The Numerical Computation of Block Method for Direct Solution of Third Order Ordinary Differential Equations

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Abstract— The computation of an accurate block method with step-length of eight for solving third order ordinary differential equations is discussed in this article. The method is developed via interpolation and collocation approach. The use of approximated power series as an interpolation polynomial and its third derivative as a collocation equation are adopted in developing the method. The new method is zero-stable, consistent, convergent and of order nine with small error constants. In addition, the method is applied as a block to third order initial value problems and the results generated are found better in accuracy when comparison was made with some existing methods.

Keywords— Power series. Interpolation, Block method. Third order initial value problems.

I. INTRODUCTION

Mathematical models in the field of science and engineering are usually developed to help in the understanding of physical phenomena. Some of these models always resulted to differential equations of the form

\[ y''' = f(x, y, y', y'') \]

as an approximate solution to the general third order of the form (1) where \( k = 8 \) is the step-length. The first, second and third derivatives of (2) are as follows

\[ y'(x) = \sum_{j=1}^{k+1} ja_j x^{j-1} \]

\[ y''(x) = \sum_{j=2}^{k} j(j-1)a_j x^{j-2} \]

\[ y'''(x) = \sum_{j=3}^{k} j(j-1)(j-2)a_j x^{j-3} = f(x, y, y', y'') \]

Equation (1) is conventionally solved by a reduction to a system of first order ordinary differential equations (ODEs) and then suitable numerical method for first order would be used to solve the system ([1], [3], [4], [6], and [11]). This method computes the numerical solution at one point at a time. However, the major setbacks for this method are computational burden which affects the accuracy of the Sch method in terms of error. In addition, the computer program to examine the accuracy of the method is always found to be complicated ([2], [7]).

In order to overcome the challenges in reduction method, scholars such as Kuboye and Omar [7], Mohammed [8], Olabode [9] and Omar [10] developed block methods for direct solution of higher order ODEs where the accuracy of the methods is better than when it is reduced to system of first order ODEs.

This paper brings improvement on numerical methods by developing new block method with step-length of eight for solving equation (1) without going through the process of reduction.

II. DERIVATION OF THE METHOD

We assume power series of the form

\[ y(x) = \sum_{j=0}^{k+1} a_j x^j \]

as an approximate solution to the general third order of the form (1) where \( k = 8 \) is the step-length. The first, second and third derivatives of (2) are as follows

\[ y'(x) = \sum_{j=1}^{k+1} ja_j x^{j-1} \]

\[ y''(x) = \sum_{j=2}^{k} j(j-1)a_j x^{j-2} \]

\[ y'''(x) = \sum_{j=3}^{k} j(j-1)(j-2)a_j x^{j-3} = f(x, y, y', y'') \]

Equation (2) is interpolated at \( x = x_{n+i}, i = 4(1)6 \) and (5) is collocated at \( x = x_{n+i}, i = 0(1)8 \).

As a result, we get

\[ AX = B \]

where
shown in APPENDIX 1.

In order to find the values of \( a \)'s in (6), Gaussian elimination method is employed and thereafter substituted into (2) to produce a continuous implicit scheme of the form

\[
y(x) = \sum_{j=0}^{\frac{k-1}{2}} \alpha_j(x) y_{x+j} + h^3 \sum_{j=0}^{k-1} \beta_j(x) f(x+j)
\]

where \( x = zh + x_n + 7h \)

Substituting (8) into (7) and simplifying, we have

\[
\alpha_j(z) = 1 + \frac{z^2}{2} + \frac{3z}{2}
\]

\[
\alpha_j(z) = -3 - 4z - z^2
\]

\[
\alpha_j(z) = 3 + \frac{5z}{2} + \frac{z^2}{2}
\]

\[
\beta_j(z) = \frac{1}{798336000} (218460 + 3551z + 53669z^2
\]

\[
-59400z^4 - 34452z^5 + 2310z^6 + 8382z^7
\]

\[
+3300z^8 + 605z^9 + 55z^{10} + 2z^{11}).
\]

\[
\beta_j(z) = \frac{1}{798336000} (-107844 - 56816z - 250041z^2
\]

\[
+277200z^3 + 158136z^4 - 12474z^5 - 38808z^6
\]

\[
-14850z^7 - 26405z^8 - 231z^{10} - 8z^{11}).
\]

\[
\beta_j(z) = \frac{1}{798336000} (-489984 - 8036z + 104660z^2
\]

\[
-116424z^3 - 64864z^4 + 61908z^5 + 16090z^6
\]

\[
+59235z^7 + 10120z^8 + 847z^9 + 28z^{10}).
\]

\[
\beta_j(z) = \frac{1}{798336000} (1370028 + 2060020z - 2599817z^2
\]

\[
+2910600z^3 + 156340z^4 - 18803z^5
\]

\[
-39309z^6 - 136950z^7 - 22220z^8 - 177z^{10}
\]

\[-56z^{11}])
\]

\[
\beta_j(z) = \frac{1}{798336000} (-2343990 - 358285z + 4450215z^2
\]

\[
-485100z^3 - 2443980z^4 + 395010z^5 + 624690z^7
\]

\[
+200475z^8 + 30525z^9 + 2310z^{10} + 70z^{11}).
\]

\[
\beta_j(z) = \frac{1}{798336000} (22270908 + 29694952z + 4271333z^2
\]

\[
+5821200z^3 + 254496z^5 - 621390z^6 - 66237z^7
\]

\[
-189750z^8 - 26840z^9 - 1925z^{10} - 56z^{11}.
\]

\]

\[
\beta_j(z) = \frac{1}{399168000} (18775416 + 40666312z + 26010886z^2
\]

\[
-5821200z^3 - 1380456z^4 + 675444z^5 + 453948z^6
\]

\[
+113025z^7 + 14740z^8 + 1001z^{10} + 28z^{11}).
\]

\[
\beta_j(z) = \frac{1}{399168000} (226644 + 3056996z + 7115229z^2
\]

\[
+6652800z^3 + 2649240z^4 + 8448z^5 - 419958z^6
\]

\[
-181368z^7 - 38610z^8 - 4620z^9 - 297z^{10} - 8z^{11}).
\]

\[
\beta_j(z) = \frac{1}{399168000} (21714 - 139405z - 308891z^2 + 415800z^4
\]

\[
+431244z^5 + 216678z^6 + 63822z^7 + 11550z^8
\]

\[
+1265z^9 + 77z^{10} - 2z^{11}).
\]

The first and second derivatives of (9) are given below

\[
\alpha_j'(z) = z + \frac{3}{2}
\]

\[
\alpha_j'(z) = -4 - 2z
\]

\[
\alpha_j'(z) = z + \frac{5}{2}
\]

\[
\beta_j'(z) = \frac{1}{399168000} (3551 + 107338z - 237600z^3 - 172260z^4
\]

\[
+13860z^5 + 58674z^6 + 26400z^7 + 5445z^8 + 550z^9
\]

\[
+22z^{10}).
\]

\[
\beta_j'(z) = \frac{1}{399168000} (-56816 - 500082z + 1108800z^3 + 790680z^4
\]

\[
-74844z^5 - 27165z^6 - 118800z^7 - 23760z^8 - 2310z^9
\]

\[-8z^{10}).
\]

\[
\beta_j'(z) = \frac{1}{399168000} (-8036 + 2093212z - 4656960z^3 - 3243240z^4
\]

\[
+371448z^5 + 1126356z^6 + 473880z^7 + 91080z^8 + 8470z^9
\]

\[
+308z^{10}).
\]

\[
\beta_j'(z) = \frac{1}{399168000} (206020 - 519634z + 1164240z^3
\]

\[
+7817040z^4 - 1128204z^5 - 2751672z^6
\]

\[
-1095600z^7 - 199980z^8 - 17710z^9 - 616z^{10}).
\]

\[
\beta_j'(z) = \frac{1}{399168000} (-358285 + 8900430z - 19404000z^3
\]

\[
-12219900z^4 + 2370060z^5 + 4372830z^6
\]

\[
+1603800z^7 + 274725z^8 + 23100z^9 + 770z^{10}).
\]

\[
\beta_j'(z) = \frac{1}{399168000} (29694952 + 8542666z + 23284800z^3
\]

\[
+12723480z^4 - 372340z^5 - 4636632z^6
\]

\[
-1518000z^7 - 241560z^8 - 19250z^9 - 616z^{10}).
\]

\[
\beta_j'(z) = \frac{1}{399168000} (40666312 + 52021772z - 23284800z^3
\]

\[
-6902280z^4 + 4052664z^5 + 3177636z^6 + 9042000z^7
\]

\[
+13266z^8 + 10010z^9 + 308z^{10}).
\]

\[
\beta_j'(z) = \frac{1}{399168000} (3056996 + 14230458z + 19958400z^3
\]

\[
+10596960z^4 + 42240z^5 - 2519748z^6 - 1269576z^7
\]

\[-30880z^8 - 41580z^9 - 88z^{10}).
\]
\[ \beta'_0(z) = \frac{1}{79833600} (-139405 - 617782z + 1663200z^3 \]
\[ +2156220z^4 + 1300068z^5 + 446754z^6 + 92400z^7 + 11385z^8 + 770z^9 - 22z^{10}) \]
\[ \alpha'_0(z) = 1 \]
\[ \alpha''_0(z) = 1 \]
\[ \beta''_0(z) = \frac{1}{79833600} (107338 - 712800z - 689040z^2 + 69300z^3 + 352044z^4 + 184800z^5 + 43560z^6 + 4950z^7 + 220z^8). \]
\[ \beta''_0(z) = \frac{1}{39916800} (-500082 + 3326400z - 3162720z^2 - 374220z^3 - 1629936z^4 - 831600z^5 - 190080z^6 - 20790z^7 - 880z^8). \]
\[ \beta''_0(z) = \frac{1}{39916800} (-5199634 + 34927200z - 3168610z^2 - 5641020z^3 - 16510032z^4 - 7669200z^5 - 1599840z^6 - 159390z^7 - 6160z^8). \]
\[ \beta''_0(z) = \frac{1}{39916800} (8900430 - 58212000z - 4887960z^2 + 11850300z^3 + 26236980z^4 + 11226600z^5 + 2197800z^6 + 207900z^7 + 770z^8). \]
\[ \beta''_0(z) = \frac{1}{39916800} (8542666 + 69854400z + 5089320z^2 - 18641700z^3 - 27819792z^4 - 10626000z^5 - 1932480z^6 - 173250z^7 - 6160z^8). \]
\[ \beta''_0(z) = \frac{1}{39916800} (5201772 - 69854400z - 27609120z^2 + 20263320z^3 + 19065816z^4 + 6329400z^5 + 1061280z^6 + 90090z^7 + 3080z^8). \]
\[ \beta''_0(z) = \frac{1}{39916800} (14230458 + 39916800z + 31790880z^2 + 168960z^3 + 12598740z^4 - 7617456z^5 - 2162160z^6 - 332640z^7 - 26730z^8 - 880z^9). \]
\[ \beta''_0(z) = \frac{1}{79833600} (-617782 + 4989600z + 8624880z^2 + 6500340z^3 + 2680524z^4 + 646800z^5 + 91080z^6 + 6930z^7 - 220z^8). \]

Equation (9) is evaluated at the non-interpolation points, i.e. \( z = -7, -6, -5, -4, 0 \) and 1 to produce the discrete schemes while equations (10) and (11) are evaluated at all the grid points, i.e. \( z = -7, -6, -5, -4, -3, -2, -1, 0, 1 \) to give the derivatives of the discrete schemes. The schemes and its derivatives are combined in a matrix form and matrix inversion is applied. This yields the block of the form

\[ A'Y_{N+1} = hC'Y_N + h^2D'Y_N^0 + h^3(E'F_{N+1} + E^0F_N) \]

where
\[ Y_{N+1} = [y_{n+1,1}, y_{n+1,2}, y_{n+1,3}, y_{n+1,4}, y_{n+1,5}, y_{n+1,6}, y_{n+1,7}]^T, \]
\[ Y_N = [y_{n,1}, y_{n,2}, y_{n,3}, y_{n,4}, y_{n,5}, y_{n,6}, y_{n,7}]^T, \]
\[ Y'_N = [y'_{n,1}, y'_{n,2}, y'_{n,3}, y'_{n,4}, y'_{n,5}, y'_{n,6}, y'_{n,7}]^T, \]
\[ Y''_N = [y''_{n,1}, y''_{n,2}, y''_{n,3}, y''_{n,4}, y''_{n,5}, y''_{n,6}, y''_{n,7}]^T, \]
\[ F_{N+1} = [f_{n+1,1}, f_{n+1,2}, f_{n+1,3}, f_{n+1,4}, f_{n+1,5}, f_{n+1,6}, f_{n+1,7}]^T, \]
\[ F_N = [f_{n,1}, f_{n,2}, f_{n,3}, f_{n,4}, f_{n,5}, f_{n,6}, f_{n,7}]^T. \]
\[
D' = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 9 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} & 8 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{2} & 5 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & 0 \\
\end{pmatrix}
\]

\[
E' = \begin{pmatrix}
409 & -483 & 674 & -298 & 353 & -106 & 338 & -165 \\
2408 & 2060 & 2285 & 1107 & 2081 & 1515 & -1249 & 19867 & 621 & 89141 \\
393 & -4423 & 2812 & -488 & 976 & 2816 & 5764 & 15283 \\
269 & 2977 & 1495 & 285 & 857 & -553 & 1077 & -173 \\
2479 & -752 & 3550 & -899 & 319 & 499 & 3997 & 902 \\
7827 & -1192 & 6471 & -2368 & 168 & 770 & 5764 & 15283 \\
890 & 247 & 703 & 297 & 667 & 957 & 2573 & 4661 \\
7571 & -247 & 6471 & -2368 & 168 & 770 & 5764 & 15283 \\
510 & 770 & 4339 & -1730 & 3324 & -2691 & 459 & -999 \\
6183 & -1183 & 3001 & 1730 & 3324 & -2691 & 459 & -999 \\
275 & 2085 & 8236 & -2999 & 3337 & 327 & 1162 & 5209 \\
5740 & 2770 & 120 & 103 & 269 & 550 & 385 & 3700 \\
181 & 567 & 227 & 138 & 236 & 331 & 780 & 13423 \\
15098 & -843 & 5426 & -17315 & 10721 & -4728 & 2425 & -313 \\
355 & -76 & 109 & 661 & 411 & 685 & 914 & 1374 \\
\end{pmatrix}
\]

\[
E'' = \begin{pmatrix}
37 & 408 & 617 & 1341 & 259 & 232 & 4410 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

The first and second derivatives of (12) yields

\[
y''_{v,3} = y''_{a} + 3h \frac{y''_{a}}{89600} (-9135f_{a,8} + 17784f_{a,7}) \\
- 73128f_{a,8} + 177264f_{a,7} = -281430f_{a,4} + 315000f_{a,3} \\
- 150624f_{a,2} + 328608f_{a,1} + 71661f_{a,0}. \\
y''_{v,4} = y''_{a} + 4h \frac{y''_{a}}{28350} (-836f_{a,8} + 7680f_{a,7}) \\
- 31552f_{a,8} + 76288f_{a,7} - 118440f_{a,6} \\
+ 160256f_{a,5} - 46400f_{a,2} + 148992f_{a,1} \\
+ 30812f_{a,0}. \\
y''_{v,5} = y''_{a} + 5h \frac{y''_{a}}{290304} (-10875f_{a,9} + 100000f_{a,8}) \\
- 412000f_{a,8} + 1020600f_{a,7} - 1283750f_{a,6} \\
+ 2294000f_{a,5} - 465000f_{a,4} + 1987000f_{a,3} \\
+ 398825f_{a,2}. \\
y''_{v,6} = y''_{a} + 6h \frac{y''_{a}}{1400} (-63f_{a,9} + 576f_{a,8} - 2268f_{a,7}) \\
+ 7200f_{a,9} - 6390f_{a,8} + 14208f_{a,7} - 2196f_{a,6} \\
+ 11808f_{a,5} - 2325f_{a,4}. \\
y''_{v,7} = y''_{a} + 7h \frac{y''_{a}}{1036800} (-8183f_{a,9} + 84168f_{a,8}) \\
- 145432f_{a,9} + 1009792f_{a,8} - 689430f_{a,7} + 1830248f_{a,6} \\
- 225008f_{a,5} + 1484112f_{a,4} + 288533f_{a,3}. \\
y''_{v,8} = y''_{a} + 8h \frac{y''_{a}}{28350} (47104f_{a,9} - 14848f_{a,8}) \\
+ 251904f_{a,9} - 145280f_{a,8} + 419840f_{a,7} \\
- 44544f_{a,6} + 329728f_{a,5} + 63296f_{a,4}. \\
y''_{v,9} = y''_{a} + 9h \frac{y''_{a}}{1069200} (-10004f_{a,9} + 92186f_{a,8}) \\
- 380447f_{a,9} + 927046f_{a,8} - 1482974f_{a,7} \\
+ 1648632f_{a,6} - 1356711f_{a,5} + 1316197f_{a,4} + 315273f_{a,3}. \\
y''_{v,10} = y''_{a} + 10h \frac{y''_{a}}{1134400} (-833f_{a,9} + 7624f_{a,8}) \\
- 31154f_{a,9} + 74728f_{a,8} - 116120f_{a,7} + 120088f_{a,6} - 42494f_{a,5} + 182584f_{a,4} + 32377f_{a,3}. \\
y''_{v,11} = y''_{a} + 11h \frac{y''_{a}}{448000} (-369f_{a,9} + 3402f_{a,8} - 14062f_{a,7}) \\
+ 34434f_{a,9} - 56160f_{a,8} + 79934f_{a,7} + 3438f_{a,6} + 70902f_{a,5} + 12881f_{a,4}. \\
y''_{v,12} = y''_{a} + 12h \frac{y''_{a}}{28350} (-214f_{a,9} + 1952f_{a,8} - 7912f_{a,7}) \\
+ 18464f_{a,9} - 18160f_{a,8} + 65504f_{a,7} + 488f_{a,6} + 45152f_{a,5} + 8126f_{a,4}. \\
y''_{v,13} = y''_{a} + 13h \frac{y''_{a}}{145152} (-1225f_{a,9} + 11450f_{a,8} - 49150f_{a,7}) \\
+ 170930f_{a,9} - 4000f_{a,8} + 318350f_{a,7} + 7550f_{a,6} + 230150f_{a,5} + 41705f_{a,4}. \\
\]
III. ESTABLISHING THE PROPERTIES OF THE BLOCK METHOD

A. Order of the Block Method

In finding the order of the block (12), \( y \) and \( f \)-functions are expanded about \( x \) using Taylor series (refer to \( J \) in APPENDIX 1).

Hence, comparing the coefficients of \( m_h \) and \( m_n \), the block has order \((9,9,9,9,9,9,9,9)\) with error constants 

\[
\begin{bmatrix}
1 & 113 & 155 & 80 & 316 & 403 \\
630939 & 111313 & 6179 & 1701 & 4176 & 3623 \\
854 & 374 & 5555 & 1847
\end{bmatrix}
\]

B. Zero-stability of the Block Method

In order to find the zero-stability of the block (12), we only put into consideration the coefficients of \( y \)-function according to Lambert [6]. That is,

\[
\rho(r) = \det[rA' - B'] = 0
\]

where \( A' \) and \( B' \) are the coefficients of \( y_{v1}, i = 1(1)8 \) and \( y_n \) in (12). This gives

\[
\det[rA^{(0)} - A^{(1)}] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \neq 0
\]

which implies \( r = 0,0,0,0,0,0,0,1 \). Hence, the method is zero stable.

C. Consistency of the Block Method

The block method (12) is consistent because it is having an order greater than one. Hence, it is convergent since it is zero-stable and consistent ([5]).

D. Region of Absolute Stability of the Block Method

The method known as boundary locus method proposed by Lambert [6] and Henrici [5] is adopted in finding the region of absolute stability of the block (12). This is given as

\[
\tilde{h}(r) = \frac{\ell(r)}{\sigma(r)}
\]

where \( \ell(r) \) and \( \sigma(r) \) are first and second characteristics polynomials respectively. The test problem of the form \( y'' = \lambda^2 y \) is substituted into the block (12).

Substituting \( r = \cos \theta + i \sin \theta \), simplifying and equating the imaginary part to zero. This gives the interval of absolute stability to be \((0, 531.18)\). This is shown in the diagram below.

IV. TEST PROBLEMS FOR THIRD ORDER ODES

The accuracy of the new block method is examined by solving third order ODEs displayed below. The same differential problems the existing methods solved are also considered for the purpose of comparison in terms of error. The results are shown in APPENDIX 2.

Problem 1:

\[
y'' = 2y' - 4, \quad y(0) = 1, y'(0) = 2, y''(0) = 6, \quad 0 \leq x \leq 1
\]

Exact Solution: \( y(x) = x^2 + e^{2x} \)

Problem 2:

\[
y'' = 8y' - 3y - 4e^x, \quad y(0) = 2, y'(0) = -2, y''(0) = 10, \quad 0 \leq x \leq 1
\]

Exact Solution: \( y(x) = e^x + xe^{-3x} \)

Problem 3:

\[
y'' = e^x, \quad y(0) = 3, y'(0) = 1, y''(0) = 5, h = 0.1 \quad 0 \leq x \leq 1
\]

Exact solution: \( y(x) = 2 + 2x^2 + e^x \)

The following notations are used in Tables 1 and 2.
S2PEB  Sequential implementation of the 2-point explicit block method
P2PEB  Parallel implementation of the 2-point explicit block method
S3PEB  Sequential implementation of the 3-point explicit block method
P3PEB  Parallel implementation of the 3-point explicit block method

V. CONCLUSION

This paper contains the development of block methods with step-length \( k = 8 \) using interpolation and collocation technique for the solution of third order initial value problems of ODEs. The properties of the methods are analyzed. The numerical results generated when the method was applied to third order problems are compared with the numerical methods of step-lengths \( k = 8 \) and \( k = 7 \). The accuracy of the new method in terms of error is better than the existing methods (refer to Tables 1 – 3 in APPENDIX 2)).

REFERENCES

APPENDIX 1

\[
A = \begin{pmatrix}
1 & x_{m+4} & x_{m+4}^2 & x_{m+4}^3 & x_{m+4}^4 & x_{m+4}^5 & x_{m+4}^6 & x_{m+4}^7 & x_{m+4}^8 & x_{m+4}^9 & x_{m+4}^{10} & x_{m+4}^{11} \\
1 & x_{m+5} & x_{m+5}^2 & x_{m+5}^3 & x_{m+5}^4 & x_{m+5}^5 & x_{m+5}^6 & x_{m+5}^7 & x_{m+5}^8 & x_{m+5}^9 & x_{m+5}^{10} & x_{m+5}^{11} \\
1 & x_{m+6} & x_{m+6}^2 & x_{m+6}^3 & x_{m+6}^4 & x_{m+6}^5 & x_{m+6}^6 & x_{m+6}^7 & x_{m+6}^8 & x_{m+6}^9 & x_{m+6}^{10} & x_{m+6}^{11} \\
0 & 0 & 0 & 6 & 24x_4 & 60x_4^2 & 120x_4^3 & 210x_4^4 & 336x_4^5 & 504x_4^6 & 720x_4^7 & 990x_4^8 \\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 & 504x_{n+1}^6 & 720x_{n+1}^7 & 990x_{n+1}^8 \\
0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 & 504x_{n+2}^6 & 720x_{n+2}^7 & 990x_{n+2}^8 \\
0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & 336x_{n+3}^5 & 504x_{n+3}^6 & 720x_{n+3}^7 & 990x_{n+3}^8 \\
0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & 336x_{n+4}^5 & 504x_{n+4}^6 & 720x_{n+4}^7 & 990x_{n+4}^8 \\
0 & 0 & 0 & 6 & 24x_{n+5} & 60x_{n+5}^2 & 120x_{n+5}^3 & 210x_{n+5}^4 & 336x_{n+5}^5 & 504x_{n+5}^6 & 720x_{n+5}^7 & 990x_{n+5}^8 \\
0 & 0 & 0 & 6 & 24x_{n+6} & 60x_{n+6}^2 & 120x_{n+6}^3 & 210x_{n+6}^4 & 336x_{n+6}^5 & 504x_{n+6}^6 & 720x_{n+6}^7 & 990x_{n+6}^8 \\
0 & 0 & 0 & 6 & 24x_{n+7} & 60x_{n+7}^2 & 120x_{n+7}^3 & 210x_{n+7}^4 & 336x_{n+7}^5 & 504x_{n+7}^6 & 720x_{n+7}^7 & 990x_{n+7}^8 \\
0 & 0 & 0 & 6 & 24x_{n+8} & 60x_{n+8}^2 & 120x_{n+8}^3 & 210x_{n+8}^4 & 336x_{n+8}^5 & 504x_{n+8}^6 & 720x_{n+8}^7 & 990x_{n+8}^8 \\
\end{pmatrix}
\]

\[
J = \begin{pmatrix}
\sum_{m=0}^{\infty} \frac{h^m}{m!} y_{m} - \frac{1}{2} \sum_{m=0}^{\infty} \frac{h^{2m}}{m!} y_{m} - \frac{3619903}{39916800} h^3 y_{m} - \sum_{m=0}^{\infty} \frac{h^{2m}}{(39916800)(m!)} y_{m} & 6779886(1)^* - 9359135(2)^* + 11774146(3)^* - 1074544(4)^* + 6771082(5)^* - 2792861(6)^* + 679110(7)^* - 73886(8)^* \\
\sum_{m=0}^{\infty} \frac{(2h)^m}{m!} y_{m} - \sum_{m=0}^{\infty} \frac{(2h)^{2m}}{m!} y_{m} & 286967(1)^* - 117864(2)^* - 1067950(4)^* + 671628(5)^* - 276634(6)^* + 67196(7)^* - 7305(8)^* \\
\sum_{m=0}^{\infty} \frac{(3h)^m}{m!} y_{m} - \sum_{m=0}^{\infty} \frac{(3h)^{2m}}{m!} y_{m} & 183384(1)^* - 413756(2)^* - 699885(4)^* + 441306(5)^* - 182043(6)^* + 44262(7)^* - 4815(8)^* \\
\sum_{m=0}^{\infty} \frac{(4h)^m}{m!} y_{m} - \sum_{m=0}^{\infty} \frac{(4h)^{2m}}{m!} y_{m} & 321172(1)^* - 513198(2)^* + 137126(4)^* + 784768(5)^* - 323744(6)^* + 87820(7)^* - 8564(8)^* \\
\sum_{m=0}^{\infty} \frac{(5h)^m}{m!} y_{m} - \sum_{m=0}^{\infty} \frac{(5h)^{2m}}{m!} y_{m} & 5253125(1)^* - 20960375(2)^* + 25537250(3)^* - 19680025(4)^* + 12920250(5)^* - 5327125(6)^* + 1295750(7)^* - 1410000(8)^* \\
\sum_{m=0}^{\infty} \frac{(6h)^m}{m!} y_{m} - \sum_{m=0}^{\infty} \frac{(6h)^{2m}}{m!} y_{m} & 74034(1)^* - 190296(2)^* - 75348(5)^* + 18360(7)^* - 1998(8)^* \\
\sum_{m=0}^{\infty} \frac{(7h)^m}{m!} y_{m} - \sum_{m=0}^{\infty} \frac{(7h)^{2m}}{m!} y_{m} & 37701874(1)^* - 54639557(2)^* + 206894170(3)^* - 122270925(4)^* + 104842066(5)^* - 35782103(6)^* + 9423582(7)^* - 1020425(8)^* \\
\sum_{m=0}^{\infty} \frac{(8h)^m}{m!} y_{m} - \sum_{m=0}^{\infty} \frac{(8h)^{2m}}{m!} y_{m} & 8169624(1)^* - 216192(2)^* + 970240(3)^* - 510560(4)^* + 508416(5)^* - 134528(6)^* + 51712(7)^* - 4440(8)^* \\
\end{pmatrix}
\]
**Table 1:** The comparison of the numerical results generated from the new block method for solving Problem 1 with Omar [10] block method $K=8$ in which maximum errors were considered.

<table>
<thead>
<tr>
<th>h-values</th>
<th>New Method</th>
<th>Omar [10]</th>
<th>Number of Steps</th>
<th>Error in new Method, $K=8$</th>
<th>Error in Omar [10], $K=8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-2}$</td>
<td>8-Step Method</td>
<td>S2PEB</td>
<td>54</td>
<td>1.267345E-02</td>
<td>1.64491E+08</td>
</tr>
<tr>
<td></td>
<td></td>
<td>P2PEB</td>
<td>54</td>
<td>1.267345E-02</td>
<td>1.64491E+08</td>
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<tr>
<td></td>
<td></td>
<td>S3PEB</td>
<td>39</td>
<td>8.063762E-04</td>
<td>7.14468E+11</td>
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<td>P3PEB</td>
<td>39</td>
<td>8.063762E-04</td>
<td>7.14468E+11</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>8-Step Method</td>
<td>S2PEB</td>
<td>504</td>
<td>3.161755E-06</td>
<td>5.58385E-03</td>
</tr>
<tr>
<td></td>
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<td>P2PEB</td>
<td>504</td>
<td>3.161755E-06</td>
<td>5.58385E-03</td>
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<td>S3PEB</td>
<td>339</td>
<td>1.736923E-07</td>
<td>9.98913E-02</td>
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<td>8-Step Method</td>
<td>S2PEB</td>
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<td>2.526122E-10</td>
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<td>S3PEB</td>
<td>3339</td>
<td>1.172396E-11</td>
<td>1.53942E-03</td>
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<td>P3PEB</td>
<td>3339</td>
<td>1.172396E-11</td>
<td>1.53942E-03</td>
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<tr>
<td>$10^{-5}$</td>
<td>8-Step Method</td>
<td>S2PEB</td>
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<td>5.236188E-09</td>
<td>3.92325E-05</td>
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<td>50004</td>
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<td>33339</td>
<td>2.625740E-10</td>
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<td>33339</td>
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**Table 2:** Comparison of the numerical results generated from the new block method $K=8$ for solving Problem 2 with Omar [10] block method $K=8$ in which maximum errors were considered.

<table>
<thead>
<tr>
<th>$h$-values</th>
<th>New Method</th>
<th>Omar [10]</th>
<th>Number of Steps</th>
<th>Error in new Method, $K=8$</th>
<th>Error in Omar [10], $K=8$</th>
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<tr>
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<td>S2PEB 54</td>
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<td>2.70995E-01</td>
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<tr>
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<td>P2PEB 54</td>
<td>5.058909E-02</td>
<td>2.70995E-01</td>
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<tr>
<td></td>
<td></td>
<td>S3PEB 39</td>
<td>3.866111E-03</td>
<td>8.01861E-01</td>
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<td>P3PEB 39</td>
<td>3.866111E-03</td>
<td>8.01861E-01</td>
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<tr>
<td>$10^{-3}$</td>
<td>8-Step Method</td>
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<td>1.19154E-02</td>
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<td>P2PEB 504</td>
<td>3.565494E-07</td>
<td>1.19154E-02</td>
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<tr>
<td></td>
<td></td>
<td>S3PEB 339</td>
<td>1.674995E-08</td>
<td>1.25684E-02</td>
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<td>P3PEB 339</td>
<td>1.674995E-08</td>
<td>1.25684E-02</td>
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<tr>
<td>$10^{-4}$</td>
<td>8-Step Method</td>
<td>S2PEB 5004</td>
<td>1.723066E-11</td>
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<tr>
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<td>P2PEB 5004</td>
<td>1.723066E-11</td>
<td>1.17030E-04</td>
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<td>P3PEB 3339</td>
<td>2.646772E-13</td>
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<td>8-Step Method</td>
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<td>P3PEB 33339</td>
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**Table 3:** Comparison of the numerical results generated from the new block method $K=8$ for solving Problem 3 with Kuboye and Omar [7] block method $K=7$.

<table>
<thead>
<tr>
<th>X</th>
<th>Exact Solution</th>
<th>Computed Solution</th>
<th>Error in New Method</th>
<th>Error in Kuboye and Omar [7]</th>
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<tbody>
<tr>
<td>0.1</td>
<td>3.125170918075647700</td>
<td>3.125170918075645500</td>
<td>2.220446E-15</td>
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<tr>
<td>0.2</td>
<td>3.301402758160169700</td>
<td>3.301402758160155500</td>
<td>1.421085E-14</td>
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<td>0.3</td>
<td>3.529858807576003300</td>
<td>3.529858807575966900</td>
<td>3.641532E-14</td>
<td>4.023448E-13</td>
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<tr>
<td>0.4</td>
<td>3.811824697641270600</td>
<td>3.811824697641202200</td>
<td>6.838974E-14</td>
<td>7.536194E-13</td>
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<tr>
<td>0.5</td>
<td>4.1487212707000128200</td>
<td>4.148721270700018900</td>
<td>1.092459E-13</td>
<td>1.212364E-12</td>
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<tr>
<td>0.6</td>
<td>4.542118800390509700</td>
<td>4.542118800390349000</td>
<td>1.607603E-13</td>
<td>1.780798E-12</td>
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<tr>
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<td>4.993752707470256300</td>
<td>2.211564E-13</td>
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<tr>
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<td>6.079603111156580500</td>
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<tr>
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<td>5.861978E-13</td>
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<tr>
<td>1.2</td>
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<td>8.200116922735819700</td>
<td>7.283063E-13</td>
<td>1.815206E-10</td>
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</tbody>
</table>