition (11) can be taken into account if

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = 0 \Leftrightarrow a\vartheta + kq\tau = 1 \tag{14}$$

The second derivation is

$$\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = a\vartheta^2\sin(\vartheta\omega) + kq\tau^2\sin(\tau\omega)\big|_{\omega=0} = 0 \tag{15}$$

Generally, any *even* n-th derivation reads

$$\begin{gathered}\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\operatorname{Im}\{m(\mathrm{j}\omega)\} = (-1)^{\frac{n}{2}-1}\left(a\vartheta^n\sin(\vartheta\omega) + kq\tau^n\sin(\tau\omega)\right)\\ \Rightarrow \frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = 0\end{gathered} \tag{16}$$

This implies that condition (11) can not be satisfied.

Third, assume that there exists a non-zero *odd* n-th, $n \ge 3$, derivation in $\omega = 0$

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} &= (-1)^{\frac{n-3}{2}}\left(a\vartheta^n\cos(\vartheta\omega) + kq\tau^n\cos(\tau\omega)\right)\Big|_{\omega=0}\\ &\overset{a\vartheta+kq\tau=1}{=} (-1)^{\frac{n-3}{2}}\left(a\vartheta^n\cos(\vartheta\omega) - (a\vartheta-1)\tau^{n-1}\cos(\tau\omega)\right)\Big|_{\omega=0}\\ &= (-1)^{\frac{n-3}{2}}\left(a\vartheta^n - (a\vartheta-1)\tau^{n-1}\right)\end{aligned} \tag{17}$$

Test the latter condition in (12), obviously

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\omega}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\substack{\omega=\Delta\\ a\vartheta+kq\tau=1}}\\ &= a\vartheta\left(\cos(\vartheta\Delta) - \cos(\tau\Delta)\right) + \cos(\tau\Delta) > 0\end{aligned} \tag{18}$$

since

$$\lim_{\Delta\to 0^+}\frac{\cos(\vartheta\omega)}{\cos(\tau\omega)} = 1 \tag{19}$$

Analogously to (10)-(12), one can easily verify that if the Mikhaylov plot passes through the fourth quadrant first, then function $\operatorname{Im}\{m(\mathrm{j}\omega)\}$ decreases in $\omega = 0$ and (9) holds.

(Sufficiency.) Consider condition (8) and verify that it satisfies (10) or (12), respectively. In the same way, formula (9) gives rise to

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\operatorname{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} < 0 \tag{20}$$

which induces the initial tendency of the Mikhaylov plot to go to the fourth quadrant. □

Lemma 4. If a, k, q are bounded, then $\operatorname{Re}\{m(\mathrm{j}\omega)\}$ is bounded for all $\omega > 0$.

Proof. Assume the following four various condition.

1) If $a > 0$ and $kq > 0$, then

$$-a - kq \le \operatorname{Re}\{m(\mathrm{j}\omega)\} = a\cos(\vartheta\omega) + kq\cos(\tau\omega) \le a + kq \tag{21}$$

2) If $a > 0$ and $kq < 0$, then

$$-a + kq \le \operatorname{Re}\{m(\mathrm{j}\omega)\} \le a - kq \tag{22}$$

3) If $a < 0$ and $kq > 0$, then

$$a - kq \le \operatorname{Re}\{m(\mathrm{j}\omega)\} \le -a + kq \tag{23}$$

4) If $a < 0$ and $kq < 0$, then

$$a + kq \le \operatorname{Re}\{m(\mathrm{j}\omega)\} \le -a - kq \tag{24}$$

It is possible to summarize and unify results (21) – (24) as

$$-(|a| + |kq|) \le \operatorname{Re}\{m(\mathrm{j}\omega)\} \le |a| + |kq| \tag{25}$$

□

Proposition 1. If (6) and (8) are satisfied simultaneously, then

$$a(\vartheta - \tau) \le 1 \tag{26}$$

Proof. Obviously,

$$a(\vartheta - \tau) \overset{kq>-a}{<} a\vartheta + kq\tau \le 1 \tag{27}$$

□

The preceding proposition also expresses that for a stable quasipolynomial (3) when the corresponding Mikhaylov plot passes the first quadrant as first, the condition (26) holds.

Proposition 2. If the following inequality holds

$$a(\vartheta - \tau) > 1 \tag{28}$$

then the corresponding Mikhaylov plot of a stable quasipolynomial (3) passes the fourth quadrant as first.

Proof. Lemma 2 states that (6) reads for stable quasipolynomial (3). Then

$$1 < a(\vartheta - \tau) \overset{kq>-a}{<} a\vartheta + kq\tau \tag{29}$$

which induces that the Mikhaylov plot goes to the fourth quadrant as first, due to Lemma 3.

Proposition 3. There always exists an intersection of the Mikhaylov curve of (3) with the imaginary axis.

Proof. The intersection exists if $\operatorname{Re}\{m(\mathrm{j}\omega)\} = 0$, i.e.

$$a\cos(\vartheta\omega) = -kq\cos(\tau\omega) \tag{30}$$

for some $\omega > 0$. Obviously, since $\vartheta > 0$, $\tau > 0$, there is $\omega > 0$ satisfying relation (30). □

The upper stability bound will now be found via some observations and a theorem. Due to highly complicated formulas (4) and (5) caused by goniometric functions, some numerical unproven observations compensate for exact analytic statements.

Definition 1. Let (6) holds. A *crossover frequency* ω_0 is an element of the set

$$\Omega_0 := \{\omega : \omega > 0, \operatorname{Re}\{m(\mathrm{j}\omega)\} = 0, \operatorname{Im}\{m(\mathrm{j}\omega)\} = 0\} \tag{31}$$

for some *crossover gain* q_0 and $a \ne 0, k, \tau, \vartheta > 0$. ■

A crossover frequency, hence, has to satisfy simultaneously these two identities

$$\begin{aligned} &a\cos(\vartheta\omega_0) + kq_0\cos(\tau\omega_0) = 0 \\ &\omega_0 - a\sin(\vartheta\omega_0) - kq_0\sin(\tau\omega_0) = 0 \end{aligned} \tag{32}$$

Relations (32) can also be expressed by transcendental equation

$$\omega_0\cos(\tau\omega_0) = a(\sin((\vartheta - \tau)\omega_0)) \tag{33}$$

Note that equation (33) is in the form suitable for utilization of numerical methods, i.e. some ratios of goniometric functions are not desirable for this purpose.

The crossover gain q_0 can be calculated from (32) as

$$q_0 = \frac{\omega_0 - a\sin(\vartheta\omega_0)}{k\sin(\tau\omega_0)} \tag{34}$$

Definition 2. Let (6) holds. The *critical frequency* ω_C is defined as

$$\omega_C := \min\left\{\omega : \omega \in \Omega_0, \underset{s=\omega\mathrm{j},\omega\in[0,\omega_C)}{\Delta\arg m(s)} = 0, \underset{s=\omega\mathrm{j},\omega\in[\omega_C,\infty)}{\Delta\arg m(s)} = \frac{\pi}{2}\right\} \tag{35}$$

for the corresponding *critical gain* q_C given by (34), where ω_C is placed instead of ω_0, and $a \ne 0, k, \tau, \vartheta > 0$. ■

Obviously, the critical frequency is the least crossover frequency for which the argument change is zero for $\omega \in [0, \omega_C)$ and consequently it equals $\pi/2$ for $\omega \in [\omega_C, \infty)$. The quasipolynomial is then on the stability border for q_C, which has to satisfy the necessary stability condition (6). There can hence exist some crossover frequencies less then the critical one which do not mean the stability border.

The difference between the crossover and the critical frequencies is clarified in Fig. 1. Whereas (a) displays the critical frequency, position (b) shows the crossover one only, because the phase shift of $m(s)$ for ω_0 is $-3\pi/2$ and there is not another ω_0.

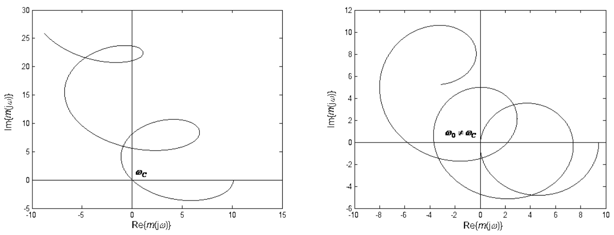

Fig. 1 The difference between ω_C (a) and $\omega_0 \neq \omega_C$ (b)

Observation 1. Let $q = q_C$, then the Mikhaylov plot of (3) circumscribes curves in the clockwise direction around the center of the rotation (like a “whirligig”). Moreover, if (8) holds, then the Mikhaylov plot of (5) initially moves to the first quadrant (as proved in Lemma 3) followed by the fourth quadrant for some frequencies $\omega > 0$. It means that although relation (8) quarantines that the plot tends to move to the first quadrant for $\omega = 0$, it immediately passes over the positive real axis to the fourth quadrant anyway. The situation is displayed in Fig. 2.

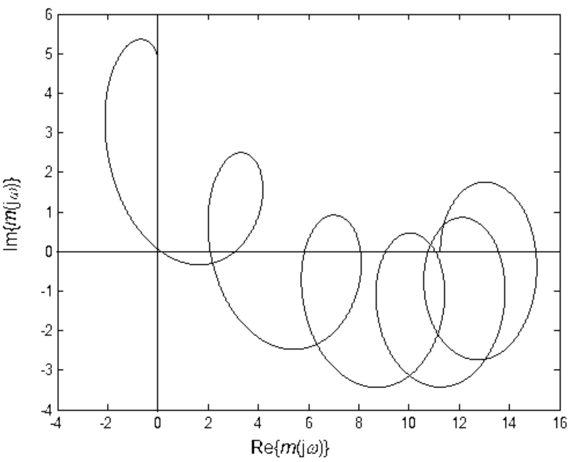

Fig. 2 Explanation of Observation 1

Remark 1. In [19] and [20] is proved a lemma which states that the spectrum of a general retarded quasipolynomial is continuous with respect to continuous changes of all its parameters. This fact implies that the Mikhaylov plot of an appropriate quasipolynomial is continuous in both axes with respect to these parameters’ changes, and viceversa.

Theorem 1. If $\sin(\tau\omega_C) > 0$, then quasipolynomial (3) is stable if and only if

$$\frac{-a}{k} < q < \frac{\omega_C - a\sin(\vartheta\omega_C)}{k\sin(\tau\omega_C)} \tag{36}$$

Contrariwise, if $\sin(\tau\omega_C) < 0$, then quasipolynomial (5) is stable if and only if

$$q > \frac{\omega_C - a\sin(\vartheta\omega_C)}{k\sin(\tau\omega_C)} \geq \frac{-a}{k} \tag{37}$$

where ω_C is the critical frequency.

Proof. (Necessity.) The Mikhaylov curve of stable quasipolynomial (3) starts on the positive real axis, and thus the left-hand side of (36) and the right-hand one of (37) hold, as proved in Lemma 2. Lemma 3 states the condition (8) guaranties that the initial change of the Mikhaylov curve in the imaginary axis is positive. i.e. the curve tends to move to the first quadrant for $\omega = 0$; however, according to Observation 1, it immediately moves to the fourth quadrant. If (9) is satisfied, the curve passes through the fourth quadrant already for an infinitesimally small ω. The critical (marginal) case is characterized by ω_C and q_C where the curve crosses the origin of the complex plane and a small change of *q* would cause the quasipolynomial stability, i.e. the overall phase change would be $\pi/2$, see Remark 1. The limit stable case thus obviously means that either when $\text{Im}\{m(\text{j}\omega_C)\} = 0$, the real part must satisfy $\text{Re}\{m(\text{j}\omega_C)\} > 0$ for some *q*, or $\text{Re}\{m(\text{j}\omega_C)\} = 0$ and $\text{Im}\{m(\text{j}\omega_C)\} > 0$. However, the former condition has one important inconvenience described in the following paragraph.

When $\vartheta = \tau$, the critical case $\text{Re}\{m(\text{j}\omega_C)\} = 0$, $q = q_C$, reads

$$(a + kq_C)\cos(\vartheta\omega_C) = 0 \tag{38}$$

Since for $\omega_C \neq 0$ and a Mikhaylov plot starting on the positive real axis, $a < -kq_C$, i.e. $\cos(\vartheta\omega_C) = 0$ (Lemma 2), then it is not possible to satisfy $\text{Re}\{m(\text{j}\omega_C)\} > 0$ for any *q*.

Therefore take the latter limit stable stability condition and apply simple calculations on (4) and (5) using (33) when $\sin(\tau\omega_C) > 0$, which yields the upper bound in (36).

Otherwise, if $\sin(\tau\omega_C) < 0$, the calculations result in the left-hand side inequality in (37). Evidently, values of *q* less then the necessary stability condition (6) can be discarded.

A case when $\sin(\tau\omega_C) = 0$ would mean that *q* reaches infinity which is not physically possible.

(Sufficiency.) Consider inequality (36) first. The lower bound means that the Mikhaylov curve initiates on the positive real axis, see Lemma 2. Lemma 3 verifies that the curve reaches infinity in the imaginary axis for $\omega \to \infty$, and Lemma 4 states that it is bounded in the real axis. Moreover, if (8) holds the Mikhaylov curve tends to move to the first quadrant and, consequently, to the fourth quadrant for $\omega = 0$; otherwise, it moves to the fourth quadrant for $\omega = \Delta$ when (9) is satisfied. For the quasipolynomial stability, expressed by the overall phase shift $\pi/2$, it is now sufficient to show that the curve does not encircle the origin of the complex plane in the clockwise direction.

Let the critical stability case be expressed by ω_C and q_C and apply the upper bound in (36) on (4) and (5) together with $\sin(\tau\omega_C) > 0$. Hence, the following conditions are satisfied simultaneously for a particular q: $q < q_C$, $\mathrm{Re}\{m(\mathrm{j}\omega_C)\} = 0$, $\mathrm{Im}\{m(\mathrm{j}\omega_C)\} > 0$. It means that the imaginary axis is crossed in the positive semi-axis first on the critical frequency and thus, with respect to Remark 1, the origin is encircled in the anti-clockwise direction with the overall phase shift $\pi/2$.

As second, the right-hand side of (37) expresses the necessary stability condition (6) which guaranties that the Mikhaylov curve starts on the positive real axis. Assume now that the left-hand side in the inequality holds. Similarly as in the previous paragraph, it is sufficient to prove that the curve encircles the origin in the complex plane in the anti-clockwise direction. Indeed, if $\sin(\tau\omega_C) < 0$, one can verify that the inequality agrees with the statement that $\mathrm{Re}\{m(\mathrm{j}\omega_C)\} = 0$ and $\mathrm{Im}\{m(\mathrm{j}\omega_C)\} > 0$ which gives rise to the stability of quasipolynomial (3). □

Remark 2. It is not always easy to check, mainly without displaying the Mikhaylov plot, whether a crossover frequency calculated by (33) is critical and thus whether it can be used in Theorem 1. Sometimes only the sufficient stability condition is searched for; in this case, it is possible to use the finding based on Observation 1. Clearly, if the Mikhaylov plot for q does not crosses the negative imaginary semi-axis, then $\omega_0 = \omega_C$ (if there is no less one). This gives rise to the sufficient condition for ω_C and, consequently, for the quasipolynomial stability according to (36) and (37).

Remark 3. Definition 2 and Theorem 1 suggest situations when the quasipolynomial stabilization by the suitable choice of q is not possible. These are two unpleasant possibilities:

1) If ω_C does not exist. Thus, although Ω_0 is non-empty set, it may not contain $\omega_0 = \omega_C$.

2) If q could not satisfy (36), i.e. if

$$\frac{\omega_C - a\sin(\vartheta\omega_C)}{k\sin(\tau\omega_C)} \le \frac{-a}{k} \tag{39}$$

This case is, however, not very likable since the continuity of the Mikhaylov curve w.r.t q supposes that there is a stabilizing q in the neighborhood of the marginal stability case $q = q_C$.

Observation 2. Numerical experiments showed that if $\sin(\tau\omega_0) < 0$, then $\omega_0 \neq \omega_C$, which might render condition (37) useless.

Observation 3. Assume that q does satisfy neither (36) nor (37); however, let (3) be stabilizable. Then the Mikhaylov plot begins either on the positive real semi-axis but the overall phase change differs from $\pi/2$ or it starts on the negative one for $\omega = 0$. In the former case, due to Observation 1, Lemma 1 and Lemma 3, the overall phase shift is

$$\Delta \arg_{s=\omega j, \omega\in[0,\infty)} m(s) = -\frac{3\pi}{2} - 2k\pi,\ k \in | \tag{40}$$

The former case yields

$$\Delta \arg_{s=\omega j, \omega\in[0,\infty)} m(s) = -\frac{\pi}{2} - 2k\pi,\ k \in | \tag{41}$$

Observation 3 is usable mainly when applying the Nyquist criterion in order to study closed loop stability for an unstable delayed controlled system with denominator (3) and a proportional controller q.

The presented approach is also usable for the analysis when respecting specific features of this class of quasipolynomials.

V. Simulation Examples

The chapter introduces examples which demonstrate the application of obtained results. All these are supported by simulations of corresponding Mikhaylov curves and/or control responses.

A. Unstabilizable Quasipolynomial

This example proposes a demonstration of Remark 3. Consider quasipolynomial (3) with $a = -5$, $\tau = 0.2$, $\vartheta = 1$, $k = 1$, which gives the following set of crossover frequencies according to (33): $\Omega_0 = \{4.663, 7.855, 10.244, 23.562, 39.27, ...\}$, giving rise to crossover gains calculated from (34) as $q_0 \in \{-0.4112, 12.855, 7.423, -18.562, 44.27, ...\}$. One can verify by drawing the appropriate Mikhaylov plot that no $\omega_0 \in \Omega_0$ is the critical frequency. For example, a pair (23.562, -18.562) results in the Mikhaylov plot pictured in Fig. 3. It is clear that $\Delta \arg_{s=\omega j, \omega\in[0,\omega_0)} m(s) = \pi$; moreover, the necessary condition (6) does not hold.

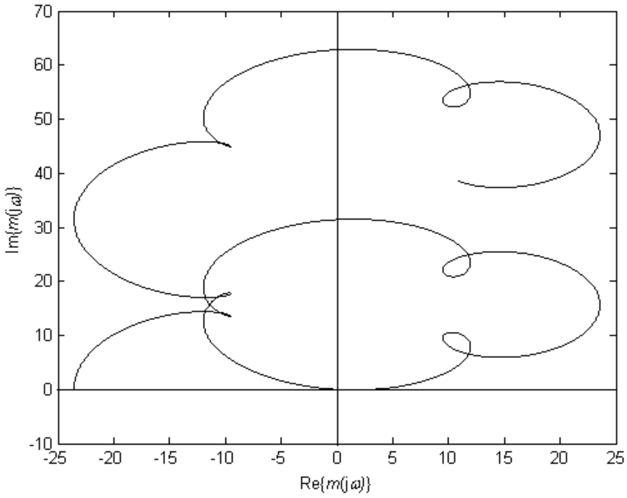

Fig. 3 Mikhaylov plot of the unstabilizable quasipolynomial – Example A

B. Coprime Factorization

A wide class of delay systems can be expressed in the form $G(s) = B(s)/A(s)$ where A and B are coprime. Then the stabilizing Diophantine equation reads $AP + BQ = 1$ where $R(s) = Q(s)/P(s)$ is the controller transfer function. P and Q then be parametrized in order to obtain desired control requirements and performance, see e.g. [21]. In case of unstable delay systems, the suitable coprime factorization is a rather difficult to find. One possibility is to set A and B so that the Diophantine equation has a simple solution, say a proper controller $P = 1$, $Q = q$.

For instance, the unstable plant

$$G(s) = \frac{\exp(-1.1s)}{s - 5\exp(-s)} \tag{42}$$

can be (coprimely) factorized as

$$G(s) = \frac{\dfrac{\exp(-1.1s)}{s - 5\exp(-s) + q\exp(-1.1s)}}{\dfrac{s - 5\exp(-s)}{s - 5\exp(-s) + q\exp(-1.1s)}} \tag{43}$$

The task is to find q so that A and B are analytic in the closed right-half plane, in particular, the denominator quasipolynomial is stable. Equation (33) enables to find the critical frequency $\omega_C = 0.953$ which yields the critical gain $q_C = 5.803$. One can verify that $\sin(\tau\omega_C) > 0$; hence, Theorem 1 results in the stabilizing interval $5 < q < 5.803$. Let $q = 5.4$, then the corresponding stabilized Mikhaylov plot is displayed in Fig. 4.

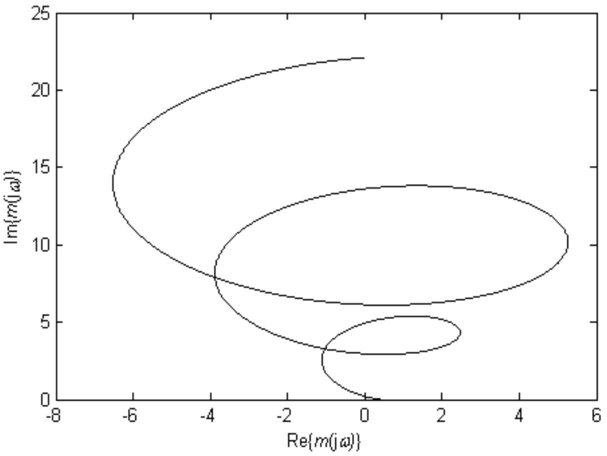

Fig. 4 Mikhaylov plot of the stabilized quasipolynomial, $\omega \in [0,15]$ - Example B

The corresponding feedback control response when using proportional controller $q = 5.4$ is in Fig. 5. Obviously, the feedback system is stabilized.

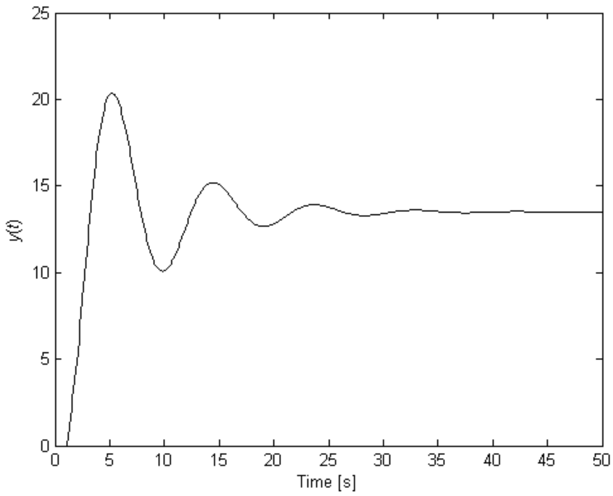

Fig. 5 Proportionally stabilized feedback response – Example B

C. Relay Autotuning

The aim of this example is to propose stabilization of a conventional high order system using a simple time delay model. Model parameters identification is based on the relay feedback test where a relay (or non-linear element in general) is introduced instead of a controller in the feedback loop. After transients perpetual oscillations appear, the amplitude and the frequency of which enables to estimate model parameters, for details the reader is referred to e.g. [22], [23].

Suppose a tenth order plant

$$G(s) = \frac{1}{(2s+1)^{10}} \tag{44}$$

which can be identified (approximated) by model

$$G_M(s) = \frac{0.065\exp(-15.3s)}{s + 0.065\exp(-6.7s)} \tag{45}$$

see [22], [23]. Again, the objective is not to preset here the whole controller design but to stabilize (44) using a proportional feedback with model (45). It is obvious that calculation gain q for model (45) brings about error in stabilizing the nominal plant (44). Therefore, one has to set q conservatively enough, for instance, according to desired (sufficiently high) gain margin.

The closed-loop characteristic quasipolynomial reads

$$m(s) = s + 0.065\exp(-6.7s) + q0.065\exp(-15.3s) \tag{46}$$

Using (33) and (34), one can find $\omega_C = 0.133$, $q_C = 1.419$, $\sin(\tau\omega_C) = 0.894$, which gives rise to stabilizing interval $-1 < q < 1.419$. Set $q = 0.473$ which means the gain margin equal to 3 (with respect to the upper bound). The Mikhaylov curve of (46) for $q = 0.473$ is displayed in Fig. 6 and the comparison of stabilized feedback responses when control model (45) and the nominal plant (44) by q introduces Fig. 7.

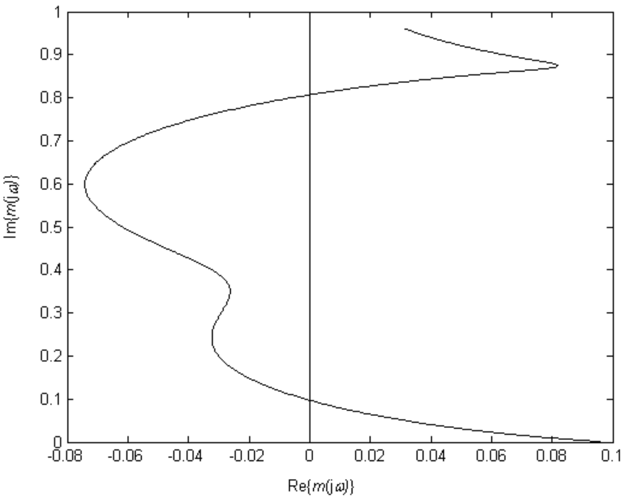

Fig. 6 Mikhaylov plot of the stabilized characteristic quasipolynomial (46)

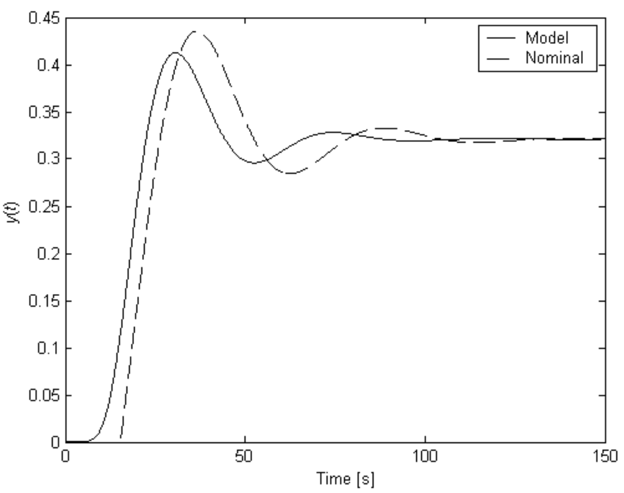

Fig. 7 Feedback response when control the nominal plant (44) and the model (45) by a proportional controller $q = 0.473$

VI. Conclusion

The issue of stabilization of a selected retarded quasipolynomial with two fixed independent delays by a non-delay real parameter has been introduced and studied in this contribution. The aim has been to derive acceptable upper and lower bounds for a non-delay real parameter so that all quasipolynomial zeros are located in the open left-half complex plane. The analysis has been based on the conventional argument principle, i.e. the Mikhaylov stability criterion, which holds also for retarded quasipolynomials in order to keep the desired shape of the Mikhaylov curve. Presented lemmas and theorems have been proved, except some hardly provable observations. The utilization of the obtained results has been then demonstrated on simulation examples solving the coprime factorization of an unstable delayed plant and the feedback stabilization of a high order system by a relay-feedback identified model with delays, respectively. All examples have been supported by simulation examples of corresponding Mikhaylov plots and feedback responses.

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