Equilibriums and Periodic Solutions of Related Systems of Piecewise Linear Difference Equations

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Abstract**—**In this paper we consider three systems of piecewise linear difference equations where the initial condition for each system is an arbitrary point on the real plane. For one system we show that there exists exactly two prime period-6 solutions, and that every solution of the system is eventually one of the two prime period-6 solutions except for equilibrium point. For the remaining two systems we show that every solution of each system is the unique equilibrium solution.

Keywords**—**periodic solution; systems of piecewise linear difference equations.

I. Introduction

In recent history there has been a surge of interest in difference equations. Rational difference equations are used in predator-prey models [1], [2], [3], and echo and reverberation models [4]. Linear difference equations are used for modeling weather patterns [2], [5], [6] and neural networks [7]. During the last three years we have been particularly interested in the global behavior of systems of piecewise linear difference equations. This paper is part of a general project which involves the following system

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ y_{n+1} = x_n + c|y_n| + d \end{cases}, n = 0,1,\ldots \qquad (\mathbf{N})$$

where the parameters $a,b,c,d \in \{-1,0,1\}$ and initial conditions $(x_0, y_0) \in \mathbb{R}^2$. There are 81 special cases. The system's number $\mathbf{N}$ is given by

$$\mathbf{N} = 27(a+1) + 9(b+1) + 3(c+1) + (d+1) + 1.$$

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This family of piecewise linear difference equation are the prototypes for more elaborate piecewise difference equation that, in many cases, exhibit complicated behavior. Interest in the area began in 1984 when Davaney published his famous paper introducing the gingerbread man map:

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ y_{n+1} = x_n \end{cases}, \ n = 0,1,\ldots$$

with parameters $a,b \in \mathbb{R}$ and initial conditions $(x_0, y_0) \in \mathbb{R}^2$. See [8], [9].

The gingerbread man map was Devaney's response to the 1978 generalized Lozi equation:

$$\begin{cases} x_{n+1} = -a|x_n| + y_n + 1 \\ y_{n+1} = bx_n \end{cases}, \ n = 0,1,\ldots$$

with parameters $a,b \in \mathbb{R}$ and initial conditions $(x_0, y_0) \in \mathbb{R}^2$. See [2], [3], [5].

The Lozi equation had been used to examine an attractor that was observed by Lorenz in the Henon map, a non-linear system of difference equation

$$\begin{cases} x_{n+1} = -ax^2 + y_n + 1 \\ y_{n+1} = bx_n \end{cases}, \ n = 0,1,\ldots$$

with parameters $a,b \in \mathbb{R}$ and initial conditions $(x_0, y_0) \in \mathbb{R}^2$ that modeled weather patterns. See [10].

For other systems of this form, see [6], [11], [12], [13], [14], [15].

II. Problem Formulation

Reconsider this family of system

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ \qquad\qquad , n = 0,1,\ldots \\ y_{n+1} = x_n + c|y_n| + d \end{cases} \qquad (\mathbf{N})$$

We will first share the results of the special case $\mathbf{N} = 1$:

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ \qquad\qquad , n = 0,1,\ldots \\ y_{n+1} = x_n - |y_n| - 1 \end{cases} \qquad (1)$$

where the initial conditions $(x_0, y_0) \in \mathbb{R}^2$. We show that every solution of System(1) is eventually one of the following period-6 cycles:

$$P_6^1 = \begin{pmatrix} x_0 = 3, & y_0 = -3 \\ x_1 = 5, & y_1 = -1 \\ x_2 = 5, & y_2 = 3 \\ x_3 = 1, & y_3 = 1 \\ x_4 = -1, & y_4 = -1 \\ x_5 = 1, & y_5 = -3 \end{pmatrix}$$

$$P_6^2 = \begin{pmatrix} x_0 = \frac{7}{5}, & y_0 = -3 \\ x_1 = \frac{17}{5}, & y_1 = -\frac{13}{5} \\ x_2 = 5, & y_2 = -\frac{1}{5} \\ x_3 = \frac{21}{5}, & y_3 = \frac{19}{5} \\ x_4 = -\frac{3}{5}, & y_4 = -\frac{3}{5} \\ x_5 = -\frac{1}{5}, & y_5 = \frac{11}{5} \end{pmatrix}$$

or the unique equilibrium solution $(1,-1)$.

We next consider the special case when $\mathbf{N} = 20$:

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ \qquad\qquad , n = 0,1,\ldots \\ y_{n+1} = x_n - |y_n| \end{cases} \qquad (20)$$

where the initial condition $(x_0, y_0) \in \mathbb{R}^2$. We will show that every solution $\{(x_n, y_n)\}_{n=5}^{\infty}$ of System (20) is the equilibrium solution, $(2,1)$.

Lastly, we consider the special case when $\mathbf{N} = 21$:

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ \qquad\qquad , n = 0,1,\ldots \\ y_{n+1} = x_n - |y_n| + 1 \end{cases} \qquad (21)$$

where the initial condition $(x_0, y_0) \in \mathbb{R}^2$. We will show that every solution $\{(x_n, y_n)\}_{n=3}^{\infty}$ of System (21) is the equilibrium solution, $(1,1)$.

III. Result

III. A Global Behavior of System (1)

Set

$Q_1 = \{(x, y): x \geq 0, y \geq 0\}$
$Q_2 = \{(x, y): x < 0, y > 0\}$
$Q_3 = \{(x, y): x \leq 0, y \leq 0\}$
$Q_4 = \{(x, y): x > 0, y < 0\}$.

Theorem 1. Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System (1) with $(x_0, y_0) \in \mathbb{R}^2$. Then, there exists a non-negative integer $N \geq 0$ such that the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ of System (1) is either the prime period-6 cycle P_6^1 or the prime period-6 cycle P_6^2 except for equilibrium point.

The proof of Theorem 1 is a direct consequence of the following lemmas.

Lemma 2. Assume that there is a positive integer N such that $x_N = y_N \geq 0$. Then the solution of System (1), $\{(x_n, y_n)\}_{n=N+1}^{\infty}$, is the prime period-6 cycle, P_6^1.

Proof: Suppose that (x_N, y_N) satisfies the hypothesis, then

$$x_{N+1} = x_N - x_N - 1 = -1$$
$$y_{N+1} = x_N - x_N - 1 = -1.$$

Hence $(x_{N+1}, y_{N+1}) = (-1, -1) \in P_6^1$. □

Lemma 3. Assume that there is a positive integer N such that $x_N = -y_N - 2 \leq 0$ and $y_N \leq 0$. Then the solution of System (1), $\{(x_n, y_n)\}_{n=N+1}^{\infty}$, is the prime period-6 cycle, P_6^1.

Proof. Suppose that (x_N, y_N) satisfies the hypothesis, then

$$x_{N+1} = y_N + 2 - y_N - 1 = 1$$
$$y_{N+1} = -y_N - 2 + y_N - 1 = -3.$$

Hence $(x_{N+1}, y_{N+1}) = (1, -3) \in P_6^1$. □

Lemma 4. Let $L := \{(x,-3) | x \in \mathbb{R}\}$. Then every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System (1) with an initial condition in L is eventually prime period-6 solution, P_6^1 or P_6^2.

Proof. Case 1: Suppose $(x_0, y_0) \in L$ and $x_0 \geq 0$.

Then $x_1 = x_0 + 2 > 0$ and $y_1 = x_0 - 4$.

If $y_1 = x_0 - 4 \geq 0$, then $(x_3, y_3) = (-1,-1) \in P_6^1$. Suppose that $y_1 = x_0 - 4 < 0$, then $x_2 = 5$ and $y_2 = 2x_0 - 3$.

Case 1.1: Suppose $y_2 = 2x_0 - 3 \geq 0$, $\left(\text{and so } \frac{3}{2} \leq x_0 < 4\right)$, then $x_3 = -2x_0 + 7$ and $y_3 = -2x_0 + 7$.

If $x_3 = y_3 = -2x_0 + 7 \geq 0, \left(\text{and so } \frac{3}{2} \leq x_0 \leq \frac{7}{2}\right)$, then by Lemma 2, $\{(x_n, y_n)\}_{n=3}^{\infty}$ is eventually prime period-6, P_6^1.

Suppose that $x_3 = y_3 = -2x_0 + 7 < 0$, $\left(\text{also, } \frac{7}{2} < x_0 < 4\right)$.

We will prove, by mathematical induction, that for $x_0 \in \left(\frac{7}{2}, 4\right)$ the solution is eventually prime period-6. For each $n \geq 0$, let P(n) be the following statement: For $x_0 \in (a_n, b_n)$,

$$x_{6n+4} = 2^{4n+2}x_0 - \delta_n$$

$$y_{6n+4} = -2^{4n+2}x_0 + \delta_n - 2 < 0.$$

If $x_0 \in (a_n, c_n]$, then $x_{6n+4} \leq 0$. So, the solution is eventually prime period-6.

If $x_0 \in (c_n, b_n)$, then $x_{6n+4} > 0$. So,

$$x_{6n+5} = 2^{4n+3}x_0 - 2\delta_n + 1 > 0$$

$$y_{6n+5} = -3,$$

$$x_{6n+6} = 2^{4n+3}x_0 - 2\delta_n + 3 > 0$$

$$y_{6n+6} = 2^{4n+3}x_0 - 2\delta_n - 3 < 0,$$

$$x_{6n+7} = 5$$

$$y_{6n+7} = 2^{4n+4}x_0 - 4\delta_n - 1.$$

If $x_0 \in [b_{n+1}, b_n)$, then $y_{6n+7} \geq 0$. So, the solution is eventually prime period-6, P_6^1.

If $x_0 \in (c_n, b_{n+1})$, then $y_{6n+7} < 0$. So

$$x_{6n+8} = -2^{4n+4}x_0 + 4\delta_n + 5 > 0$$

$$y_{6n+8} = 2^{4n+4}x_0 - 4\delta_n + 3 > 0,$$

$$x_{6n+9} = -2^{4n+5}x_0 + 8\delta_n + 1$$

$$y_{6n+9} = -2^{4n+5}x_0 + 8\delta_n + 1.$$

If $x_0 \in (c_n, a_{n+1}]$, then $x_{6n+9} = y_{6n+9} \geq 0$. So, $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually prime period-6 solution, P_6^1.

If $x_0 \in (a_{n+1}, b_{n+1})$, then

$x_{6n+9} = y_{6n+9} = -2^{4n+5}x_0 + 8\delta_n + 1 < 0$, where

$$a_n = \frac{19 \times 2^{4n+1} - 3}{5 \times 2^{4n+1}}, b_n = \frac{19 \times 2^{4n} + 1}{5 \times 2^{4n}},$$

$$c_n = \frac{19 \times 2^{4n+2} - 1}{5 \times 2^{4n+2}}, \delta_n = \frac{19 \times 2^{4n+2} - 1}{5}.$$

We shall show that P(0) is true . For $x_0 \in (a_0, b_0) = \left(\frac{7}{2}, 4\right)$ and

$x_3 = y_3 = -2x_0 + 7 < 0$. Thus, we have

$$x_{6(0)+4} = x_4 = 4x_0 - 15 = 2^{4(0)+2}x_0 - \delta_0,$$

$$x_{6(0)+4} = y_4 = -4x_0 + 13$$

$$= -2^{4(0)+2}x_0 + \delta_0 - 2 < 0.$$

If $x_0 \in (a_0, c_0] = \left(\frac{7}{2}, \frac{15}{4}\right]$, then $x_4 = 4x_0 - 15 \leq 0$. By Lemma 3, $\{(x_n, y_n)\}_{n=4}^{\infty}$ is eventually prime period-6 solution P_6^1.

If $x_0 \in (c_0, b_0) = \left(\frac{15}{4}, 4\right)$, then $x_4 = 4x_0 - 15 > 0$. Thus, we have

$$x_{6(0)+5} = 8x_0 - 29 = 2^{4(0)+3}x_0 - 2\delta_0 + 1 > 0$$

$$y_{6(0)+5} = -3,$$

$$x_{6(0)+6} = 8x_0 - 27 = 2^{4(0)+3}x_0 - 2\delta_0 + 3 > 0$$

$$y_{6(0)+6} = 8x_0 - 33 = 2^{4(0)+3}x_0 - 2\delta_0 - 3 < 0,$$

$$x_{6(0)+7} = 5$$

$$y_{6(0)+7} = 16x_0 - 61 = 2^{4(0)+4}x_0 - 4\delta_0 - 1.$$

If $x_0 \in (b_1, b_0] = \left[\frac{61}{16}, 4\right)$, then $x_{6(0)+7} = 16x_0 - 61 \geq 0$, so $x_{6(0)+8} = y_{6(0)+8} = -16x_0 + 65 > 0$. By Lemma 2, $\{(x_n, y_n)\}_{n=9}^{\infty}$ is prime period-6.

If $y_0 \in (c_0, b_1) = \left(\frac{15}{4}, \frac{61}{16}\right)$, then $x_{6(0)+7} = 16x_0 - 61 < 0.$ Thus, we have

$$x_{6(0)+8} = -16x_0 + 65 = -2^{4(0)+4}x_0 + 4\delta_0 + 5 > 0$$

$$y_{6(0)+8} = 16x_0 - 57 = 2^{4(0)+4}x_0 - 4\delta_0 + 3 > 0,$$

$$x_{6(0)+9} = -32x_0 + 121 = -2^{4(0)+5}x_0 + 8\delta_0 + 1$$

$$y_{6(0)+9} = -32x_0 + 121 = -2^{4(0)+5}x_0 + 8\delta_0 + 1.$$

If $y_0 \in (c_0, a_1] = \left(\frac{15}{4}, \frac{121}{32}\right]$. Then $x_{6(0)+9} = y_{6(0)+9} - 32x_0 + 121 \geq 0$. So, by Lemma 2 $\{(x_n, y_n)\}_{n=10}^{\infty}$ is eventually prime period-6, P_6^1.

If $y_0 \in (a_1, b_1) = \left(\frac{121}{32}, \frac{61}{16}\right)$, then

$$x_{6(0)+9} = y_{6(0)+9} = -32y_0 + 121 < 0.$$

Hence P(0) is true.

Next, we assume that P(N) is true for some positive integer $N \geq 1$. We shall show that P(N+1) is true. Since P(N) is true, $x_{6N+9} = y_{6N+9} = -2^{4N+5}x_0 + 8\delta_N + 1 < 0,$ where $x_0 \in (a_{N+1}, b_{N+1}) = \left(\frac{19 \times 2^{4N+5} - 3}{5 \times 2^{4N+5}}, \frac{19 \times 2^{4N+4} + 1}{5 \times 2^{4N+4}}\right).$

Then

$$\begin{aligned} x_{6(N+1)+4} &= x_{6N+10} = 2^{4N+6}x_0 - 16\delta_N - 3 \\ &= 2^{4(N+1)+2}x_0 - \delta_{N+1} \end{aligned}$$

$$\begin{aligned} y_{6(N+1)+4} &= y_{6N+10} = -2^{4N+6}x_0 + 16\delta_N + 1 \\ &= -2^{4(N+1)+2}x_0 + \delta_{N+1} - 2 \\ &= -2^{4N+6}x_0 + \left(\frac{19 \times 2^{4N+6} - 11}{5}\right) < 0. \end{aligned}$$

Note that

$$\begin{aligned} \delta_{N+1} &= \frac{19 \times 2^{4N+6} - 1}{5} = \frac{19 \times 2^{4N+6} - 16}{5} + \frac{15}{5} \\ &= 16\delta_N + 3. \end{aligned}$$

If

$$x_0 \in (a_{N+1}, c_{N+1}] = \left(\frac{19 \times 2^{4N+5} - 3}{5 \times 2^{4N+5}}, \frac{19 \times 2^{4N+6} - 1}{5 \times 2^{4N+6}}\right],$$

then $x_{6N+10} = 2^{4N+6}x_0 - \left(\frac{19 \times 2^{4N+6} - 1}{5}\right) \leq 0.$

By Lemma 3, $\{(x_n, y_n)\}_{n=6N+10}^{\infty}$ is eventually prime period-6, P_6^1.

If $x_0 \in (c_{N+1}, b_{N+1}) = \left(\frac{19 \times 2^{4N+6} - 1}{5 \times 2^{4N+6}}, \frac{19 \times 2^{4N+4} + 1}{5 \times 2^{4N+4}}\right),$

then $x_{6N+10} = 2^{4N+6}x_0 - \left(\frac{19 \times 2^{4N+6} - 1}{5}\right) > 0.$

Thus, we have

$$\begin{aligned} x_{6(N+1)+5} &= x_{6N+11} \\ &= 2^{4(N+1)+3}x_0 - 2\delta_{N+1} + 1 > 0 \end{aligned}$$

$$y_{6(N+1)+5} = x_{6N+11} = -3,$$

$$\begin{aligned} x_{6(N+1)+6} &= x_{6N+12} \\ &= 2^{4(N+1)+3}x_0 - 2\delta_{N+1} + 3 > 0 \end{aligned}$$

$$\begin{aligned} y_{6(N+1)+6} &= y_{6N+12} \\ &= 2^{4(N+1)+3}x_0 - 2\delta_{N+1} - 3 \\ &= 2^{4N+7}x_0 - \left(\frac{19 \times 2^{4N+7} + 13}{5}\right) < 0, \end{aligned}$$

$$x_{6(N+1)+7} = x_{6N+13} = 5$$

$$y_{6(N+1)+7} = y_{6N+13} = 2^{4(N+1)+4} x_0 - 4\delta_{N+1} - 1.$$

If

$$y_0 \in \left[b_{N+2}, b_{N+1}\right) = \left[\frac{19\times 2^{4N+8}+1}{5\times 2^{4N+8}}, \frac{19\times 2^{4N+4}+1}{5\times 2^{4N+4}}\right),$$

then

$$y_{6N+13} = 2^{4(N+1)+4} x_0 - 4\delta_{N+1} - 1 = 2^{4N+8} x_0 - \left(\frac{19\times 2^{4N+8}+1}{5}\right) \geq 0,$$

$$x_{6N+14} = -2^{4N+8} x_0 + 4\delta_{N+1} + 5 = -2^{4N+8} x_0 + \left(\frac{19\times 2^{4N+8}+21}{5}\right) > 0$$

$y_{6N+14} = -2^{4N+8} x_0 + 4\delta_{N+1} + 5 > 0.$ By Lemma 2, $\left\{(x_n, y_n)\right\}_{n=6N+14}^{\infty}$ is eventually prime period-6.

If

$$x_0 \in \left(c_{N+1}, b_{N+2}\right) = \left(\frac{19\times 2^{4N+6}-1}{5\times 2^{4N+6}}, \frac{19\times 2^{4N+8}+1}{5\times 2^{4N+8}}\right),$$

then

$$y_{6N+13} = 2^{4(N+1)+4} x_0 - 4\delta_{N+1} - 1 = 2^{4N+8} x_0 - \left(\frac{19\times 2^{4N+8}+1}{5}\right) < 0.$$

Thus, we have

$$x_{6(N+1)+8} = x_{6N+14} = -2^{4(N+1)+4} x_0 + 4\delta_{N+1} + 5 > 0$$

$$y_{6(N+1)+8} = y_{6N+14} = 2^{4(N+1)+4} x_0 - 4\delta_{N+1} + 3 = 2^{4N+8} x_0 - \left(\frac{19\times 2^{4N+8}-19}{5}\right) > 0,$$

$$x_{6(N+1)+9} = x_{6N+15} = -2^{4(N+1)+5} x_0 + 8\delta_{N+1} + 1$$

$$y_{6(N+1)+9} = y_{6N+15} = -2^{4(N+1)+5} x_0 + 8\delta_{N+1} + 1.$$

If

$$x_0 \in \left(c_{N+1}, a_{N+2}\right] = \left(\frac{19\times 2^{4N+6}-1}{5\times 2^{4N+6}}, \frac{19\times 2^{4N+9}-3}{5\times 2^{4N+9}}\right],$$

then

$$x_{6N+15} = y_{6N+15} = -2^{4(N+1)+5} x_0 + 8\delta_{N+1} + 1 = -2^{4N+9} + \left(\frac{19\times 2^{4N+9}-3}{5}\right) \geq 0.$$

By Lemma 2, $\left\{(x_n, y_n)\right\}_{n=6N+15}^{\infty}$ is eventually prime period-6, P_6^1.

If

$$x_0 \in \left(a_{N+2}, b_{N+2}\right) = \left(\frac{19\times 2^{4N+9}-3}{5\times 2^{4N+9}}, \frac{19\times 2^{4N+8}+1}{5\times 2^{4N+8}}\right),$$

then

$$x_{6N+15} = y_{6N+15} = -2^{4(N+1)+5} x_0 + 8\delta_{N+1} + 1 = -2^{4N+9} x_0 + \left(\frac{19\times 2^{4N+9}-3}{5}\right) < 0$$

Hence, P(N+1) is true. By mathematical induction, P(n) is true for all $n \geq 0$.

Note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \frac{19}{5}$.

We also note that if $\left(x_0, y_0\right) = \left(\frac{19}{5}, -3\right)$, then $\left(x_3, y_3\right) = \left(-\frac{3}{5}, -\frac{3}{5}\right) \in P_6^2$.

Case 1.2: Suppose $y_2 = 2x_0 - 3 < 0, \left(\text{and so } 0 \leq x_0 < \frac{3}{2}\right)$, then $x_4 = -4x_0 + 5$ and $y_4 = -4x_0 + 5$.

If $x_0 \in \left(0, \frac{5}{4}\right]$, then $x_4 = y_4 = -4x_0 + 5 \geq 0$. So we apply Lemma 2 and $\left\{(x_n, y_n)\right\}_{n=4}^{\infty}$ is eventually prime period-6, P_6^1.

Suppose that $x_0 \in \left(\frac{5}{4}, \frac{3}{2}\right)$. We will prove, for $x_0 \in \left(\frac{5}{4}, \frac{3}{2}\right)$, that the solution is eventually prime period-6 by mathematical induction. For each $n \geq 0$, let Q(n) be the following statement: For $x_0 \in \left(d_n, e_n\right)$,

$$x_{6n+5} = 2^{4n+3} x_0 - \sigma_n$$

$$y_{6n+5} = -2^{4n+3} x_0 + \sigma_n - 2 < 0.$$

If $x_0 \in (d_n, f_n]$, then $x_{6n+5} \le 0$. So, $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually prime period-6, P_6^1.

If $x_0 \in (f_n, e_n)$, then $x_{6n+5} > 0$. So

$x_{6n+6} = 2^{4n+4} x_0 - 2\sigma_n + 1 > 0$

$y_{6n+6} = -3,$

$x_{6n+7} = 2^{4n+4} x_0 - 2\sigma_n + 3 > 0$

$y_{6n+7} = 2^{4n+4} x_0 - 2\sigma_n - 3 < 0,$

$x_{6n+8} = 5$

$y_{6n+8} = 2^{4n+5} x_0 - 4\sigma_n - 1.$

If $x_0 \in [e_{n+1}, e_n)$, then $y_{6n+8} \ge 0$, So the solution is eventually prime period-6, P_6^1.

If $x_0 \in (f_n, e_{n+1})$, then $y_{6n+8} < 0$. Thus, we have

$x_{6n+9} = -2^{4n+5} x_0 + 4\sigma_n + 5 > 0$

$y_{6n+9} = 2^{4n+5} x_0 - 4\sigma_n + 3 > 0,$

$x_{6n+10} = -2^{4n+6} x_0 + 8\sigma_n + 1$

$y_{6n+10} = -2^{4n+6} x_0 + 8\sigma_n + 1.$

If $x_0 \in (f_n, d_{n+1}]$, then $x_{6n+10} = y_{6n+10} \ge 0$, and so the solution is eventually prime period-6 solution, P_6^1.

If $x_0 \in (d_{n+1}, e_{n+1})$, then $x_{6n+10} = y_{6n+10} < 0$, where

$$d_n = \frac{7\times 2^{4n+2} - 3}{5\times 2^{4n+2}},\quad e_n = \frac{7\times 2^{4n+1} + 1}{5\times 2^{4n+1}},$$

$$f_n = \frac{7\times 2^{4n+3} - 1}{5\times 2^{4n+3}},\quad \sigma_n = \frac{7\times 2^{4n+3} - 1}{5}.$$

The proof is similar to the previous case. We can conclude that $Q(n)$ is true for all $n \ge 0$.

Note that $\lim_{n\to\infty} d_n = \lim_{n\to\infty} e_n = \lim_{n\to\infty} f_n = \frac{7}{5}$.

We also note that $\left(\frac{7}{5}, -3\right) \in P_6^2$.

Case 2: Let $(x_0, y_0) \in L$ and $x_0 < 0$. Then $y_0 = -3$. Thus, $x_2 = -2x_0 + 5 > 0$ and $y_2 = -3$.

We see that $(x_2, y_2) \in L$ and $x_2 > 0$. By Case 1, every solution $\{(x_n, y_n)\}_{n=2}^{\infty}$ is eventually prime period-6, P_6^1 or P_6^2. □

Lemma 5. Every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (1) with initial condition in Q_1 is eventually prime period-6, P_6^1 or P_6^2.

Proof. Let $(x_0, y_0) \in Q_1$ Then $x_0 \ge 0$ and $y_0 \ge 0$. Hence,

$x_1 = x_0 - y_0 - 1$

$y_1 = x_0 - y_0 - 1.$

If $x_1 = y_1 = x_0 - y_0 + 1 \ge 0$, then by Lemma 2, $(x_2, y_2) \in P_6^1$.

If $x_1 = y_1 = x_0 - y_0 + 1 < 0$, then

$x_2 = -2x_0 + 2y_0 + 1$

$y_2 = 2x_0 - 2y_0 - 3 < 0.$

If $x_2 = -2x_0 + 2y_0 + 1 = -y_2 - 2 \le 0$, then by Lemma 3, $(x_2, y_2) \in P_6^1$.

If $x_2 = -2x_0 + 2y_0 + 1 > 0$, then

$x_3 = -4x_0 + 4y_0 + 3,$

$y_3 = -3$. Hence by Lemma 4, the solution is eventually prime period-6, P_6^1 or P_6^2. □

Lemma 6. Every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System (1) with initial condition in Q_3 is eventually prime period-6, P_6^1 or P_6^2.

Proof. Let $(x_0, y_0) \in Q_3$. Then $x_0 \le 0$ and $y_0 \le 0$. Hence,

$x_1 = -x_0 - y_0 - 1$

$y_1 = x_0 + y_0 - 1 < 0,$

$x_2 = -3.$

By Lemma 4, the solution is eventually prime period-6, P_6^1 or P_6^2. □

Lemma 7. Every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System (1) with an initial condition in Q_2 is eventually prime period-6 solution period-6, P_6^1 or P_6^2.

Proof. Let $(x_0, y_0) \in Q_2$. Then $x_0 < 0$ and $y_0 < 0$. Hence,

$x_1 = -x_0 - y_0 - 1$

$y_1 = x_0 - y_0 - 1 < 0.$

Case 1: Suppose $x_1 = -x_0 - y_0 - 1 \le 0$ then,

$x_2 = 2y_0 + 1 > 0$

$y_2 = -2y_0 - 3 < 0,$

$y_3 = -3.$

By Lemma 4, the solution is eventually prime period-6, P_6^1 or P_6^2.

Case 2: Suppose $x_1 = -x_0 - y_0 - 1 > 0$ and so $-x_0 > 1$. Then,

$x_2 = -2x_0 - 1 > 0$

$y_2 = -2y_0 - 3 < 0,$

$x_3 = -2x_0 + 2y_0 + 1 > 0$

$y_3 = -2x_0 - 2y_0 - 5.$

If $y_3 = -2x_0 - 2y_0 - 5 \ge 0$ then

$x_4 = 4y_0 + 5 > 0$

$y_4 = 4y_0 + 5 > 0.$

By Lemma 2, the solution is eventually prime period-6, P_6^1.

If $y_3 = -2x_0 - 2y_0 - 5 < 0,$ then

$x_4 = 4y_0 + 5 > 0$

$y_4 = -4x_0 - 5.$

If $y_4 = -4x_0 - 5 \le 0, \left(\text{and so} -\frac{5}{4} \le x_0 < -1\right),$ then

$x_5 = 4x_0 + 4y_0 + 9 > 0$

$y_5 = -4x_0 + 4y_0 - 1 > 0.$

By Lemma 5, the solution is eventually prime period-6, P_6^1 or P_6^2. If $y_4 = -4x_0 - 5 > 0, \left(\text{and so } x_0 < -\frac{5}{4}\right),$ then

$x_5 = 4x_0 + 4y_0 + 9$

$y_5 = 4x_0 + 4y_0 + 9.$

If $x_5 = y_5 \ge 0,$ then by Lemma 5, the solution is eventually prime period-6, P_6^1 or P_6^2.

If $x_5 = y_5 < 0,$ then by Lemma 6, the solution is eventually prime period-6, P_6^1 or P_6^2. □

Lemma 8. Every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System (1) with an initial condition in Q_4 is eventually prime period-6, P_6^1 or P_6^2.

Proof. Let $(x_0, y_0) \in Q_4$, $x_0 > 0$ and $y_0 < 0$. Then,

$x_1 = |x_0| - y_0 - 1 = x_0 - y_0 - 1$

$y_1 = x_0 - |y_0| - 1 = x_0 + y_0 - 1.$

If $(x_1, y_1) \in \mathbb{R}^2 \setminus Q_4$, we apply the above lemmas to conclude that the solution is eventually prime period-6. For the case that $x_1 = x_0 - y_0 - 1 > 0$ and $y_1 = x_0 + y_0 - 1 < 0$, then

$x_2 = |x_1| - y_1 - 1 = -2y_0 - 1$

$y_2 = x_1 - |y_1| - 1 = 2x_0 - 3.$

If $(x_2, y_2) \in \mathbb{R}^2 \setminus Q_4$, we apply the above lemmas to conclude that the solution is eventually prime period-6. For the case that $x_2 = -2y_0 - 1 > 0$ and $y_2 = 2x_0 - 3 < 0$, $\left(\text{and so } x_0 < \frac{3}{2} \text{ and } y_0 < -\frac{1}{2}\right)$, then

$x_3 = |x_2| - y_2 - 1 = -2x_0 - 2y_0 + 1$

$y_3 = x_2 - |y_2| - 1 = 2x_0 - 2y_0 - 5.$

If $(x_3, y_3) \in \mathbb{R}^2 \setminus Q_4$, we apply the above lemmas to conclude that the solution is eventually prime period-6. For the case that $x_3 = -2x_0 - 2y_0 + 1 > 0$ and $y_3 = 2x_0 - 2y_0 - 5 < 0,$ then

$x_4 = |x_3| - y_3 - 1 = -4x_0 + 5$

$y_4 = x_3 - |y_3| - 1 = -4y_0 - 5.$

If $(x_4, y_4) \in \mathbb{R}^2 \setminus Q_4$, we apply the above lemmas to conclude that the solution is eventually prime period-6. For the case that $x_4 = -4x_0 + 5 > 0$ and $y_4 = -4y_0 - 5 < 0 \left(\text{and so } x < \frac{5}{4} \text{ and } -\frac{5}{4} < y < -\frac{1}{2}\right)$, then

$x_5 = |x_4| - y_4 - 1 = -4x_0 + 4y_0 + 9$

$y_5 = x_4 - |y_4| - 1 = -4x_0 - 4y_0 - 1.$

If $(x_5, y_5) \in \mathbb{R}^2 \setminus Q_4$, we apply the above lemmas to conclude that the solution is eventually prime period-6. For the case that $x_5 = -4x_0 + 4y_0 + 9 > 0$ and $y_5 = -4x_0 - 4y_0 - 1 < 0$, then

$$x_6 = |x_5| - y_5 - 1 = 8y_0 + 9$$
$$y_6 = x_5 - |y_5| - 1 = -8x_0 - 7.$$

If $(x_6, y_6) \in \mathbb{R}^2 \setminus Q_4$, we apply the above lemmas to conclude that the solution is eventually prime period-6. For the case that $x_6 = 8y_0 + 9 > 0$ and $y_6 = -8x_0 + 7 < 0$, $\left(\text{and so } \frac{7}{8} < x_0 < \frac{5}{4} \text{ and } -\frac{9}{8} < y_0 < -\frac{1}{2}\right)$.

We will prove that for $x_0 \in \left(\frac{7}{8}, \frac{5}{4}\right)$ and $y_0 \in \left(-\frac{9}{8}, -\frac{1}{2}\right)$ the solution of System (1) is eventually prime period-6, P_6^1 or P_6^2 by Mathematical induction.

For each $n \geq 0$, let P(n) be the following statement: For $x_0 \in (a_n, b_n)$ and $y_0 \in (c_n, d_n)$

$$x_{8n+7} = 2^{4n+3} x_n + 2^{4n+3} y_0 + 1$$
$$y_{8n+7} = -2^{4n+3} x_n + 2^{4n+3} y_0 + \delta_n,$$

such that x_{8n+7} is positive and y_{8n+7} is negative. Otherwise, the solution is eventually prime period-6, P_6^1 or P_6^2. If x_{8n+7} is positive and y_{8n+7} is negative, then

$$x_{8n+8} = 2^{4n+4} x_0 - \delta_n$$
$$y_{8n+8} = 2^{4n+4} y_0 + \delta_n,$$

such that x_{8n+8} is positive and y_{8n+8} is negative when $x_0 \in (e_n, b_n)$ and $y_n \in (c_n, -e_n)$. Otherwise, the solution is eventually prime period-6, P_6^1 or P_6^2.

If x_{8n+8} is positive and y_{8n+8} is negative, then

$$x_{8n+9} = 2^{4n+4} x_0 - 2^{4n+4} y_0 - (2\delta_n + 1)$$
$$y_{8n+9} = 2^{4n+4} x_0 + 2^{4n+4} y_0 - 1,$$

such that x_{8n+9} is positive and y_{8n+9} is negative. Otherwise, the solution is eventually prime period-6, P_6^1 or P_6^2. If x_{8n+9} is positive and y_{8n+9} is negative, then

$$x_{8n+10} = -2^{4n+5} y_0 - (2\delta_n + 1)$$
$$y_{8n+10} = 2^{4n+5} x_0 - (2\delta_n + 3),$$

such that x_{8n+10} is positive and y_{8n+10} is negative when $x_0 \in (e_n, f_n)$ and $y_0 \in (c_n, d_{n+1})$. Otherwise, the solution is eventually prime period-6, P_6^1 or P_6^2.

If x_{8n+10} is positive and y_{8n+10} is negative, then

$$x_{8n+11} = -2^{4n+5} x_0 - 2^{4n+5} y_0 + 1$$
$$y_{8n+11} = 2^{4n+5} x_0 - 2^{4n+5} y_0 - (4\delta_n + 5),$$

such that x_{8n+11} is positive and y_{8n+11} is negative. Otherwise, the solution is eventually prime period-6, P_6^1 or P_6^2.

If x_{8n+11} is positive and y_{8n+11} is negative, then

$$x_{8n+12} = -2^{4n+6} x_0 + (4\delta_n + 5)$$
$$y_{8n+12} = -2^{4n+6} y_0 - (4\delta_n + 5),$$

such that x_{8n+12} is positive and y_{8n+12} is negative when $x_0 \in (e_n, b_{n+1})$ and $y_0 \in (-b_{n+1}, d_{n+1})$. Otherwise, the solution is eventually prime period-6, P_6^1 or P_6^2.

If x_{8n+12} is positive and y_{8n+12} is negative, then

$$x_{8n+13} = -2^{4n+6} x_0 + 2^{4n+6} y_0 + (8\delta_n + 9)$$
$$y_{8n+13} = -2^{4n+6} x_0 - 2^{4n+6} y_0 - 1.$$

such that x_{8n+13} is positive and y_{8n+13} is negative. Otherwise, the solution is eventually prime period-6, P_6^1 or P_6^2.

If x_{8n+13} is positive and y_{8n+13} is negative, then

$$x_{8n+14} = 2^{4n+7} y_0 + (8\delta_n + 9)$$
$$y_{8n+14} = -2^{4n+7} x_0 + (8\delta_n + 7).$$

such that x_{8n+14} is positive and y_{8n+14} is negative when $x_0 \in (a_{n+1}, b_{n+1})$ and $y_0 \in (c_{n+1}, d_{n+1})$. Otherwise, the solution is eventually prime period-6, P_6^1 or P_6^2 where

$a_n = \frac{2^{4n+3}-1}{2^{4n+3}}, b_n = \frac{2^{4n+2}+1}{2^{4n+2}}, c_n = \frac{-2^{4n+3}-1}{2^{4n+3}},$

$d_n = \frac{-2^{4n+3}+1}{2^{4n+3}}, e_n = \frac{2^{4n+4}-1}{2^{4n+4}}, f_n = \frac{2^{4n+5}+1}{2^{4n+5}},$ and

$\delta_n = 2^{4n+4} - 1.$

We shall show that P(0) is true. For $x_0 \in (a_0, b_0) = \left(\frac{7}{8}, \frac{5}{4}\right)$, $y_0 \in (c_0, d_0) = \left(-\frac{9}{8}, -\frac{1}{2}\right)$ and $x_6 = 8y_0 + 9 > 0$, $y_6 = -8x_0 + 7 < 0$. Then

$$x_7 = 8x_0 + 8y_0 + 1 = 2^{4(0)+3} x_0 + 2^{4(0)+3} y_0 + 1$$

$$y_7 = -8x_0 + 8y_0 + 15 = -2^{4(0)+3} x_0 + 2^{4(0)+3} y_0 + \delta_1,$$

where x_7 is positive and y_7 is negative. Otherwise, we apply the above lemmas to conclude that the solution is eventually a prime period-6.

If x_7 is positive and y_7 is negative, then

$$x_8 = 16x_0 - 15 = 2^{4(0)+4} x_0 + \delta_1$$

$$y_8 = 16y_0 + 15 = 2^{4(0)+4} y_0 + \delta_1,$$

where x_8 is positive and y_8 is negative when $x_0 \in (e_0, b_0)$ and $y_n \in (c_0, -e_0)$. Otherwise, we apply the above lemmas to conclude that the solution is eventually a prime period-6.

If x_8 is positive and y_8 is negative is negative, then

$$x_9 = 16x_0 - 16y_0 - 31 = 2^{4(0)+4} x_0 - 2^{4(0)+4} y_0 - (2\delta_1 + 1)$$

$$y_9 = 16x_0 + 16y_0 - 1 = 2^{4(0)+4} x_0 + 2^{4(0)+4} y_0 - 1,$$

where x_9 is positive and y_9 is negative. Otherwise, we apply the above lemmas to conclude that the solution is eventually prime period-6 solution.

If x_9 is positive and y_9 is negative, then

$$x_{10} = -32y_0 - 31 = -2^{4(0)+5} y_0 - (2\delta_1 + 1)$$

$$y_{10} = 32x_0 - 33 = 2^{4(0)+5} x_0 - (2\delta_1 + 3),$$

where x_{10} is positive and y_{10} is negative when $x_0 \in (e_0, f_0)$ and $y_n \in (c_0, d_1)$. Otherwise, we apply the above lemmas to conclude that the solution is eventually prime period-6.

If x_{10} is positive and y_{10} is negative, then

$$x_{11} = -32x_0 - 32y_0 + 1 = -2^{4(0)+5} x_0 - 2^{4(0)+5} y_0 + 1$$

$$y_{11} = 32x_0 - 32y_0 - 65 = 2^{4(0)+5} x_0 - 2^{4(0)+5} y_0 - (4\delta_1 + 5),$$

where x_{11} is positive and y_{11} is negative. Otherwise, we apply the above lemmas to conclude that the solution is eventually prime period-6.

If x_{11} is positive and y_{11} is negative, then

$$x_{12} = -64x_0 + 65 = -2^{4(0)+6} x_0 + (4\delta_0 + 5)$$

$$y_{12} = -64y_0 - 65 = -2^{4(0)+6} y_0 - (4\delta_0 + 5),$$

where x_{12} is positive and y_{12} is negative when $x_0 \in (e_0, b_1)$ and $y_n \in (-b_1, d_1)$. Otherwise, we apply the above lemmas to conclude that the solution is eventually prime period-6.

If x_{12} is positive and y_{12} is negative, then

$$x_{13} = -64x_0 + 64y_0 + 129 = -2^{4(0)+6} x_0 + 2^{4(0)+6} y_0 + (8\delta_0 + 9)$$

$$y_{13} = -64x_0 - 64y_0 - 1 = -2^{4(0)+6} x_0 - 2^{4(0)+6} y_0 - 1,$$

where x_{13} is positive and y_{13} is negative. Otherwise, we apply the above lemmas to conclude that the solution is eventually prime period-6.

If x_{13} is positive and y_{13} is negative, then

$$x_{14} = 128y_0 + 129 = 2^{4(0)+7} y_0 + (8\delta_0 + 9)$$

$$y_{14} = -128x_0 + 127 = -2^{4(0)+7} x_0 + (8\delta_0 + 7),$$

where x_{14} is positive and y_{14} is negative when $x_0 \in (a_1, b_1)$ and $y_n \in (c_1, d_1)$. Otherwise, we apply the above lemmas to conclude that the solution is eventually prime period-6. Hence P(0) is true.

Next, we assume that P(N) is true for some $N \in \mathbb{N}$. We shall show that P(N+1) is true.

Since P(N) is true, for

$$x_0 \in (a_{N+1}, b_{N+1}) = \left(\frac{2^{4N+7}-1}{2^{4N+7}}, \frac{2^{4N+6}+1}{2^{4N+6}} \right),$$

$$y_0 \in (c_{N+1}, d_{N+1}) = \left(\frac{-2^{4N+7}-1}{2^{4N+7}}, \frac{-2^{4N+7}+1}{2^{4N+3}} \right),$$

$x_{8n+14} = 2^{4n+7} y_0 + (8\delta_n + 9) > 0$, and

$y_{8n+14} = -2^{4n+7} x_0 + (8\delta_n + 7) < 0$. Then

$$x_{8(N+1)+7} = x_{8N+15} = 2^{4N+7} x_0 + 2^{4N+7} y_0 + 1 = 2^{4(N+1)+3} x_0 + 2^{4(N+1)+3} y_0 + 1,$$

$$y_{8(N+1)+7} = y_{8N+15} = -2^{4N+7} x_0 + 2^{4N+7} y_0 + (16\delta_N + 15) = -2^{4(N+1)+3} x_0 + 2^{4(N+1)+3} y_0 + \delta_{N+1}.$$

If $\left(x_{8(N+1)+7}, y_{8(N+1)+7}\right) \in \mathbb{R}^2 \setminus Q_4$, then we apply the above lemmas to conclude that the solution is eventually prime period-6. For the case that

$\left(x_{8(N+1)+7}, y_{8(N+1)+7}\right) \in Q_4$, then

$$x_{8(N+1)+8} = x_{8N+16} = 2^{4(N+1)+4} x_0 - \delta_{N+1}$$

$$y_{8(N+1)+8} = y_{8N+16} = 2^{4(N+1)+4} y_0 + \delta_{N+1}.$$

If $\left(x_{8(N+1)+7}, y_{8(N+1)+7}\right) \in \mathbb{R}^2 \setminus Q_4$, then we apply the above lemmas to conclude that the solution is eventually prime period-6. The case when

$$x_0 \in \left(e_{N+1}, b_{N+1}\right) = \left(\frac{2^{4N+8}-1}{2^{4N+8}}, \frac{2^{4N+6}+1}{2^{4N+6}}\right), \text{ and}$$

$$y_0 \in \left(c_{N+1}, -e_{N+1}\right) = \left(\frac{-2^{4N+7}-1}{2^{4N+7}}, \frac{-2^{4N+8}+1}{2^{4N+8}}\right), \text{ we}$$

have $x_{8(N+1)+8} > 0$ and $y_{8(N+1)+8} < 0$, then

$$x_{8(N+1)+9} = 2^{4(N+1)+4} x_0 - 2^{4(N+1)+4} y_0 - \left(2\delta_{N+1} + 1\right)$$

$$y_{8(N+1)+9} = 2^{4(N+1)+4} x_0 + 2^{4(N+1)+4} y_0 - 1.$$

If $\left(x_{8(N+1)+9}, y_{8(N+1)+9}\right) \in \mathbb{R}^2 \setminus Q_4$, then we apply the above lemmas to conclude that the solution is eventually prime period-6.

If $\left(x_{8(N+1)+9}, y_{8(N+1)+9}\right) \in Q_4$, then

$$x_{8(N+1)+10} = -2^{4(N+1)+5} y_0 - \left(2\delta_{N+1} + 1\right)$$

$$y_{8(N+1)+10} = 2^{4(N+1)+5} x_0 - \left(2\delta_{N+1} + 3\right).$$

If $\left(x_{8(N+1)+10}, y_{8(N+1)+10}\right) \in \mathbb{R}^2 \setminus Q_4$, then we apply the above lemmas to conclude that the solution is eventually prime period-6. The case when

$$x_0 \in \left(e_{N+1}, f_{N+1}\right) = \left(\frac{2^{4N+8}-1}{2^{4N+8}}, \frac{2^{4N+9}+1}{2^{4N+9}}\right), \text{ and}$$

$$y_0 \in \left(c_{N+1}, d_{N+1}\right) = \left(\frac{-2^{4N+7}-1}{2^{4N+7}}, \frac{-2^{4N+11}+1}{2^{4N+11}}\right), \text{ we}$$

have $x_{8(N+1)+10} > 0$ and $y_{8(N+1)+10} < 0$. Then

$$x_{8(N+1)+11} = -2^{4(N+1)+5} x_0 - 2^{4(N+1)+5} y_0 + 1$$

$$y_{8(N+1)+11} = 2^{4(N+1)+5} x_0 - 2^{4(N+1)+5} y_0 - \left(4\delta_{N+1} + 5\right)$$

If $\left(x_{8(N+1)+11}, y_{8(N+1)+11}\right) \in \mathbb{R}^2 \setminus Q_4$, then we apply the above lemmas to conclude that the solution is eventually prime period-6.

If $\left(x_{8(N+1)+11}, y_{8(N+1)+11}\right) \in Q_4$, then

$$x_{8(N+1)+12} = -2^{4(N+1)+6} x_0 + \left(4\delta_{N+1} + 5\right)$$

$$y_{8(N+1)+12} = -2^{4(N+1)+6} y_0 - \left(4\delta_{N+1} + 5\right).$$

If $\left(x_{8(N+1)+12}, y_{8(N+1)+12}\right) \in \mathbb{R}^2 \setminus Q_4$, then we apply the above lemmas to conclude that the solution is eventually prime period-6. The case when

$$x_0 \in \left(e_{N+1}, b_{N+2}\right) = \left(\frac{2^{4N+8}-1}{2^{4N+8}}, \frac{2^{4N+10}+1}{2^{4N+10}}\right), \text{ and}$$

$$y_0 \in \left(-b_{N+2}, d_{N+2}\right) = \left(\frac{-2^{4N+10}-1}{2^{4N+10}}, \frac{-2^{4N+11}+1}{2^{4N+11}}\right),$$

we have $x_{8(N+1)+12} > 0$ and $y_{8(N+1)+12} < 0$. Then

$$x_{8(N+1)+13} = -2^{4(N+1)+6} x_0 + 2^{4(N+1)+6} y_0 + \left(8\delta_{N+1} + 9\right)$$

$$y_{8(N+1)+13} = -2^{4(N+1)+6} x_0 - 2^{4(N+1)+6} y_0 - 1$$

If $\left(x_{8(N+1)+13}, y_{8(N+1)+13}\right) \in \mathbb{R}^2 \setminus Q_4$, then we apply the above lemmas to conclude that the solution is eventually prime period-6.

If $\left(x_{8(N+1)+13}, y_{8(N+1)+13}\right) \in Q_4$, then

$$x_{8(N+1)+14} = 2^{4(N+1)+7} y_0 + \left(8\delta_{N+1} + 9\right)$$

$$y_{8(N+1)+14} = -2^{4(N+1)+7} x_0 + \left(8\delta_{N+1} + 7\right).$$

If $\left(x_{8(N+1)+14}, y_{8(N+1)+14}\right) \in \mathbb{R}^2 \setminus Q_4$, then we apply the above lemmas to conclude that the solution is eventually prime period-6. The case when

$$x_0 \in \left(a_{N+2}, b_{N+2}\right) = \left(\frac{2^{4N+11}-1}{2^{4N+11}}, \frac{2^{4N+10}+1}{2^{4N+10}}\right),$$

$$y_0 \in \left(c_{N+2}, d_{N+2}\right) = \left(\frac{-2^{4N+11}-1}{2^{4N+11}}, \frac{-2^{4N+11}+1}{2^{4N+11}}\right), \text{ we}$$

have $x_{8(N+1)+14} > 0$ and $y_{8(N+1)+14} < 0$.

Hence $\mathrm{P}(N+1)$ is true. By mathematical induction $\mathrm{P}(n)$ is true for all $n \geq 0$.

Note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} f_n = 1$$

and $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = \lim_{n\to\infty} -b_n = \lim_{n\to\infty} -e_n = -1$.

The proof is complete. □

III. B GLOBAL BEHAVIOR OF SYSTEM (20)

In this sub-section, we consider the system:

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ y_{n+1} = x_n - |y_n| \end{cases}, \quad n = 0,1,\ldots \tag{20}$$

where the initial condition $(x_0, y_0) \in \mathbb{R}^2$. We will show that every solution $\{(x_n, y_n)\}_{n=5}^{\infty}$ of System (20) is the equilibrium solution, $(2,1)$.

Theorem 2. Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be the solution of System (20).

Then $\{(x_n, y_n)\}_{n=5}^{\infty}$ is equilibrium point (2, 1).

Proof. Suppose $x_0, y_0 \in \mathbb{R}$.

We will first show that $x_2 \ge |y_2|$. (20.1)

By the triangle inequality

$$\begin{aligned} |y_1| - |x_1| &\le |y_1 - x_1| \\ &= \big|x_0 - |y_0| - (|x_0| - y_0 + 1)\big| \\ &= \big||x_0| - x_0 + |y_0| - y_0 + 1\big| \\ &= |x_0| - x_0 + |y_0| - y_0 + 1 \\ &= x_1 - y_1. \end{aligned}$$

We see that

$$|y_1| + y_1 \le x_1 + |x_1|.$$

So

$$|x_1| + x_1 \ge |y_1| + y_1 - 1.$$

Thus

$$|x_1| - y_1 + 1 \ge -x_1 + |y_1|.$$

We have $x_2 \ge -y_2$.

It is clear that for any $x_1, y_1 \in \mathbb{R}$.

$|x_1| - x_1 + |y_1| - y_1 \ge 0$, and so

$-x_1 + |y_1| \ge y_1 - |x_1|$, which mean

$x_1 - |y_1| \le |x_1| - y_1 + 1$, that is

$y_2 \le x_2$.

Since $x_2 \ge -y_2$ and $x_2 \ge y_2$, we know $x_2 \ge |y_2|$, as required.

Next we show that $x_3 \ge y_3 \ge 0$. (20.2)

By condition (20.1), we know

$$y_3 = x_2 - |y_2| \ge 0.$$

We also know that $x_2 \ge 0$ and $x_2 \ge y_2$.

So $x_3 = |x_2| - y_2 + 1$

$= x_2 - y_2 + 1 \ge 0.$

Clearly, $x_3 \ge y_3 \ge 0$ as require.

Next we show that $x_4 \ge y_4 + 1$ and $y_4 > 0.$ (20.3)

By condition (20.2), we know

$$x_4 = |x_3| - y_3 + 1 = x_3 - y_3 + 1 \ge 0$$

$$y_4 = x_3 - |y_3| = x_3 - y_3 \ge 0.$$

So $x_4 = y_4 + 1 \ge 0$, as require.

Finally, we show that $x_5 = 2$ and $y_5 = 1$.

By condition (20.2), we know

$$x_5 = |x_4| - y_4 + 1 = (y_4 + 1) - y_4 + 1 = 2$$

$$y_5 = x_4 - |y_4| = y_4 + 1 - y_4 = 1,$$

and the proof is complete. □

III.C GLOBAL BEHAVIOR OF SYSTEM (21)

In this sub-section, we consider the system:

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, \quad n = 0,1,\ldots \tag{21}$$

where the initial condition $(x_0, y_0) \in \mathbb{R}^2$. We will show that every solution $\{(x_n, y_n)\}_{n=5}^{\infty}$ of System (21) is the equilibrium solution, $(1,1)$.

Theorem 3. Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be the solution of System (21). Then $\{(x_n, y_n)\}_{n=5}^{\infty}$ is the equilibrium point (1, 1).

Proof. Suppose $x_0, y_0 \in \mathbb{R}$.

We will first show that

$$|x_1| + x_1 + 3 \ge |y_1| + y_1. \tag{21.1}$$

By the triangle inequality

$$\begin{aligned}|y_1| - |x_1| &\le |y_1 - x_1| \\ &= \left|x_0 - |y_0| + 1 - |x_0| + y_0 - 1\right| \\ &= \left|x_0 - |x_0| + y_0 - |y_0|\right| \\ &= \left||x_0| - x_0 + |y_0| - y_0\right| \\ &= |x_0| - x_0 + |y_0| - y_0.\end{aligned}$$

This implies that

$$\left|x_0 - |y_0| + 1\right| - \left||x_0| - y_0 + 1\right| \le |x_0| - x_0 + |y_0| - y_0 + 3,$$

and so

$$\left||x_0| - y_0 + 1\right| + |x_0| - y_0 + 1 + 3 \ge \left|x_0 - |y_0| + 1\right| + x_0 - |y_0| + 1.$$

Hence $|x_1| + x_1 + 3 \ge |y_1| + y_1$, as required.

Next we show that $|y_2| \le x_2 + 1$. (21.2)

Clearly, for any $x, y \in \mathbb{R}$,

$|x_1| - x_1 + |y_1| - y_1 \ge 0$, and so

$x_1 - |y_1| + 1 \le |x_1| - y_1 + 2.$

By condition (21.1), we have

$x_1 - |y_1| + 1 \ge -|x_1| + y_1 - 2.$

Hence $\left|x_1 - |y_1| + 1\right| \le |x_1| - y_1 + 2$ and so

$|y_2| \le x_2 + 1$, as required.

Next we show that $x_3 \ge y_3 \ge 0.$ (21.3)

By condition (21.2), we have

$|y_2| \le x_1 + 1$ and so

$y_2 \le |y_2| \le x_2 + 1 \le |x_2| + 1.$

Then $x_3 = |x_2| - y_2 + 1 \ge 0$

$y_3 = x_2 - |y_2| + 1 \ge 0.$

For any $x_2, y_2 \in \mathbb{R}$ we have

$|x_2| - x_2 + |y_2| - y_2 \ge 0.$

So we know

$|x_2| - y_2 \ge x_2 - |y_2|$ and so

$x_3 = |x_2| - y_2 + 1 \ge x_2 - |y_2| + 1 = y_3.$

Hence $x_3 \ge y_3 \ge 0$, as required. □

Next we show that $x_4 \ge y_4 \ge 0.$ (21.4)

By condition (21.3), we have

$x_4 = |x_3| - y_3 + 1 = x_3 - y_3 + 1 \ge 0$

$y_4 = x_3 - |y_3| + 1 = x_3 - y_3 + 1 \ge 0.$

Hence $x_4 = y_4 \ge 0$, as required.

Finally, we will show that $x_5 = y_5 = 1.$

By condition (21.4), we have

$x_5 = |x_4| - y_4 + 1 = x_4 - y_4 + 1 = 1$

$y_5 = x_4 - |y_4| + 1 = x_4 - y_4 + 1 = 1.$

Hence $(x_5, y_5) = (1, 1)$, the equilibrium solution.

IV. Conclusion and Discussion

We utilized mathematical induction, and direct computations to show that every solution of System (1) is eventually either one of the two prime period-6 solution or the equilibrium solution, and that every solution of System (20) and System (21) is the unique equilibrium solution.

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