Convergence of Iterative Schemes in Spaces with Two Metrics

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Abstract— The purpose of this paper is to study the stability of the Jungck- Mann and Jungck-Ishikawa iterative schemes for mappings satisfying general contractive conditions. We obtain the stability results for the maps in complete *b*-metric spaces with one and two metrics. Our results generalize the recent results of Olatinwo [12], Prasad and Sahni [20] and Singh and Prasad [26]. An example is also given to justify the need of the Jungck-Ishikawa iterative scheme.

Keywords— *b*-Metric space, stability, Jungck-Mann iteration, Jungck-Ishikawa iteration.

I. Introduction

Most of the physical problems of applied sciences and engineering are usually formulated as functional equations. Such equations can be written in the form of fixed point equations in an easy manner. To solve these equations, we generally use an iteration procedure with some specific initial choice and successively obtain a sequence of iterates converging towards the solution of the equation. But in many situations, this may not be the actual solution of the equation because of the rounding off or discretization in the function. We obtain an approximate sequence which may not converge towards the same value as the actual sequence. Here is the role of the stability of iterative procedures as observed by Urabe [28] in fifties of the twentieth century. However, Harder and Hicks [8]-[9] were the first to define formally the stability of iterative procedures in metric spaces in the following manner:

Let T be a self map on X and $\{x_n\}$ be a sequence in X converging to a fixed point u of T. Let $\{y_n\}$ be an arbitrary sequence in X, and define

$$\varepsilon_n = d\,(y_{n+1}, f(T, y_n)),\ \text{n} = 0, 1, 2, \dots .$$

If $\lim_{n\to\infty}\varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = u$, than the iteration procedure is said to be T-stable or stable with respect to T.

Following Harder and Hicks [op. cit], a number of authors have extended and generalized the stability theory of iterative procedures in various ways in different settings see, among others, Berinde [1], Imoru and Olatinwo [10] Olatinwo [12], Osilike [14]-[15], Prasad and Sahni [20], Rhoades [22]-[23], Rhoades and Solutz [24], Singh [25], Singh and Prasad [26], Singh et al [26]-[27] and several references thereof. Recently Olatinwo [12] obtained some stability results in complete metric spaces
for Picard iteration process. We obtain stability results of a more general iterative scheme in the *b*-metric spaces for one and two metrics.

II. Preliminaries

There are a number of iterative procedures in the literature used in various settings. The most popular iterative procedure called Picard iteration is defined by

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, \dots \tag{1}$$

The first result on the stability of iterative procedures in the metric spaces for Picard iteration was given by Ostrowski [16]. He proved the following:

Theorem 2.1[16]. Let (X,d) be a complete metric space and $T : X \to X$ a Banach contraction with contraction constant k, i.e., $d(Tx,Ty) \le kd(x,y)$ for all $x, y \in X$, where $0 \le k < 1$. Let $u \in X$ be the fixed point of T. Let $x_0 \in X$ and $x_{n+1} = Tx_n$, $n = 0,1,2,\dots$ .Suppose that $\{y_n\}$ is a sequence in X and $\varepsilon_n = d(y_{n+1}, Ty_n)$. Then

$$d(u, y_{n+1}) \le d(u, x_{n+1}) + k^{n+1} d(x_0, y_0) + \sum_{r=0}^{n} k^{n-r}\varepsilon_r$$

Moreover, $\lim_n y_n = u \ \mathit{iff}\ \lim_n \varepsilon_n = 0$.

Indeed the Picard iteration has been widely used to solve the functional equations obtained out of the modeled physical problems (see, for instance, Berinde [1], Dobritoiu et al [3]-[4], Dobritoiu and Dobritoiu [4]-[5], Fang and Fneg [7]).

When the contractive conditions are slightly weaker the Picard iteration (1) need not converge to a fixed point of the operator under consideration and some other iterative procedures should be considered. The Mann iteration scheme is defined in the following manner:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n \tag{2}$$

where $\alpha_n \subset [0,1]$ and $n = 0,1,2,...$.

If the iterative scheme is defined as:

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Tz_n \\ z_n &= (1-\beta_n)x_n + \beta_n Tx_n \end{aligned}\right\}, \ n = 0,1,... \tag{3}$$

It is called Ishikawa iteration scheme.

After Ostrowski [16], Harder and Hicks [9] obtained interesting stability results for some of these iteration procedures using various contractive definitions. Thereafter, a number of papers appeared in the literature generalizing the above concept in various ways (see, for example [1], [12]-[15], [20]-[21], [22]-[27] and references thereof). Rhoades [22]-[23] generalized the results of Harder and Hicks [9] to a more general contractive mappings. Osilike [14] further extended and generalized their results by employing the following contractive condition:

$$d(Tx,Ty) \le L\, d(x,Tx) + a\, d(x,y) \tag{4}$$

where $L \ge 0$ and $a \in [0,\, 1)$.

Condition (4) is more general than those of Rhoades [22]-[23] and Harder and Hicks [9].

Recently, Imoru and Olatinwo [10] obtained some stability results for Picard and Mann iteration procedures by using a condition more general than that of (4). Olatinwo [12] used the following contractive definition:

Let $\varphi : R_+ \to R_+$ be a monotone increasing function with $\varphi(0) = 0$ and $a \in [0,\, 1)$ such that

$$d(Tx,Ty) \le \varphi(\, d(x,Tx)) + a\, d(x,y) \tag{5}$$

We use this condition to obtain the stability results in general metric space

We need the following definitions for our results.

Definition 2.1 [27]. Let $S,T : Y \to X$, $T(Y) \subseteq S(Y)$ and z a coincidence point of T and S , that is, $Sz = Tz = p$ (say). For any $x_0 \in Y$, let the sequence $\{Sx_n\}$, generated by the iterative procedure (1), converges to p. Let $\{sy_n\} \subset X$ be an arbitrary sequence, and set $\varepsilon_n = d(Sy_{n+1}, f(T,y_n)), n = 0,1,2,...$ Then the iterative procedure $f(T,x_n)$ will be called (S,T) - stable if and only if $\lim_n \varepsilon_n = 0$ implies that $\lim_n Sy_n = p$.

Definition 2.2. Let $S : X \to X$ and $T(X) \subseteq S(X)$. Define

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tx_n \tag{6}$$

where $\{\alpha_n\}$ satisfies

$(i)\, \alpha_0 = 1,\ (ii)\, 0 \le \alpha_n \le 1,\ n \ge 0,$

$(iii) \sum \alpha_n = \infty, (iv) \sum_{j=0}^{n} \prod_{i=j+1}^{n} \{1-\alpha_i + a\alpha_i\}$ converges.

It is called Jungck-Mann iteration.

Notice that on putting $\alpha_n = 1$, (6) becomes the Jungck iteration [11].

Definition 2.3 [13]. Let $S : X \to X$ and $T(X) \subseteq S(X)$. the Jungck-Ishikawa iteration scheme is defined in the following manner:

$$\left.\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sx_n + \alpha_n Tz_n \\ Sz_n &= (1-\beta_n)Sx_n + \beta_n Tx_n \end{aligned}\right\}, \ n = 0,1,... \tag{7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfies

$(i)\, \alpha_0 = 1,\ (ii)\, 0 \le \alpha_n, \beta_n \le 1,\ n \ge 0,$

$(iii) \sum \alpha_n = \infty, (iv) \sum_{j=0}^{n} \prod_{i=j+1}^{n} \{1-\alpha_i + a\alpha_i\}$ converges.

Notice that when $S = id$, the identity map, it is called Ishikawa iteration.

Definition 2.4 [2]. Let X be a set and $c \ge 1$ be a given real number. A function $d : X \times X \to R_+$ is said to be a *b*-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $d(x,y) = 0$ *iff* $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,z) \le c[d(x,y) + d(y,z)]$.

A pair (X,d) is called a *b*-metric space.

The class of *b*-metric spaces is effectively larger than that of metric spaces, since a *b*-metric space is a metric space when $c = 1$ in condition (iii) as mentioned above. After Czerwik [2], a number of authors have studied these spaces and obtained interesting results about fixed points and approximate fixed points of the maps defined on these spaces (see for instance [17]-[21] and [26]).

The following example of Singh and Prasad [26] shows that a b-metric on X need not be a metric on X.

Example 2.1 [26]. Let $X = \{x_1, x_2, x_3, x_4\}$ and
$d(x_1,x_2) = k \geq 2,\ d(x_1,x_3) = d(x_1,x_4)$
$= d(x_2,x_3) = d(x_2,x_4) = d(x_3,x_4) = 1,$
$d(x_i,x_j) = d(x_j,x_i)$
for all $i, j = 1,2,3,4$ and $d(x_i,x_i) = 0,\ i = 1,2,3,4.$ Then

$$d(x_i,x_j) \leq \frac{k}{2}\left[d(x_j,x_n) + d(x_n,x_j)\right]$$

for $n,i,j = 1,2,3,4$ and if $k > 2$, the ordinary triangle inequality does not hold.

III. STABIITY RESULTS INVOLVING ONE METRIC

Theorem 3.1. Let (X,d) be a b-metric space and S,T maps on an arbitrary set Y with values in X such that $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of X. Let z be a coincidence point of T and S, i.e., $Sz = Tz = p$.
Let $x_0 \in Y$ and the sequence $\{Sx_n\}$ generated by (6) converges to p. Let $\{Sy_n\} \subset X$ and define

$$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ty_n),\quad n \geq 0$$

If the pair (S,T) satisfies (5), then

$$\begin{aligned} d(p,Sy_{n+1}) &\leq cd(p,Sx_{n+1}) \\ &+ c^{n+1}\prod_{i=0}^{n}(1-\alpha_i + a\alpha_i)\ d(Sx_0,Sy_0) \\ &+ \varphi\sum_{j=0}^{n}\alpha_j c^{j+2}\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\ d(Sx_i,Tx_i) \\ &+ \sum_{j=0}^{n}c^{j+2}\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\varepsilon_j \end{aligned}$$

Further

(ii) $\lim_{n\to\infty} Sy_n = p$ if and only if $\lim_{n\to\infty}\varepsilon_n = 0$.

Proof. By the triangle inequality

$$\begin{aligned} d(p,Sy_{n+1}) &\leq c[d(p,Sx_{n+1}) + d(Sx_{n+1},Sy_{n+1})] \\ &\leq cd(p,Sx_{n+1}) + c[d((1-\alpha_n)Sx_n + \alpha_n Tx_n, Sy_{n+1})] \\ &\leq cd(p,Sx_{n+1}) + c^2[d((1-\alpha_n)Sx_n + \\ &\alpha_n Tx_n, (1-\alpha_n)Sy_n + \alpha_n Ty_n) \\ &+ d((1-\alpha_n)Sy_n + \alpha_n Ty_n, Sy_{n+1})] \\ &\leq cd(p,Sx_{n+1}) + c^2(1-\alpha_n)d(Sx_n,Sy_n) + \\ &c^2\alpha_n d(Tx_n,Ty_n) + c^2\varepsilon_n \\ &\leq cd(p,Sx_{n+1}) + c^2(1-\alpha_n)d(Sx_n,Sy_n) + \\ &c^2\alpha_n[ad(Sx_n,Sy_n) + \varphi d(Sx_n,Tx_n)] + c^2\varepsilon_n \end{aligned}$$

Also,

$$\begin{aligned} d(Sx_n,Sy_n) &= d[(1-\alpha_{n-1})Sx_{n-1} + \alpha_{n-1}Tx_{n-1}, Sy_n] \\ &\leq cd[(1-\alpha_{n-1})Sx_{n-1} + \alpha_{n-1}Tx_{n-1}, \\ &(1-\alpha_{n-1})Sy_{n-1} + \alpha_{n-1}Ty_{n-1}] \\ &+ cd[(1-\alpha_{n-1})Sy_{n-1} + \alpha_{n-1}Ty_{n-1}, Sy_n] \\ &\leq c(1-\alpha_{n-1})d(Sx_{n-1},Sy_{n-1}) + \\ &c\alpha_{n-1}d(Tx_{n-1},Ty_{n-1}) + c\varepsilon_{n-1} \\ &\leq c(1-\alpha_{n-1})d(Sx_{n-1},Sy_{n-1}) + \\ &c\alpha_{n-1}[ad(Sx_{n-1},Sy_{n-1}) + \\ &\varphi d(Sx_{n-1},Tx_{n-1})] + c\varepsilon_{n-1} \\ &\leq [c(1-\alpha_{n-1}) + ac\alpha_{n-1}]d(Sx_{n-1},Sy_{n-1}) + \\ &c\alpha_{n-1}\varphi d(Sx_{n-1},Tx_{n-1}) + c\varepsilon_{n-1} \end{aligned}$$

Therefore,

$$\begin{aligned} d(p,Sy_{n+1}) &\leq cd(p,Sx_{n+1}) \\ &+ [c^2(1-\alpha_n) + c^2\alpha_n a]\{[c(1-\alpha_{n-1}) \\ &+ ac\alpha_{n-1}]d(Sx_{n-1},Sy_{n-1}) \\ &+ c\alpha_{n-1}\varphi d(Sx_{n-1},Tx_{n-1}) + c\varepsilon_{n-1}\} \\ &+ c^2\alpha_n\varphi d(Sx_n,Tx_n) + c^2\varepsilon_n \\ &\leq cd(p,Sx_{n+1}) + [c^2(1-\alpha_n) + c^2\alpha_n a] \\ &[c(1-\alpha_{n-1}) + ac\alpha_{n-1}]d(Sx_{n-1},Sy_{n-1}) \\ &+ [c^2(1-\alpha_n) + c^2\alpha_n a]c\alpha_{n-1}\varphi d(Sx_{n-1},Tx_{n-1}) \\ &+ c^2\alpha_n\varphi d(Sx_n,Tx_n) \\ &+ [c^2(1-\alpha_n) + c^2\alpha_n a]c\varepsilon_{n-1} + c^2\varepsilon_n \end{aligned}$$

This process when repeated $n-1$ times, yields (i).

To prove (ii), suppose that $\lim_{n\to\infty} Sy_n = p$. Then,

$$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ty_n)$$
$$\leq cd(Sy_{n+1}, p) + cd(p, (1-\alpha_n)Sy_n + \alpha_n Ty_n)$$
$$\leq cd(Sy_{n+1}, p) + c(1-\alpha_n)d(p, Sy_n) + c\alpha_n d(p, Ty_n)$$
$$\leq cd(Sy_{n+1}, p) + c(1-\alpha_n)d(p, Sy_n) + c\alpha_n d(Tp, Ty_n)$$
$$\leq cd(Sy_{n+1}, p) + c(1-\alpha_n)d(p, Sy_n) +$$
$$c\alpha_n[ad(Sp, Sy_n) + \varphi d(Sp, Tp)] \to 0 \quad as \ n \to \infty$$

Now suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. Let A denote the lower triangular matrix with entries

$$b_{nj} = \prod_{i=j+1}^{n} (1-\alpha_i + a\alpha_i) .$$

Hence, the condition $\sum \alpha_n = \infty$ implies that this product diverges. Hence, $\lim_{n\to\infty} b_{nj} = 0$ for each j . Now $\sum_{j=0}^{n} \alpha_j \prod_{i=j+1}^{n} \{1-\alpha_i + a\alpha_i\}$ implies that the limit of the row sum exists. Therefore, A is multiplicative.

Also, $\lim_{n\to\infty} Sx_n = p$ implies that $\lim_{n\to\infty} d(Sx_n, Tx_n) = 0$.

Thus,

$$\lim_{n\to\infty} \varphi \sum_{j=0}^{n} \alpha_j c^{j+2} \prod_{i=j+1}^{n} (1-\alpha_i + a\alpha_i) \ d(Sx_i, Tx_i) = 0$$

$$\lim_{n\to\infty} \sum_{j=0}^{n} c^{j+2} \prod_{i=j+1}^{n} (1-\alpha_i + a\alpha_i)\varepsilon_j = 0 .$$

This completes the proof.

Theorem 3.2. Let (X, d) be a b-metric space and S, T maps on an arbitrary set Y with values in X such that $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of X . Let z be a coincidence point of T and S , i.e., $Sz = Tz = p$. Let $x_0 \in Y$ and the sequence $\{Sx_n\}$ generated by (7) converges to p . Let $\{Sy_n\} \subset X$ and define

$$Ss_n = (1-\beta_n)Sy_n + \beta_n Ty_n, \ n \geq 0$$
$$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n), \quad n \geq 0$$

If the pair (S, T) satisfies (5), then

(i)

$$d(p, Sy_{n+1}) \leq cd(p, Sx_{n+1}) + c^2 \prod_{i=0}^{n} (1-\alpha_i + a\alpha_i) \ d(Sx_0, Sy_0)$$
$$+ \varphi c^2 a \sum_{j=0}^{n} c^{n-i} \alpha_j \beta_j \prod_{i=j+1}^{n} (1-\alpha_i + a\alpha_i) \ d(Sx_j, Tx_j)$$
$$+ \varphi c^2 \sum_{j=0}^{n} c^{n-i} \alpha_i \prod_{i=j+1}^{n} (1-\alpha_i + a\alpha_i) \ d(Sz_j, Tz_j)$$
$$+ \sum_{j=0}^{n} c^{j+2} \prod_{i=j+1}^{n} (1-\alpha_i + a\alpha_i)\varepsilon_j$$

Further

(ii) $\lim_{n\to\infty} Sy_n = p$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$

Proof. By the triangle inequality

$$d(p, Sy_{n+1}) \leq c[d(p, Sx_{n+1}) + d(Sx_{n+1}, Sy_{n+1})]$$
$$\leq cd(p, Sx_{n+1}) + cd[(1-\alpha_n)Sx_n + \alpha_n Tz_n, Sy_{n+1}]$$
$$\leq cd(p, Sx_{n+1}) + c^2 d[(1-\alpha_n)Sx_n + \alpha_n Tz_n,$$
$$(1-\alpha_n)Sy_n + \alpha_n Ts_n]$$
$$+ c^2 d[(1-\alpha_n)Sy_n + \alpha_n Ts_n, Sy_{n+1}]$$
$$\leq cd(p, Sx_{n+1}) + c^2(1-\alpha_n)d(Sx_n, Sy_n) +$$
$$c^2\alpha_n d(Tz_n, Ts_n) + c^2\varepsilon_n$$
$$\leq cd(p, Sx_{n+1}) + c^2(1-\alpha_n)d(Sx_n, Sy_n) +$$
$$c^2\alpha_n[ad(Sz_n, Ss_n) + \varphi d(Sz_n, Tz_n)] + c^2\varepsilon_n$$

But,

$$d(Sz_n, Ss_n) = d[(1-\beta_n)Sx_n + \beta_n Tx_n,$$
$$(1-\beta_n)Sy_n + \beta_n Ty_n]$$
$$\leq (1-\beta_n)d(Sx_n, Sy_n) + \beta_n d(Tx_n, Ty_n)$$
$$\leq (1-\beta_n)d(Sx_n, Sy_n) + \beta_n[ad(Sx_n, Sy_n) +$$
$$\varphi(d(Sx_n, Tx_n))]$$
$$\leq (1-\beta_n)d(Sx_n, Sy_n) + a\beta_n d(Sx_n, Sy_n) +$$
$$\varphi\beta_n d(Sx_n, Tx_n)$$
$$\leq d(Sx_n, Sy_n) + \varphi\beta_n d(Sx_n, Tx_n)$$

Therefore,

$$d(p,Sy_{n+1}) \le cd(p,Sx_{n+1}) + c^2(1-\alpha_n)d(Sx_n,Sy_n)$$
$$+ac^2\alpha_n d(Sx_n,Sy_n) + \varphi\beta_n d(Sx_n,Tx_n)] +$$
$$\varphi c^2\alpha_n d(Sz_n,Tz_n) + c^2\varepsilon_n$$

Also,

$$d(Sx_n,Sy_n) = d((1-\alpha_{n-1})Sx_{n-1} + \alpha_{n-1}Tz_{n-1},Sy_n)$$
$$\le cd((1-\alpha_{n-1})Sx_{n-1} + \alpha_{n-1}Tz_{n-1},$$
$$(1-\alpha_{n-1})Sy_{n-1} + \alpha_{n-1}Ts_{n-1})$$
$$+cd((1-\alpha_{n-1})Sy_{n-1} + \alpha_{n-1}Ts_{n-1},Sy_n)$$
$$\le cd((1-\alpha_{n-1})Sx_{n-1} + \alpha_{n-1}Tz_{n-1},$$
$$(1-\alpha_{n-1})Sy_{n-1} + \alpha_{n-1}Ts_{n-1}) + c\varepsilon_{n-1}$$
$$\le c(1-\alpha_{n-1})d(Sx_{n-1},Sy_{n-1}) +$$
$$c\alpha_{n-1}d(Tz_{n-1},Ts_{n-1}) + c\varepsilon_{n-1}$$
$$\le c(1-\alpha_{n-1})d(Sx_{n-1},Sy_{n-1}) +$$
$$c\alpha_{n-1}[ad(Sz_{n-1},Ss_{n-1}) + \varphi(d(Sz_{n-1},Tz_{n-1}))] + c\varepsilon_{n-1}$$
$$\le c(1-\alpha_{n-1})d(Sx_{n-1},Sy_{n-1}) + ac\alpha_{n-1}d(Sz_{n-1},Ss_{n-1})$$
$$+c\alpha_{n-1}\varphi(d(Sz_{n-1},Tz_{n-1})) + c\varepsilon_{n-1}$$

Now observe that,

$$d(Sz_{n-1},Ss_{n-1}) = d[(1-\beta_{n-1})Sx_{n-1} +$$
$$\beta_{n-1}Tx_{n-1},(1-\beta_{n-1})Sy_{n-1} + \beta_{n-1}Ty_{n-1}]$$
$$\le (1-\beta_{n-1})d(Sx_{n-1},Sy_{n-1}) + \beta_{n-1}d(Tx_{n-1},Ty_{n-1})$$
$$\le (1-\beta_{n-1})d(Sx_{n-1},Sy_{n-1}) + \beta_{n-1}[ad(Sx_{n-1},Sy_{n-1})$$
$$+\varphi(d(Sx_{n-1},Tx_{n-1}))]$$
$$\le (1-\beta_{n-1})d(Sx_{n-1},Sy_{n-1}) + a\beta_{n-1}d(Sx_{n-1},Sy_{n-1})$$
$$+\varphi\beta_{n-1}d(Sx_{n-1},Tx_{n-1})$$
$$\le d(Sx_{n-1},Sy_{n-1}) + \varphi\beta_{n-1}d(Sx_{n-1},Tx_{n-1})$$

So,

$$d(Sx_n,Sy_n) \le c(1-\alpha_{n-1})d(Sx_{n-1},Sy_{n-1}) +$$
$$ac\alpha_{n-1}[d(Sx_{n-1},Sy_{n-1})$$
$$+\varphi\beta_{n-1}d(Sx_{n-1},Tx_{n-1})] +$$
$$c\alpha_{n-1}\varphi d(Sz_{n-1},Tz_{n-1}) + c\varepsilon_{n-1}$$

Thus,

$$d(p,Sy_{n+1}) \le cd(p,Sx_{n+1}) + [c^2(1-\alpha_n) +$$
$$ac^2\alpha_n]d(Sx_n,Sy_n)$$
$$+ac^2\alpha_n\varphi\beta_n d(Sx_n,Tx_n) + \varphi c^2\alpha_n d(Sz_n,Tz_n)$$
$$+c^2\varepsilon_n$$

$$\le cd(p,Sx_{n+1}) + [c^2(1-\alpha_n) + ac^2\alpha_n]$$
$$[c(1-\alpha_{n-1})d(Sx_{n-1},Sy_{n-1})$$
$$+ac\alpha_{n-1}[d(Sx_{n-1},Sy_{n-1})$$
$$+ac\alpha_{n-1}\varphi\beta_{n-1}d(Sx_{n-1},Tx_{n-1})$$
$$+c\alpha_{n-1}\varphi(d(Sz_{n-1},Tz_{n-1})) + c\varepsilon_{n-1}\ ]$$
$$+ac^2\alpha_n\varphi\beta_n d(Sx_n,Tx_n) + \varphi c^2\alpha_n d(Sz_n,Tz_n) + c^2\varepsilon_n$$
$$\le cd(p,Sx_{n+1}) + [c^2(1-\alpha_n) + ac^2\alpha_n]$$
$$[c((1-\alpha_{n-1}) + ac\alpha_{n-1}]d(Sx_{n-1},Sy_{n-1})$$
$$+[c^2(1-\alpha_n) + ac^2\alpha_n](ac\alpha_{n-1}\varphi\beta_{n-1})d(Sx_{n-1},Tx_{n-1})$$
$$+[c^2(1-\alpha_n) + ac^2\alpha_n](c\alpha_{n-1}L)d(Sz_{n-1},Tz_{n-1})$$
$$+[c^2(1-\alpha_n) + ac^2\alpha_n]c\varepsilon_{n-1} + c^2\alpha_n a\varphi\beta_n d(Sx_n,Tx_n)$$
$$+\varphi c^2\alpha_n d(Sz_n,Tz_n) + c^2\varepsilon_n$$

This process when repeated $n-1$ times, yields (i).

To prove (ii), suppose that $\lim_{n\to\infty} Sy_n = p$. Then,

$$\varepsilon_n = d(Sy_{n+1},(1-\alpha_n)Sy_n + \alpha_n Ts_n)$$
$$\le cd(Sy_{n+1},p) + cd(p,(1-\alpha_n)Sy_n + \alpha_n Ts_n)$$
$$\le cd(Sy_{n+1},p) + c(1-\alpha_n)d(p,Sy_n) + c\alpha_n d(p,Ts_n)$$
$$\le cd(Sy_{n+1},p) + c(1-\alpha_n)d(p,Sy_n) + c\alpha_n d(Tp,Ts_n)$$
$$\le cd(Sy_{n+1},p) + c(1-\alpha_n)d(p,Sy_n) +$$
$$c\alpha_n[ad(Sp,Ss_n) + \varphi(d(Sp,Tp))] \to 0 \quad as\ n\to\infty$$

Now suppose that $\lim_{n\to\infty}\varepsilon_n = 0$. Let A denote the lower triangular matrix with entries

$$a_{nj} = \alpha_j \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)$$

Then A is multiplicative, so that

$$\lim_{n\to\infty}\varphi c^2\sum_{j=0}^{n} c^{n-i}\alpha_i \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\ d(Sz_i,Tz_i) = 0$$

and

$$\lim_{n\to\infty}\varphi c^2 a\sum_{i=0}^{n} c^{n-i}\alpha_i\beta_i \prod_{j=i+1}^{n}(1-\alpha_j + a\alpha_j)\ d(Sx_i,Tx_i) = 0$$

Let B be the lower triangular matrix with entries

$$b_{nj} = \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)$$

Condition (iv) of iterative scheme implies that B is multiplicative, and hence

$$\lim_{n\to\infty}\sum_{j=0}^{n} c^{j+2}\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\varepsilon_j = 0$$

Finally, condition (iii) of iterative scheme implies that

$$\lim_{n\to\infty}\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i) = 0 \;.$$

Hence, it follows from inequality that $\lim_{n\to\infty} Sy_n = p$. This completes the proof.

If we set $\beta_n = 1$ for all $n \ge 0$, $\varphi(t) = L$ for all $t \in R_+$ and $c = 1$ in above theorem, we obtain the following result of Singh and Prasad [26]:

Corollary 3.1 [26]. Let S,T maps on an arbitrary set Y with values in X such that $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of X. Let z be a coincidence point of T and S, i.e., $Sz = Tz = p$. Let $x_0 \in Y$ and the sequence $\{Sx_n\}$ generated by (6). Let $\{Sy_n\} \subset X$ and define

$$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ty_n), \quad n \ge 0.$$

Then,

$$(I) \qquad d(p, Sy_{n+1}) \le d(p, Sx_{n+1}) + \prod_{i=0}^{n}(1-\alpha_i + a\alpha_i)\, d(Sx_0, Sy_0) + L\sum_{j=0}^{n}\alpha_j \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\, d(Sx_i, Tx_i) + \sum_{j=0}^{n}\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\varepsilon_j$$

where the product is 1 when $j = n$.
Further

(II) $\lim_{n\to\infty} Sy_n = p$ if and only if $\lim_{n\to\infty}\varepsilon_n = 0$

If we set $\varphi(t) = L$ for all $t \in R_+$ and $c = 1$ in above theorem, we obtain the following result of Prasad and Sahni [20]:

Corollary 3.2 [20]. Let (X,d) be a metric space and S,T maps on an arbitrary set Y with values in X such that $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of X. Let z be a coincidence point of T and S, i.e., $Sz = Tz = p$. Let $x_0 \in Y$ and the sequence $\{Sx_n\}$ generated by (2) converges to p. Let $\{Sy_n\} \subset X$ and define

$$Ss_n = (1-\beta_n)Sy_n + \beta_n Ty_n, \; n \ge 0$$
$$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n), \quad n \ge 0$$

If the pair (S,T) satisfies

$$d(Tx,Ty) \le ad(Sx,Sy) + Ld(Sx,Tx), \; a \in (0,1), \; L \ge 0$$

Then

(i)

$$d(p, Sy_{n+1}) \le d(p, Sx_{n+1}) + \prod_{i=0}^{n}(1-\alpha_i + a\alpha_i)\, d(Sx_0, Sy_0) + La\sum_{j=0}^{n}\alpha_j\beta_j \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\, d(Sx_j, Tx_j) + L\sum_{j=0}^{n}\alpha_i \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\, d(Sz_j, Tz_j) + \sum_{j=0}^{n}\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\varepsilon_j$$

Further

(ii) $\lim_{n\to\infty} Sy_n = p$ if and only if $\lim_{n\to\infty}\varepsilon_n = 0$.

IV. Stability Results Involving Two Metrics d and ρ

Theorem 4.1. Let $S,T : Y \to X$ such that $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of X. Let d and ρ be two b-metric on Y. Let z be a coincidence point of T and S. Let $x_0 \in Y$ and the sequence $\{Sx_n\}$ generated by Jungck-Mann iteration converges to p. Let $\{Sy_n\} \subset X$ and define

$$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ty_n), \; n \ge 0$$

Suppose that

(i) there exists $c_1 > 0$, and a monotone increasing function $\varphi_1 : R_+ \to R_+$ with $\varphi_1(0) = 0$ such that

$$d(Tx,Ty) \le \varphi_1(\rho(Sx,Tx)) + c_1\rho(Sx,Sy),\ \forall x,y \in Y$$

(ii) $T:(Y,\rho) \to (X,\rho)$ satisfies the contractive condition

$$\rho(Tx,Ty) \le \varphi_2(\rho(Sx,Tx)) + \psi(\rho(Sx,Sy)),\ \forall x,y \in Y$$

Where $\psi^k : R_+ \to R_+$, $k = 1,2,...,$ are continuous comparison functions (ψ^k is the kth iterate of ψ) and $\varphi_2 : R_+ \to R_+, K = 1,2,...,$ is a monotone increasing function such that $\varphi_2(0) = 0$. Then the Jungck-Mann iteration process with $T:(Y,d) \to (X,d)$ is (*S*, *T*)-stable.

Proof. Suppose $\lim_{n\to\infty} \varepsilon_n = 0.$ Then we shall establish that $\lim_{n\to\infty} Sy_n = p$, using conditions (i) and (ii) and triangle inequality.

$$\begin{aligned}
d(Sy_{n+1},p) &\le c[d(Sy_{n+1},(1-\alpha_n)Sy_n + \alpha_n Ty_n)) \\
&+ d((1-\alpha_n)Sy_n + \alpha_n Ty_n),p)] \\
&\le cd((1-\alpha_n)Sy_n + \alpha_n Ty_n),Tz) + c\varepsilon_n \\
&\le c(1-\alpha_n)d(Sy_n,Tz) + c\alpha_n d(Ty_n,Tz) + c\varepsilon_n \\
&\le c^2(1-\alpha_n)[d(Sy_n,Sz) + d(Sz,Tz)] + \\
&c\alpha_n[\varphi_1(\rho(Sz,Tz) + c\rho(Sz,Sy_n)] + c\varepsilon_n \\
&\le c^2(1-\alpha_n)d(Sy_n,p) + c\alpha_n c_1\rho(p,Sy_n) + c\varepsilon_n
\end{aligned}$$

But,

$$\begin{aligned}
\rho(p,Sy_n) &= \rho(Tz,(1-\alpha_{n-1})Sy_{n-1} + \alpha_{n-1}Ty_{n-1}) \\
&\le (1-\alpha_{n-1})\rho(Tz,Sy_{n-1}) + \alpha_{n-1}\rho(Tz,Ty_{n-1}) \\
&\le (1-\alpha_{n-1})\rho(p,Sy_{n-1}) + \alpha_{n-1}[\varphi_2(\rho(Sz,Tz)) + \psi(\rho(Sz,Sy_{n-1})] \\
&\le (1-\alpha_{n-1})\rho(p,Sy_{n-1}) + \alpha_{n-1}\psi(\rho(p,Sy_{n-1}) \\
&\dots \\
&\le (1-\alpha_0)\rho(p,Sy_0) + \alpha_0\psi^{n-1}(\rho(p,Sy_0) \to \infty \ as\ n \to \infty
\end{aligned}$$

So,

$$\begin{aligned}
d(Sy_{n+1},p) &\le c^2(1-\alpha_n)d(Sy_n,p) + \\
&c\alpha_n c_1[(1-\alpha_0)\rho(p,Sy_0) + \alpha_0\psi^{n-1}(\rho(p,Sy_0))] + c\varepsilon_n
\end{aligned}$$

Taking limit on both sides, we get $\lim_{n\to\infty} Sy_n = p.$

Conversely, let $\lim_{n\to\infty} Sy_n = p,$ then

$$\begin{aligned}
\varepsilon_n &= d(Sy_{n+1},(1-\alpha_n)Sy_n + \alpha_n Ty_n) \\
&\le cd(Sy_{n+1},p) + cd(p,(1-\alpha_n)Sy_n + \alpha_n Ty_n) \\
&\le cd(Sy_{n+1},p) + c(1-\alpha_n)d(p,Sy_n) + c\alpha_n d(Tz,Ty_n) \\
&\le cd(Sy_{n+1},p) + c(1-\alpha_n)d(p,Sy_n) + \\
&c\alpha_n[\varphi_1(\rho(Sz,Tz) + c_1\rho(Sz,Sy_n)] \\
&\le cd(Sy_{n+1},p) + c(1-\alpha_n)d(p,Sy_n) + c\alpha_n c_1\rho(p,Sy_n) \\
&\le cd(Sy_{n+1},p) + c(1-\alpha_n)d(p,Sy_n) + \\
&c\alpha_n c_1[(1-\alpha_0)\rho(p,Sy_0) + \alpha_0\psi^{n-1}(\rho(p,Sy_0)] \to 0 \\
&as\ n \to \infty.
\end{aligned}$$

Theorem 4.2. Let $S,T : Y \to X$ such that $T(Y) \subseteq S(Y)$ and *S*(*Y*) or *T*(*Y*) is a complete subspace of *X*. Let d and ρ be two *b*-metric on *Y*. Let *z* be a coincidence point of *T* and *S*. Let $x_0 \in Y$ and the sequence $\{Sx_n\}$ generated by J-Ishikawa iteration converges to *p*. Let $\{Sy_n\} \subset X$ and define

$$\varepsilon_n = d(Sy_{n+1},(1-\alpha_n)Sy_n + \alpha_n Ts_n),\ n \ge 0$$

and

$$Ss_n = (1-\beta_n)y_n + \beta_n Ty_n,\ n \ge 0 .$$

Suppose that

(i) there exists $c_1 > 0$, and a monotone increasing function $\varphi_1 : R_+ \to R_+$ with $\varphi_1(0) = 0$ such that

$$d(Tx,Ty) \le \varphi_1(\rho(Sx,Tx)) + c_1\rho(Sx,Sy),\ \forall x,y \in Y$$

(ii) $T:(Y,\rho) \to (X,\rho)$ satisfies the contractive condition

$$\rho(Tx,Ty) \le \varphi_2(\rho(Sx,Tx)) + \psi(\rho(Sx,Sy)),\ \forall x,y \in Y$$

where $\psi^k : R_+ \to R_+$, $k = 1,2,...,$ are continuous comparison functions (ψ^k is the kth iterate of ψ) and $\varphi_2 : R_+ \to R_+, K = 1,2,...,$ is a monotone increasing function such that $\varphi_2(0) = 0$. Then the Jungck-Ishikawa iteration process with $T(Y,d) \to (X,d)$ is (*S* ,*T*)-stable.

Proof. Suppose $\lim_{n\to\infty} \varepsilon_n = 0.$ Then we shall establish that $\lim_{n\to\infty} Sy_n = p$, using conditions (i), (ii) and triangle inequality.

$$\begin{aligned}
d(Sy_{n+1}, p) &\le c[d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n)) + \\
&\quad d((1-\alpha_n)Sy_n + \alpha_n Ts_n), p)] \\
&\le cd((1-\alpha_n)Sy_n + \alpha_n Ts_n), Tz) + c\varepsilon_n \\
&\le c(1-\alpha_n)d(Sy_n, Tz) + c\alpha_n d(Ts_n, Tz) + c\varepsilon_n \\
&\le c^2(1-\alpha_n)[d(Sy_n, Sz) + d(Sz, Tz)] + \\
&\quad c\alpha_n[\varphi_1(\rho(Sz, Tz) + c_1\rho(Sz, Ss_n)] + c\varepsilon_n \\
&\le c^2(1-\alpha_n)d(Sy_n, p) + c\alpha_n c_1\rho(p, Ss_n) + c\varepsilon_n
\end{aligned}$$

But,

$$\begin{aligned}
\rho(p, Ss_n) &= \rho(Tz, (1-\beta_n)Sy_n + \beta_n Ty_n) \\
&\le (1-\beta_n)\rho(Tz, Sy_n) + \beta_n\rho(Tz, Ty_n) \\
&\le (1-\beta_n)\rho(p, Sy_n) + \beta_n[\varphi_2(\rho(Sz, Tz)) \\
&\quad + \psi(\rho(Sz, Sy_n)] \\
&\le (1-\beta_n)\rho(p, Sy_n) + \beta_n\psi(\rho(p, Sy_n) \\
&\quad \ldots \\
&\le (1-\beta_0)\rho(p, Sy_0) + \beta_0\psi^n(\rho(p, Sy_0) \to \infty \\
&\quad as\ n \to \infty
\end{aligned}$$

So,

$$\begin{aligned}
d(Sy_{n+1}, p) &\le c^2(1-\alpha_n)d(Sy_n, p) + \\
&\quad c\alpha_n c_1[(1-\beta_0)\rho(p, Sy_0) + \beta_0\psi^n(\rho(p, Sy_0)] + c\varepsilon_n
\end{aligned}$$

Taking limit on both sides, we get $\lim_{n\to\infty} Sy_n = p.$

Conversely, let $\lim_{n\to\infty} Sy_n = p,$ then

$$\begin{aligned}
\varepsilon_n &= d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n) \\
&\le cd(Sy_{n+1}, p) + cd(p, (1-\alpha_n)Sy_n + \alpha_n Ts_n) \\
&\le cd(Sy_{n+1}, p) + c(1-\alpha_n)d(p, Sy_n) + c\alpha_n d(Tz, Ts_n) \\
&\le cd(Sy_{n+1}, p) + c(1-\alpha_n)d(p, Sy_n) + \\
&\quad c\alpha_n[\varphi_1(\rho(Sz, Tz) + c_1\rho(Sz, Ss_n)] \\
&\le cd(Sy_{n+1}, p) + c(1-\alpha_n)d(p, Sy_n) + c\alpha_n c_1\rho(p, Ss_n) \\
&\le cd(Sy_{n+1}, p) + c(1-\alpha_n)d(p, Sy_n) + \\
&\quad c\alpha_n c_1[(1-\beta_0)\rho(p, Sy_0) + \beta_0\psi^n(\rho(p, Sy_0)] \to 0\ as \\
&\quad n \to \infty.
\end{aligned}$$

If we set $c = \alpha_n = \beta_n = 1$, we obtained following corollary.

Corollary 4.1 [12]. Let X be a nonempty set and Y an arbitrary set. Let d and ρ two metrics on Y and $S, T : Y \to X$ such that $T(Y) \subseteq S(Y)$ and $S(Y)$ a complete subspace of X. Suppose that:

(i) S and T have a coincidence point z;
(ii) there exists $c_1 > 0$, and a monotone increasing function $\varphi_1 : R_+ \to R_+$ with $\varphi_1(0) = 0$ such that

$$d(Tx, Ty) \le \varphi_1(\rho(Sx, Tx)) + c_1\rho(Sx, Sy),\ \forall x, y \in Y$$

(iii) (X, d) is a complete metric space.
(iv) $T : (X, \rho) \to (X, \rho)$ satisfies the contractive condition

$$\rho(Tx, Ty) \le \varphi_2(\rho(Sx, Tx)) + \psi(\rho(Sx, Sy)),\ \forall x, y \in Y$$

where $\psi^k : R_+ \to R_+$, $k = 1, 2, \ldots,$ are continuous comparison functions (ψ^k is the kth iterate of ψ) and $\varphi_2 : R_+ \to R_+, K = 1, 2, \ldots,$ is a monotone increasing function such that $\varphi_2(0) = 0$. Then the Jungck Picard iteration process with $T(Y, d) \to (X, d)$ is (S, T)-stable.

By setting $\varphi_1(t) = L, \forall t \in R_+$, we obtained the following corollary.

Corollary 4.2 [12]. Let X be a nonempty set and Y an arbitrary set. Let d and ρ two metrics on Y and $S, T : Y \to X$ such that $T(Y) \subseteq S(Y)$ and $S(Y)$ a complete subspace of X. Suppose that:

(i) S and T have a coincidence point z;
(ii) there exists $c_1 > 0$, $L \ge 0$ such that

$$d(Tx, Ty) \le L\rho(Sx, Tx) + c_1\rho(Sx, Sy),\ \forall x, y \in Y$$

(iii) (X, d) is a complete metric space.
(iv) $T : (X, \rho) \to (X, \rho)$ satisfies the contractive condition

$$\rho(Tx, Ty) \le \varphi_2(\rho(Sx, Tx)) + \psi(\rho(Sx, Sy)),\ \forall x, y \in Y$$

where $\psi^k : R_+ \to R_+,\ k = 1, 2, ...,$ are continuous comparison functions (ψ^k is the kth iterate of ψ) and $\varphi_2 : R_+ \to R_+, K = 1, 2, ...,$ is a monotone increasing function such that $\varphi_2(0) = 0$. Then the Jungck Picard iteration process with $T(Y, d) \to (X, d)$ is (*S*, *T*)-stable.

Now, we illustrate the convergence of Jungck- Picard and Jungck-Ishikawa iteration scheme for the example of Singh [25].

Example 4.1 [25]. To find a root (lying between 0 and 1 indeed nearer $x > 0.5$) of the equation

$$1/2\, Sin^{-1} x - 1/2\, x (1 - x^2)^{1/2} + x^2\, Cos^{-1} x = \pi / 8$$

(or $4\,Sin^{-1} x - 4x (1 - x^2)^{1/2} + 8x^2 Cos^{-1} x = \pi$)

Rewrite it as $Sx = Tx$, where $Sx = 4\,Sin^{-1} x$ and $Tx = \pi + 4x(1 - x^2)^{1/2} - 8x^2 Cos^{-1} x$.

The following table shows the convergence of Jungck-Picard and Jungck-Ishikawa iterative procedure.

	Jungck-Ishikawa Iteration			Jungck-Picard Iteration		
n	Tx_n	Sx_{n+1}	x_{n+1}	Tx_n	Sx_{n+1}	x_{n+1}
0	2.70326	2.65167	0.61542	2.70326	2.70326	0.625533
1	2.33108	2.36314	0.557013	2.29216	2.29216	0.542189
2	2.55951	2.53987	0.593151	2.61743	2.61743	0.60865
3	2.41775	2.42996	0.570808	2.35731	2.35731	0.555802
4	2.50539	2.49784	0.58466	2.56425	2.56425	0.598047
5	2.45101	2.4557	0.57608	2.39861	2.39861	0.564355
6	2.48469	2.48179	0.581399	2.53072	2.53072	0.591307
7	2.46381	2.46561	0.578103	2.42496	2.42496	0.569782
8	2.47675	2.47563	0.580146	2.50941	2.50941	0.587004
.	.	.	.	.	.	.
.	.	.	.	.	.	.
24	2.4718	2.4718	0.579365	2.47285	2.47285	0.579579
25	2.47179	2.47179	0.579364	2.47095	2.47095	0.579193
26	2.47179	2.47179	0.579364	2.47247	2.47247	0.579501
27	2.47179	2.47179	0.579364	2.47126	2.47126	0.579255
28	2.47179	2.47179	0.579364	2.47222	2.47222	0.579452
.	.	.	.	.	.	.
.	.	.	.	.	.	.
52	2.47179	2.47179	0.579364	2.4718	2.4718	0.579365
53	2.47179	2.47179	0.579364	2.47179	2.47179	0.579364
54	2.47179	2.47179	0.579364	2.4718	2.4718	0.579365
55	2.47179	2.47179	0.579364	2.47179	2.47179	0.579364
56	2.47179	2.47179	0.579364	2.47179	2.47179	0.579364

We remark that the convergence is achieved in only 24th iteration by using Jungck-Ishikawa iterative scheme while the same is achieved at 56th iteration of Jungck Picard scheme.

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REFERENCES

[1] V. Berinde, *Iterative Approximation of Fixed Points*, Springer Verlag, Lectures Notes in Mathematics, 2007.

[2] S. Czerwik, "Nonlinear set-valued contraction mappings in b-metric spaces", Atti Sem. Mat. Fis. Univ. Modena vol. 46, no. 2, pp. 263-276, 1998.

[3] M. Dobriţoiu, W. W. Kecs and A. Toma, "The differentiability of the solution of a nonlinear integral equation", *Proceedings of the 8th WSEAS International Conference on Mathematical Methods and Computational Techniques in Electrical Engineering, Bucharest, Romania*, ISSN 1790-5117, ISBN: 960–8457–54–8, pp. 155–158, 2006.

[4] M. Dobriţoiu, W. W. Kecs and A. Toma, "An application of the fiber generalized contractions theorem", *WSEAS Transactions on Mathematics*, Issue 12, vol. 5, ISSN 1109–2769, pp. 1330–1335,2006.

[5] M. Dobriţoiu and A. M. Dobriţoiu, "A generalization of some integral equations", *WSEAS conference proceeding, Latest Trends on Computers*, ISSN: 1792-4251, ISBN: 978-960-474-201-1, pp. 182-186, 2010.

[6] M. Dobriţoiu and A. M. Dobriţoiu, "A functional-integral equation via weakly picard operators, Proceedings of the 13th WSEAS International Conference on Computers", ISSN: 1790-5109, ISBN: 978-960-474-099-4, pp. 159-162, 2009.

[7] D. Fang and X. L. Fneg, "An application of L-system and IFS in 3D fractal simulation", *WSEAS Transactions on Systems*, ISSN: 1109-2777, Issue 4, vol. 7, pp. 352-361, 2008.

[8] A. M. Harder and T. L. Hicks, "A stable iteration procedure for nonexpansive mappings", Math. Japonica. vol. 33 , pp. 687–692, 1988.

[9] A. M. Harder and T. L. Hicks, "Stability results for fixed point iteration procedures", Math. Japonica, vol. 33, pp. 693-706, 1988.

[10] C. O. Imoru and M. O. Olatinwo, "On the stability of Picard and Mann iteration processes". Carpathian J. Math., 2, pp. 155–160, 2003.

[11] G. Jungck, "Commuting mappings and fixed points", Amer.Math. Monthly, vol.. 83 no. 4, pp. 261– 263, 1976.

[12] M. O. Olatinwo, "Some stability results in complete metric space", Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica 48, pp. 83–92, 2009.

[13] M. O. Olatinwo and C. O. Imoru, "Some convergence results for the Jungck-Mann and Jungck-Ishikawa processes in the class of generalized Zamfirescu operators", Acta Math. Univ. Comenianae, vol. LXXVII, no. 2, pp. 299-304, 2008.

[14] M. O. Osilike, "Stability results for fixed point iteration procedures" J. Nigerian Math. Soc., vol. 14/15, pp. 17–29, 1995/96.

[15] M. O. Osilike, "Stability results for the Ishikawa fixed point procedures", Indian J Pure Appl Math, vol. 26, no.5, pp. 937 – 945, 1995.

[16] A. M. Ostrowski, "The round –off stability of iterations", Z.Angew. Math. Mech. vol. 47 , no. 1, pp. 77-81, 1967.

[17] B. Prasad, B. Singh and R. Sahni, "Some approximate fixed-point theorems", Int. J. Math. Anal.,vol. 3, no. 5, pp. 203-210, 2009.

[18] B. Prasad, B. Singh and R. Sahni, "Common fixed point theorems with integral inequality", Applied Mathematical Sciences, vol. 4, no. 48, pp. 2369 – 2377, 2010.

[19] B. Prasad, P. Pradhan and R. Sahni, "Approximate fixed points of some general contractions", Int. J. Math. Sci. & Engg. Appls., vol. 4, no. 3, pp. 159-163, 2010.

[20] B. Prasad and R. Sahni, "Some general iterative algorithm", *WSEAS conference proceeding, Selected Topics in Applied Computer Science*, 1SSN:1792-4863, ISBN: 978-960-474-231-8, pp. 216-221, 2010.

[21] B. Prasad and R. Sahni, "A convergence theorem for Jungck-Ishikawa iteration", *WSEAS conference proceeding, Recent Researches in Artificial Intelligence, Knowledge Engineering and Data Bases*, 1SSN:1792-8117, ISBN: 978-960-474-273-8, pp. 79-84, 2011.

[22] B. E. Rhoades, "Fixed point theorems and stability results for fixed point iteration procedures", Indian J. Pure Appl. Math., vol. 21, no. 1, pp. 1–9, 1990.

[23] B. E. Rhoades, "Fixed point theorems and stability results for fixed point iteration procedures", Indian J. Pure Appl. Math., vol. 24, no. 11, pp. 691–703, 1993.

[24] B. E. Rhoades and S. M. Soltuz, "The equivalence between the T-stabilities of Mann and Ishikawa iterates", J. Math. Anal. Appl., vol. 318, pp. 472-475, 2006.

[25] S. L. Singh, "A new approach in numerical praxix", Prog. of Maths, vol. 32, no. 2, pp. 75-89, 1998.

[26] S. L. Singh, and B. Prasad, "Some coincidence theorems and stability of iterative procedures", Comput. Math. Appl. vol. 55, no. 11, pp. 2512-2520, 2008.

[27] S. L. Singh, C. Bhatnagar and S. N. Mishra, "Stability of Jungck-type iterative procedures", Int. J. Math. Sci, vol. 2005, pp. 3035-3043, 2005.

[28] M. Urabe, "Convergence of numerical iteration in solution of equations", J. Sci. Hiroshima Univ. Sér A. 19, pp. 479-489, 1956.

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