An analysis of a problem related to decision-making applying computational geometry techniques

J. Rodrigo, M.D. López, S. Lantarón, R. Caro

***Abstract*—**This paper introduces a problem related to decision-making and the shaping of political strategies in the course of one term of office. First it is assumed than one of the parties, say the one in the government, does not modify its position and then the other one, say the opposition, searches for the best proposal within a circular neighborhood of flexibility. Next it is assumed that the government and the opposition shape their proposals for action on two issues that are relevant for the citizens and a variable component is considered regarding both the relevance of the issues to be dealt with and the strategies that the parties are presumed to adopt. This component is reflected in the consideration of elliptic neighborhoods of flexibility for both parties. In addition, it is considered that the process is dynamic because the proposals are intended to be modified taking into account the other party's foreseen action in order to get the maximum number of votes. The contribution of this article lies in this approach, as well as in its taking into account variable components. The problem is dealt with from a geometric point of view, and a search algorithm to find optimum strategies is developed.

***Keywords*—** Computational geometry, Operations research, Search algorithms.

I. Introduction

Models of point location have been studied in various fields such as industrial organization, image processing, movement of robots, location of tourist elements and location-aided routing protocol [6], [25], [26]. Optimum strategy models have also been proposed for locating parties in political economy research [14], [15]. Most of these models consider the population as a continuum [3]. Novel elements of this research include working with a discrete population, applying techniques and results from computational geometry as adapted to the problem, and considering weighted distances and neighborhoods as well as uncertainty parameters (as an example of how to introduce uncertainty parameters by means of techniques of the fuzzy logic, see [13], [21])

This research tries to solve a particular kind of political economy problem [18], [20] using geometric tools. The points of a plane, called here the policy plane, represent various political options with respect to two important topics. It is assumed that the distances between points can be used to represent the affinity of the citizens to the policies represented by the points [16], [17], [13], [1], [2]. In a first setting it is considered the Euclidean distance as the work distance. In a second approximation it is considered that the relevance of the topics to be dealt with need not be the same and may not even be perfectly determined, so it is proposed to attach a weighting parameter to each topic. This parameter is introduced by means of weighted Euclidean distance.

The aim of this study is to find optimum strategies to be followed by the two majority parties of a country (government and opposition), while still allowing them to fine-tune their proposals to a certain extent. In addition, the process is sequential, because the proposals are intended to be modified taking into account the other party's foreseen action. The contribution of this research lies in these approaches, as well as in the consideration of subjective components.

The article is structured as follows: the model representing the two cases aforementioned and the search strategies to find optimum positions are developed in Sections II and III. An example of the second case is developed in Section IV.

II. First Case: Usual Euclidean Distance

A. The model

At a first stage to establish the model it is necessary to define the political space. Because politics is highly complex, it is necessary to simplify the model by restricting the analysis to a small number of representative issues. It is known that societies develop low-dimensional mental models of political decisions, so this restriction is appropriate for this study. In spite of this, the one-dimensional models of the space where the preferences of voters are distributed, are excessively simplistic, since representing all the political areas of a party

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by a single parameter does not illustrate the reality. Indeed, it is possible to extrapolate voters' positions on certain issues from their positions on other issues, and therefore a policy space of two or three dimensions should suffice [4].

Concretely, this research focuses on two topics which are relevant to the citizenry. A policy plane is defined by these two currently relevant issues, with the two political parties represented by points t_1 and t_2 and the location of the n voters represented by points $v_1,\ldots,v_n$ [20]. By considering the appropriate perpendicular bisector at the defined distance, it is possible to calculate the number of voters that will choose each party by determining their proximity or affinity to its policy. It is accepted that each party can change its policy within a certain neighborhood with the objective of obtaining the greatest possible number of voters. First it is considered the case in which one of the parties does not change its position and the other one moves. Next, the case of movement for the two parties is studied. In both cases, the objective is to find the optimum location for one party in this neighborhood, that is to say, the location that ensures the greatest number of supporters. In the second case, by assuming that this party will choose one of these positions, it is intended to determine the other party's reply, that is, the optimum strategy to prepare for the other party's possible proposal. This second case will be developed in section III.

The essence of this problem can be seen as a discrete version of the Voronoi game. In this game, two players locate several points in the plane in an effort to claim the greatest possible area [11], [9], [19], [7]. In the present problem, two points are located so as to earn the greatest possible number of points (voters) instead of the greatest area.

The consideration of restrictions on the movement of the parties represents an alternative to the analysis presented by [4], in which a continuous model with more than two parties but without restriction of neighborhoods is considered, or that of [22], in which a simplified model on a discrete rather than a continuous real space is discussed.

Let us now formalize the model by representing the political stance adopted by the government and the opposition on each of the two topics as points $t_1=\left(t_1^1,t_2^1\right)$ and $t_2=\left(t_1^2,t_2^2\right)$ and letting $v_i=\left(v_1^i,v_2^i\right)$, i=1,…, n, be the coordinates which represent the preferences on these topics for the n voters of a certain population.

In the game presented here, the utility function of a policy t_j for each voter v_i is defined as:

$$\gamma\left(t_j,v_i\right)=-d\left(t_j,v_i\right)^2,$$

where $d(t_j,v_i)$ represents the Euclidean distance between political position t_j and voter v_i:

$$d\left(t_j,v_i\right)=\sqrt{(v_1^i-t_1{}^j)^2+(v_2^i-t_2{}^j)^2}$$

The payoff functions in the game are given by:

$\Pi^1\left(t_1,t_2\right)=$ number of points v_i such that $d\left(v_i,t_1\right)\le d\left(v_i,t_2\right)$

$\Pi^2\left(t_1,t_2\right)=$ number of points v_i such that $d\left(v_i,t_1\right)> d\left(v_i,t_2\right)=$ $n-\Pi^1\left(t_1,t_2\right)$

if $t_1\ne t_2$

The set of voters siding with the first party will be the subset of voters that are closer to position t_1 than to the position of the second party. To locate these voters, we use the bisector of the line between t_1 and t_2, which is given by:

$$\{(x,y)\in\Re^2\ /(x-t_1^1)^2+(y-t_2^1)^2=(x-t_1^2)^2+(y-t_2^2)^2\}$$

The flexibility neighborhoods must also be stated. They are defined as follows:

Definition 1. Starting with the initial position of one of the parties $\left(x_0^i,y_0^i\right)$, i=1, 2, its flexibility neighborhood is defined as:

$N_i=\{(x,y)\in\Re^2\ /(x-x_0^i)^2+(y-y_0^i)^2\le R^2\ \}$. This is the disk centered on $\left(x_0^i,y_0^i\right)$ with radius R

B. *Algorithm of resolution*

We look for the best situation for one of the parties within its neighbourhood, the one that approaches it to the greatest number of voters.

Proposition 1: An optimum position t_1 for the first party, given a position t_2 for the second party, is found on the boundary of N_1 in the arc of the disk located between the two points p', p'' of the tangent lines from t_2 to the disk (the part of N_1 which is visible from t_2), as shown in Figure 1

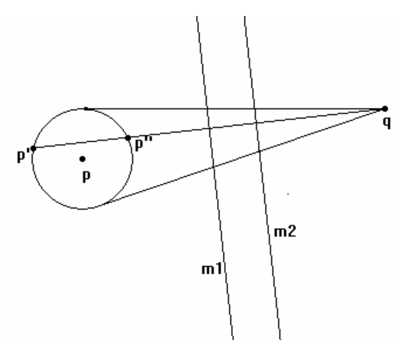

Fig. 1: The arc delimits an area with maximum votes for t_1.

We develop the algorithm that permits us to obtain the region from the visible part from t_2 that procures the maximum number of voters for the first party.

Algorithm:

Let C denote the arc defined in proposition 1, which is the visible part from t_2 of N_1.

For all points $\left(v_1^i, v_2^i\right)$, compute D_i, a disk centered at $\left(v_1^i, v_2^i\right)$ whose boundary contains t_2. Compute the intersection of D_i with C. Let C_i be such intersection.

Compute the points on C that are contained in the largest number of arcs C_i. This can be done by sorting the begin and endpoints of the arcs C_i´s, and then scanning this sorted list. (Figure 2)

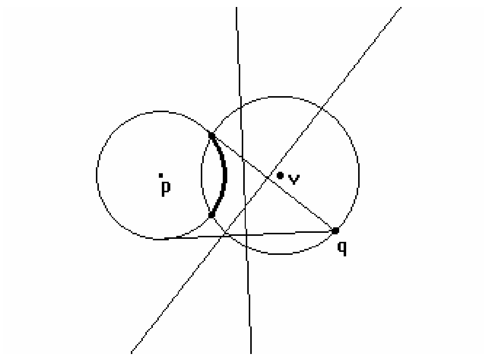

Fig. 2: Captation zone for *p* of the point *v*

The complexity of the algorithm in the worst case is *O(n log n)*. In fact, first part of the algorithm is linear in the number of points, because the operation for each point requires constant time. The complexity is dominated by the last step, which requires *O(n log n)* for the sorting of the extreme points of the arcs. The final scanning can be done in linear time thus obtaining not only the best solution, but the number of voters for this solution. For that it is only necessary to add a counter to the scanning process indicating the number of overlapping arcs.

III. SECOND CASE: GENERALITION TO WEIGTHED EUCLIDEAN DISTANCE

A. Modification of the model

A generalization of the model presented in section II must now be defined to fit the reality in the following two ways:

1) Politics is a dynamic process in which each party reacts to proposals presented by its adversary

2) The relevance of the topics to be dealt with need not be the same and may not even be perfectly determined.

Given the proposals made by a country's majority parties on particular topics, the governing party searches for an optimum strategy that brings it closer to the maximum number of citizens within a neighborhood that represents its ideological flexibility. On the other hand, the opposition party is ready for this reaction by the government and prepares a different strategy in its flexibility neighborhood with a view to finding the best reply to any of the possible optimum stances taken by the government. Thus, the parties' positions evolve within their flexibility neighborhoods. These kinds of sequential games have already been presented in the continuous case by [20]. In this paper a different approach to query 1) is showed in subsections *C* and *D*

With respect to the second query, it is proposed to attach a weighting parameter to each topic. This parameter is introduced by means of weighted Euclidean distance:

Definition 2. The weighted Euclidean distance between political position t_j and voter v_i:

$$d\left(t_j, v_i\right) = \sqrt{\alpha(v_1^i - t_1{}^j)^2 + (1-\alpha)(v_2^i - t_2{}^j)^2}\ .$$

Parameter $\alpha \in (0,1)$ indicates the importance of each of the topics to be dealt with.

The perpendicular bisector and flexibility neighborhoods will also be affected by the weighting assigned to each topic. They are defined as follows:

Definition 3. The bisector of the line between the positions t_1 and t_2, is given by:

$$\{(x,y) \in \Re^2 / \alpha(x - t_1^1)^2 + (1-\alpha)(y - t_2^1)^2 =$$
$$\alpha(x - t_1^2)^2 + (1-\alpha)(y - t_2^2)^2\}$$

Definition 4. Starting with the initial position of each party $\left(x_0^i, y_0^i\right)$, i=1,2, its flexibility neighborhood is defined as:

$N_i = \{(x,y) \in \Re^2 / \alpha(x - x_0^i)^2 + (1-\alpha)(y - y_0^i)^2 \le R_i^2\ \}$, where R_i, i=1,2, represents the degree of flexibility of each party, that is, N_i is the inner region of the ellipse centered on the initial position taken by the party, whose semi-axes are $\frac{Ri}{\sqrt{\alpha}}$ and $\frac{R_i}{\sqrt{1-\alpha}}$.

It can be seen that the lower the relevance of one of the topics to be dealt with, say the first (α closer to zero), the greater the flexibility that is granted to the party to handle it. This behavior is logical because the parties must stay closer to their initial ideological stances when dealing with a highly relevant topic.

Throughout the section, it is assumed that the neighborhoods of both parties have an empty intersection, that is, $N_1 \cap N_2 = \varnothing$.

B. Classifying voters by region

The flexibility neighborhood of each party ensures it a certain number of voters, no matter which location its adversary occupies within its neighborhood. Therefore, the points in the voter set can be classified into three regions:

- Sure voters for the first party.
- Sure voters for the second party.
- Undecided voters who can be captured by the party that places itself in the appropriate area of each voter's neighborhood. These voters are decisive, and the campaign and political proposals must be oriented towards them.

Certain regions can then be determined and interpreted as described following:

a) Points that the first party always captures:

Points (v_1^i, v_2^i) that belong to the set:

$$\{(x,y)/ \max\{d[(x,y),(c_1,c_2): with\,(c_1,c_2)\in N_1]\} \le\ d[(x,y),N_2]\}.$$

The boundary of this set is:

$$\sqrt{\alpha(x-x_0^2)^2+(1-\alpha)(y-y_0^2)^2}-\sqrt{\alpha(x-x_0^1)^2+(1-\alpha)(y-y_0^1)^2}$$
$$=R_1+R_2$$

The sure voters for the first party are those located in the region bounded by this curve where the party is located.

b) Points that the second party always captures:

Points (v_1^i, v_2^i) that belong to the set:

$$\{(x,y)/ \max\{d[(x,y),(c_1,c_2): with\,(c_1,c_2)\in N_2]\} <\ d[(x,y),N_1]\}.$$

The boundary of the set is:

$$\sqrt{\alpha(x-x_0^1)^2+(1-\alpha)(y-y_0^1)^2}-\sqrt{\alpha(x-x_0^2)^2+(1-\alpha)(y-y_0^2)^2}$$
$$=R_1+R_2$$

The sure voters for the second party are those located in the region bounded by this curve where the party is located.

The undecided voters are those located between the two curves determined above.

The curves bounding the regions that classify the voters keep changing, and so do the regions captured, depending on the weights.

Figure 3 illustrates these regions in the particular case in which both parties' initial positions agree on one of the topics. In this example, $(x_0^1, y_0^1)=(0,0)$, $(x_0^2, y_0^2)=(2,0)$, $R_1=R_2=1/2$, and α varies from 0.2 to 0.9 in steps of 0.1. In these cases, one of the topics (here, the second topic) is not relevant because both parties initially agree on it. In this case, a greater weight and consequently a greater relevance for the first topic means that the region of sure voters for each party increases and contains the regions corresponding to lower weights. Thus, the number of undecided voters decreases depending on the importance of the topic being considered.

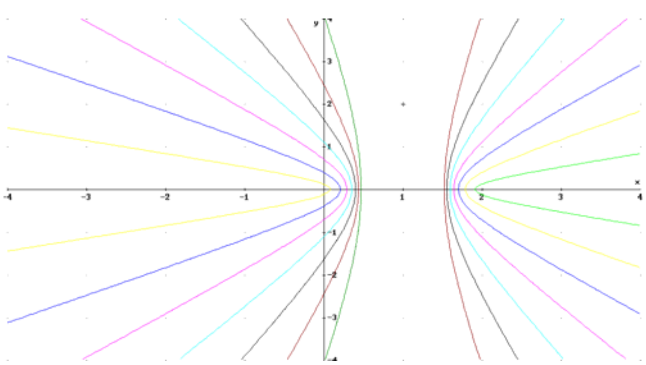

Fig. 3: Regions for capturing the maximum number of voters when the parties agree on one of the policies they offer

C. Search for an optimum position for the governing party

The following proposition makes it possible to search for an optimum position for the first party:

Proposition 2: An optimum position t_1 for the first party, given a position t_2 for the second party, is found on the boundary of N_1 in the arc of the ellipse located between the two points p', p'' of the tangent lines from t_2 to the ellipse (the part of N_1 which is visible from t_2), as shown in Figure 4.

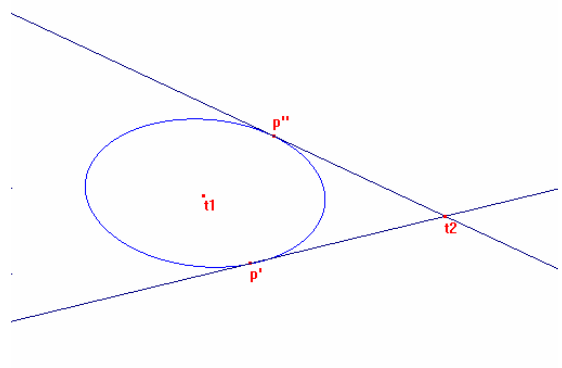

Fig. 4: The arc delimits an area with maximum votes for t_1.

Inside the arc defined in Proposition 2, let *A* be the region which captures the maximum number of votes for the first party. The procedure to be followed to determine *A* is based on calculating the area of maximum intersection between this arc and the sets $\{(x,y)\in\Re^2 / \alpha(x-v_1^i)^2+(1-\alpha)(y-v_2^i)^2 \le \alpha(t_1^2-v_1^i)^2+(1-\alpha)(t_2^2-v_2^i)^2\}$ with i=1,...,n, as shown in Figure 5.

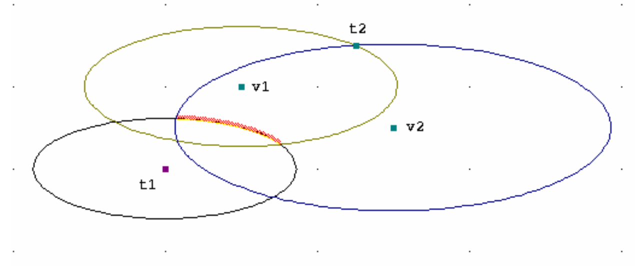

Fig. 5 The marked arc represents the area on the border of N_1 with maximum votes for the first party given points v_1, v_2.

- Algorithm for the calculation of region *A*

The algorithm for obtaining *A* results from a generalization of the algorithm presented in section II to the case of weighted Euclidean distances. The complexity of the algorithm in the worst case is therefore *O(n log n)* [6]. The procedure is detailed in Appendix A.

D. Reply strategies for the opposition

The opposition party realizes that, taking its initial position at a starting point, the government will choose a position, located in region *A*, which guarantees it the maximum number of supporters. Now the opposition must prepare an appropriate reply strategy. To this end, it is assumed that the opposition party will follow the most conservative stance; that is, it must find the optimum location, the one that ensures it the maximum number of followers, whatever the position of the first party in this optimum region. Proposition 3 determines this optimum location under this assumption.

Proposition 3: Within N_2, let B be the region which captures the maximum number of votes for the second party, assuming that the first party is located in A. Region B is calculated as the maximum intersection in N_2 of the neighborhoods centered on the voters with a radius equal to the minimum distance between these voters and A:

$\{(x,y)\in\Re^2 / \alpha(x-v_1^i)^2+(1-\alpha)(y-v_2^i)^2 \le (d(v_i,A))^2\}$, with i=1,...,n.

Remark: The distance from an external point to a neighborhood is attained in the intersection of the segment that joins the point with the center of the neighborhood and the border of this neighborhood. Thus, if the segment that joins v_i with the center of N_1 intersects A (part of the border of the neighborhood), the distance from v_i to A is attained at this intersection point. Otherwise, it is attained at one of the extreme points of A.

- Algorithm for the calculation of region B

The following discussion develops the algorithm to determine the region B that provides the maximum number of voters for the second party for any locations of the first party in A. The algorithm is based on the following idea:

For all points (v_1^i, v_2^i), compute D_i, an elliptical neighborhood centered at (v_1^i, v_2^i) and with a radius equal to the distance from (v_1^i, v_2^i) to A. Compute the maximum intersection of D_i, i=1,...n, that intersects with N_2. One way to determine this maximum intersection is the algorithm developed by the authors, for which the procedure is detailed in Appendix A.

This algorithm consists of applying n times the algorithm for calculating region A. Given that the latter has complexity $O(n\log n)$, the complexity of the whole algorithm is $O(n^2\log n)$. It can be seen that the procedure presented here has the same complexity as that developed by [5] for calculating the maximum intersection of a set of circles, thus representing an alternative to that algorithm in the case of ellipses. In any case, there exists an algorithm that calculates the region of maximum intersection of a circle arrangement slightly faster [10], [21]. In spite of this, the algorithm developed here is easier to implement, so it is preferable for practical purposes

IV. Simulation of an Example from the National Politics of Spain

The game developed here has been simulated using an example of political competition in Spain. Because the game is two-dimensional, it was necessary to define a two-dimensional political space, and so two relevant issues had to be chosen on which the parties could adopt policies. In this respect, it was decided that education and health were two issues that nowadays concern the citizenry in Spain (a deeper explanation of how to construct a political space is given in [12]).

To find the best positions for the parties in this simulation, the algorithms described in section III were implemented in the C programming language.

Following the ideas enunciated in Section III, the following inputs were provided for the implementation:

- The political plane was determined using the percentages of expenses committed to education and health as derived from the Consolidated General Government Budget of Spain (1997–2006) [8]. These quantities were extracted from the expense statements (chapters 1 to 8).
- The initial policies chosen by the first party (PSOE: Partido Socialista Obrero Español) and the second party (PP: Partido Popular) were determined using the mean percentages of expenditures dedicated to these two policies, as calculated from the total mean expenditure during two years of PSOE party government (2005, 2006) and eight years of PP party government (1997-2004): $\left(x_0^1, y_0^1\right)$=(0.6,1.4), $\left(x_0^2, y_0^2\right)$=(1.6,8.9).
- The radius of the neighborhoods of political flexibility for the parties: R_1, R_2.
- The parameter α .
- The 2276 voters and their positions v_i, i=1,...,2276. The location of the voters on the political plane was simulated according to the ideological spectrum of Spain.

For the initial policies used in this example, it was found that the voting intentions would give the victory to the PSOE (1277 voters) over the PP (999 voters).

After applying the algorithm to these inputs, the outputs obtained are:

- Region A.
- Numbers of voters obtained by the PSOE party by locating in A.
- Region B.
- Numbers of voters obtained by the PP party by locating in B.

Twelve simulation cases were developed by considering three values for the parameter α : $(\frac{1}{3},\frac{1}{2},\frac{2}{3})$ and four pairs of degrees of flexibility: (R_1=3, R_2=1), (R_1=1, R_2=3), (R_1=1, R_2=1), (R_1=2, R_2=2). It should be noted that the greater the value of α , the more important is the first issue. Moreover, if the flexibility of a party increases, then it can capture more voters.

The optimum regions A, B and the percentages of votes obtained for each party in each case are presented in Table 1 (in Appendix B) and Figures 4–8. The results obtained using this example show that varying the degree of flexibility of party policies and the parameter that weights the relevance of the issues significantly affects the maximum number of voters that each party can capture.

Figure 6 shows that despite the greater degree of flexibility for the government party, the increment of voters captured when this party chooses a position is less than the increment obtained by the opposition party when it chooses a response if the first issue becomes more relevant (that is to say, if the value of α is increased).

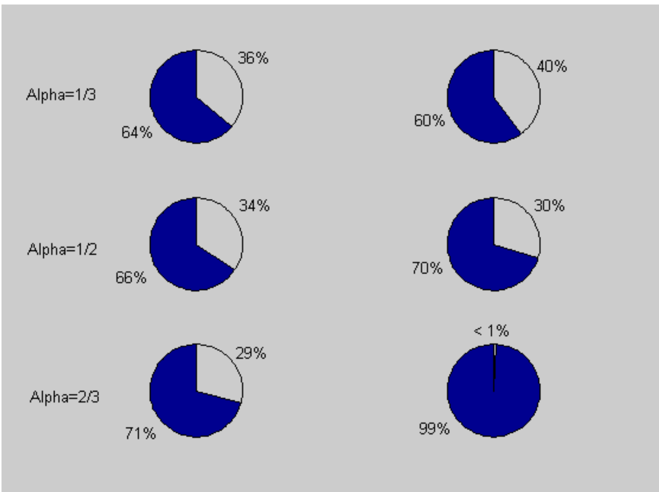

Fig. 6 Percentage increments in vote capture for R_1=3, R_2=1: the first column represents the percentage increments in vote capture when the government party chooses a position, and the second column those obtained when the opposition responds. The shaded region represents the party that changes position.

Figure 7 shows that when the degree of flexibility of the government party is less than that of the opposition party, the increment in voters when the government party chooses a policy is insensitive to the weight given to the issues, but when the opposition party responds, the number of voters it captures increases with an increase in the degree of relevance for the first issue.

Finally, as Figures 8 and 9 indicate, when the radii are equal for the two parties, the larger the radii, the greater is the increment in the number of voters captured for larger α , no matter which party chooses a position. Similarly to the previous cases, the responding party obtains an increment of voters that is greater than that obtained by the first party that chooses a position.

As a general conclusion, the party that responds to the movement of the other obtains more votes because it has previous knowledge of the region where the other party must locate. On the one hand, for low degrees of flexibility, there is little sensitivity to the degree of importance of the issues for either party. On the other hand, for small values of α , the response of the opposition is not as profitable for it as when α is large.

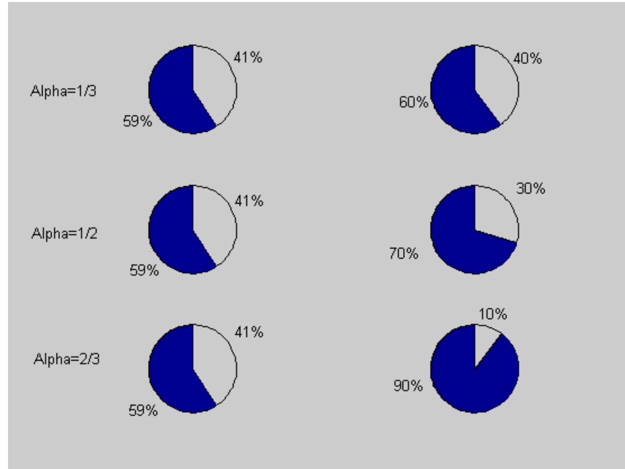

Fig. 7 Percentage increments in vote capture for R_1=1, R_2=3: the first column represents the percentage increments in vote capture when the government party chooses a position, and the second column those obtained when the opposition responds. The shaded region represents the party that changes position.

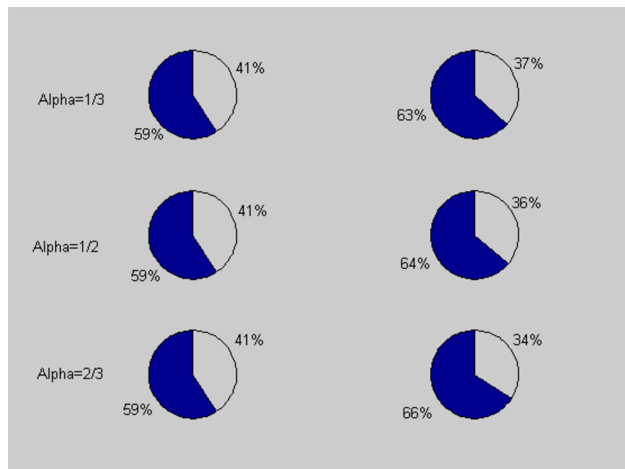

Fig. 8 Percentage increments in vote capture for R_1=1, R_2=1: the first column represents the percentage increments in vote capture when the government party chooses a position, and the second column those obtained when the opposition responds. The shaded region represents the party that changes position.

Consequently, knowledge of the methodology presented here by political parties may be useful for them to prepare their strategies.

It should be noted that this example has been developed to illustrate the theoretical model. It does not claim to be an exact description of the political reality of Spain.

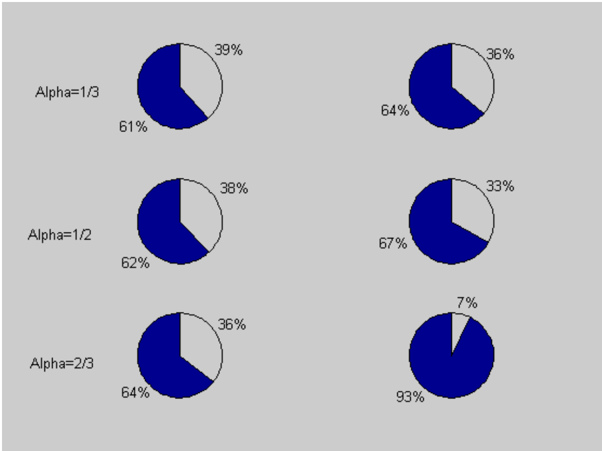

Fig. 9 Percentage increments in vote capture for R_1=2, R_2=2: the first column represents the percentage increments in vote capture when the government party chooses a position, and the second column those obtained when the opposition responds. The shaded region represents the party that changes position.

V. CONCLUSIONS

This research has dealt with the design of optimum political strategies in the face of significant changes in the situation that calls for making the requirements more flexible. This study was performed using a discrete geometric model based on the techniques of computational geometry. The result is not only a scientific description, but also one which is helpful in making decisions.

It is known that the concepts of computational geometry applied to economics also make it possible to solve problems in the area of political economy, for example, the problem of winning the greatest number of votes. In this research, the techniques used have provided a new vision whose most relevant contributions are, on the one hand, the consideration of the variable importance of political issues in a country at a given moment, and on the other hand, the consideration of neighborhoods representing the degree of political flexibility which the parties may allow themselves with respect to particular issues. This flexibility is also affected by the importance factor for these issues, which may be indeterminate or imprecise.

The techniques developed in this research enable political parties to design reply strategies by considering the options that their political adversary is likely to choose. Furthermore, algorithms are provided that obtains optimum solutions for maximum vote winning for a variety of proposals.

The algorithms developed here make it possible to consider a wide range of cases illustrating various real situations, such as the importance of the issues considered, the flexibility of parties' political programs with the intention of winning elections, and the assessment of the possible strategies that parties can follow. This is a general model that can be applied to other location problems, not only in the field of political competition.

APPENDIX A

a) Procedure to obtain the region A:

Step 1: Find p' and p'' (Proposition 2) and define a counter c' with initial value $c' = 0$.

Step 2: Let L be an empty list, and let m be another counter with initial value m=0. For each point v_i, find the points of intersection between N_1 and the elliptical neighborhood centered on v_i that goes through t_2.

- **2.1.** If there is no such intersection because N_1 is entirely contained in the elliptical neighborhood centered on v_i that goes through t_2, increase m by one.
- **2.2.** If there is no intersection because the ellipses are disjoint, keep the same value of m.
- **2.3.** If there are two points of intersection outside the part of N_1 visible from t_2, increase m by one unit.
- **2.4**. Otherwise:
  - **2.4.1.** If both points belong to the part of N_1 visible from t_2, then include both points in list L.
  - **2.4.2**. If x'_i belongs to the part of N_1 visible from t_2 and x''_i does not, then include x'_i in L.
  - **2.4.3**. If x''_i belongs to the part of N_1 visible from t_2 and x'_i does not, then include x''_i in L and increase c' by one unit.

Step 3. Sort the points in L according to their angle with respect to t_1 (clockwise).

Step 4. Let $c \leftarrow c'+m$ and $x \leftarrow p'$. Go through list L, proceeding as follows for each element:

- **4.1**. If it is an x'_i element, let $c \leftarrow c+1$, and if c>m, then let $m \leftarrow c$ and $x \leftarrow x'_i$;
- **4.2.** If it is an x''_i element, let $c \leftarrow c-1$.

  Remark: If x'_i and x''_i coincide because the corresponding ellipses are tangent, x'_i is considered to be previous to x''_i in list L.

When the algorithm has completed execution, the counter m indicates the maximum number of points v_i that the first party can win. The initial extreme point of the arc of N_1 where the first party must be located is the point stored in the variable x, and the final extreme point is the point following that point in list L.

b) Procedure to obtain the region B:

Input: neighborhood N_2 and n ellipses, D_1,...,D_n, which are assumed not to be tangent to N_2, none of them contained in N_2 (this case is excluded from this approach), not all of them disjoint from N_2, nor all of them containing N_2 (in these two cases, the area of maximum intersection will be N_2 and the maximum intersection 0 and n respectively):

Step 1: Intersect the borders of D_1 and N_2 and find the two points of intersection a_1, b_1 (if the borders are disjoint, go to Step 3).

Step 2: Use the algorithm in a) to find the area of maximum intersection of $D_2, \ldots, D_n$ in the arc of D_1 which joins a_1 with b_1 within N_2, and the number k_1 of ellipses that intersect in this arc.

Step 3: Repeat Steps 1 and 2 for $D_2, \ldots, D_n$.

Step 4: Among all the arcs found in Step 2, select those where $\max_{i=1,\ldots,j} k_i$ is reached, j being the number of ellipses which have a non-disjoint border with N_2. To this end, a list L is created with points α_i, β_i, the extreme points of the arcs obtained in Step 2. The arcs in list L are not arranged in any particular order.

Step 5: The intention is to obtain all the regions bounded by the arcs obtained in Step 4 or the regions bounded by the arcs obtained in Step 4 and the border of N_2. There are two possibilities:

5.1: The area to be found is solely bounded by the arcs obtained in Step 4.

5.1.1. Select the extreme points that delimit the first unused arc of list L (call them p1, p2). The arc is marked as used.

5.1.2. Search for point p1 between the extreme points of the rest of the unused arcs in the list. Given the arc with extreme points α_i, β_i where p1 is located, with for instance p1=α_i, then a new value is assigned to p1 (p1 $\leftarrow \beta_i$).

5.1.3. The search process is repeated as many times as necessary until p1=p2 (closed region).

5.2: The area to be found is bounded by the arcs obtained in Step 4 and by neighborhood N_2.

5.2.1. Select the extreme points that delimit the first unused arc of list L (call them p1, p2). The arc is marked as used (as in 5.1.1)

5.2.2. Search for point p1 between the extreme points of the rest of the unused arcs in the list. If p1 is found in one extreme point of the arc, the procedure in 5.1.2 is followed until p1 is not located at any extreme point of an unused arc; then p1 belongs to neighborhood N_2 (call it pB1), and p1 is assigned the value p2 (p1 ← p2).

5.2.3. Once again, search for point p1 between the extreme points of the rest of the unused arcs in the list, according to the procedure in 5.2.2. At this moment, point p1 is a point of neighborhood B (call it pB2). The arc (pB1, pB2) in neighborhood N_2 belongs to the area of maximum intersection (closed region).

Output: The regions found in Step 5 are the areas of maximum intersection in N_2 of the set of ellipses, while the maximum intersection is: $\max_{i=1,\ldots,j} k_i + 1$.

APPENDIX B

TABLE I
BEST POSITIONS FOR THE PARTIES IN THE VARIOUS CASES

Input	Movement of the government party: arc on which it should locate (P1, P2: boundary points of the arc)	Response of the opposition party: region in which it should locate (extreme points of the arcs enclosing the region indicated below)
R_1=3 R_2=1 α=1/3	P1=(3.169501,4.593561) P2=(3.174827, 4.591416)	Ellipse 306 p1 → p2
		Ellipse 1550 p2 → p3
		Arc in the boundary of N2 p1 → p3
		p1=(2.418245 , 7.820538) p2=(2.416555, 7.821072)
		p3=(2.405915, 7.815910)
		Ellipse 306 is centered on (0.564425,4.889248) and has radius 2.621804. *Ellipse 1550 has center (8.715210,1.321082) and radius 6.433578.*
		Ellipse 505 p1 → p2
		Ellipse 1671 p2 → p3
		Arc in the boundary of N2 p1 → p3
		p1=(2.337616, 7.791866) p2=(2.330703, 7.793956)
		p3=(2.247139, 7.763952)
		Ellipse 505 is centered on (0.530702,4.811335) and has radius 2.647771. *Ellipse 1671 has center (3.496017,6.098041) and radius 1.539504.*
R_1=3 R_2=1 α=1/2	P1=(3.743043,4.249786) P2=(3.757103, 4.234202) Another arc which gives the maximum number of voters obtained: P1'=(3.512932, 4.484611) P2'=(3.546029, 4.453017)	Ellipse 32 p1 → p2
		Ellipse 1671 p2 → p3
		Arc in the boundary of N2 p1 → p3
		p1=(1.694445,7.488944) p2=(1.688137, 7.490324)
		p3=(1.686692,7.488446)
		Ellipse 32 is centered on (0.365218,1.428697) and has radius 4.387108. *Ellipse 1671 has center (3.496017,6.098041) and radius 1.613518.*
R_1=3 R_2=1 α=2/3	P1=(2.596293, 5.762297) P2=(2.599790, 5.759092) Other arcs which give the maximum number of voters obtained: P1'=(2.535108, 5.817093) P2'=(2.543070.	Ellipse 1132 p1 → p2
		Ellipse 1607 p2 → p3
		Ellipse 1364 p3 → p4
		Arc in the boundary of N2 p1 → p4
		p1=(2.359936, 7.541695) p2=(2.283496, 7.603309)
		p3=(2.207489, 7.543660) p4=(2.136771, 7.343159)

	5.810098) P1''=(2.545452, 5.807996) P2''=(2.577728, 5.779176)	 *Ellipse 1132 has center (0.979778,4.242790) and radius 2.213025.* *Ellipse 1607 has center (2.663161,6.509064) and radius 0.703720.* *Ellipse 1364 has center (5.771352,4.904492) and radius 3.284684.*
R_1=1 R_2=3 α=1/3	P1=(1.489680, 2.450826) P2=(1.499298, 2.446724)	Ellipse 743 p1 → p2 Ellipse 1172 p2 → p3 Arc in the boundary of N2 p1 → p3 p1=(2.286557, 5.257979) p2=(2.266290, 5.268999) p3=(2.213969, 5.251504) *Ellipse 743 is centered on (0.631504,3.751030) and has radius 1.557882.* *Ellipse 1172 has center (4.873116,1.323127) and radius 3.555998.*
R_1=1 R_2=3 α=1/2	P1=(1.544457, 2.452616) P2=(1.550957, 2.446748)	Ellipse 589 p1 → p2 Ellipse 2076 p2 → p3 Arc in the boundary of N2 p1 → p3 p1=(1.925686, 4.669878) p2=(1.919273, 4.676218) p3=(1.907953, 4.668551) *Ellipse 589 is centered on (0.637529,3.373318) and has radius 1.292366.* *Ellipse 2076 has center (4.179266,1.327515) and radius 2.856692.* Ellipse 1088 p1 → p2 Ellipse 1259 p2 → p3 Arc in the boundary of N2 p1 → p3 p1=(1.742155, 4.659742) p2=(1.740639, 4.661013) p3=(1.738912, 4.659634) *Ellipse 1088 has center (0.589114,3.286370) and radius 1.268001.* *Ellipse 1259 has center (4.201648,1.577914) and radius 2.789451.*
R_1=1 R_2=3 α=2/3	P1=(1.572131, 2.453529) P2=(1.575781, 2.446759)	Ellipse 562 p1 → p2 Ellipse 1172 p2 → p3 Arc in the boundary of N2 p1 → p3 p1=(0.977010, 3.779085) p2=(0.976192, 3.779317) p3=(0.976182, 3.779287) *Ellipse 562 is centered on (0.642039,1.412198) and has radius 1.393624.* *Ellipse 1172 has center (4.873116,1.323127) and radius 3.483528.*
R_1=1 R_2=1 α=1/3	P1=(1.489680, 2.450826) P2=(1.499298, 2.446724)	Ellipse 418 p1 → p2 Ellipse 2175 p2 → p3 Arc in the boundary of N2 p1 →

		p3 p1=(2.006948, 7.709539) p2=(1.952059, 7.721772) p3=(1.884102, 7.691843) *Ellipse 418 is centered on (0.682940,4.806818) and has radius 2.490286.* *Ellipse 2175 has center (7.491842,1.378939) and radius 6.086933.*
R_1=1 R_2=1 α=1/2	P1=(1.544457 2.452616) P2=(1.550957 2.446748)	Ellipse 700 p1 → p2 Ellipse 1793 p2 → p3 Arc in the boundary of N2 p1 → p3 p1=(2.379556, 7.720046) p2=(2.375580, 7.722411) p3=(2.328396, 7.687796) *Ellipse 700 is centered on (0.596188,4.726646) and has radius 2.463823.* *Ellipse 1793 has center (7.033595,1.323640) and radius 5.596489.*
R_1=1 R_2=1 α=2/3	P1=(1.572131, 2.453529) P2=(1.575781, 2.446759)	Ellipse 28 p1 → p2 Ellipse 1138 p2 → p3 Arc in the boundary of N2 p1 → p3 p1=(2.116767, 7.329680) p2=(2.116462, 7.330045) p3=(2.115713, 7.328988) *Ellipse 28 is centered on (0.328474,4.341029) and has radius 2.260385.* *Ellipse 1138 has center (6.274364,1.439683) and radius 4.805298.*
R_1=2 R_2=2 α=1/3	P1=(2.627055, 3.386335) P2=(2.717063, 3.338820)	Ellipse 89 p1 → p2 Ellipse 1307 p2 → p3 Arc in the boundary of N2 p1 → p3 p1=(2.392977, 6.515552) p2=(2.345330, 6.533560) p3=(2.274134, 6.497341) *Ellipse 89 is centered on (0.585031,4.164312) and has radius 2.185202.* *Ellipse 1307 has center (7.419395,1.493408) and radius 5.051481.*
R_1=2 R_2=2 α=1/2	P1=(2.095262, 3.800873) P2=(2.109216, 3.792126)	Ellipse 1006 p1 → p2 Ellipse 1855 p2 → p3 Arc in the boundary of N2 p1 → p3 p1=(2.183080, 6.132326) p2=(2.177297, 6.137852) p3=(2.165887, 6.128760) *Ellipse 1006 has center (0.551302,4.430235) and radius 1.667306.* *Ellipse 1855 has center (5.258916,2.258812) and radius 3.503093.*
R_1=2 R_1=2 α=2/3	P1=(2.628924, 3.340859) P2=(2.639543, 3.318469)	Ellipse 421 p1 → p2 Ellipse 1628 p2 → p3 Arc in the boundary of N2 p1 →

		p3
		p1=(2.143013, 5.522090) p2=(2.141720, 5.523311)
		p3=(2.140787, 5.521376)
		Ellipse 421 is centered on (0.689233,2.445692) and has radius 2.136288. *Ellipse 1628 has center (6.381564,1.429795) and radius 4.191635.*
		Ellipse 1145 p1 → p2
		Ellipse 1888 p2 → p3
		Arc in the boundary of N2 p1 → p3
		p1=(2.118679, 5.514451) p2=(2.116147, 5.515986)
		p3=(2.114849, 5.513282)
		Ellipse 1145 *has center* (0.894372,1.480718) *and radius* 2.534354. *Ellipse* 1888 *has center* (6.421616,1.381399) *and radius* 4.249272.
		Ellipse 1228 p1 → p2
		Ellipse 2154 p2 → p3
		Arc in the boundary of N2 p1 → p3
		p1=(2.113209, 5.512784) p2=(2.102852, 5.518395)
		p3=(2.100366, 5.508943)
		Ellipse 1228 *has center* (0.979189,1.348329) *and radius* 2.576476. *Ellipse* 2154 *has center* (6.276131,3.317988) *and radius* 3.636588.

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