The Numerical Computation of Block Method for Direct Solution of Third Order Ordinary Differential Equations

[1] J. O. Kuboye and [2] Z. Omar

[1, 2,] Department of Mathematics, School of Quantitative Sciences, Universiti Utara Malaysia, Malaysia

Abstract— The computation of an accurate block method with step-length of eight for solving third order ordinary differential equations is discussed in this article. The method is developed via interpolation and collocation approach. The use of approximated power series as an interpolation polynomial and its third derivative as a collocation equation are adopted in developing the method. The new method is zero-stable, consistent, convergent and of order nine with small error constants. In addition, the method is applied as a block to third order initial value problems and the results generated are found better in accuracy when comparison was made with some existing methods.

Keywords— Power series. Interpolation, Block method. Third order initial value problems.

I. Introduction

Mathematical models in the field of science and engineering are usually developed to help in the understanding of physical phenomena. Some of these models always resulted to differential equations of the form

$$y''' = f(x, y, y', y''),\quad y(x_0) = y_0,\ y'(x_0) = y_1,\ y''(x_0) = y_2 \qquad (1)$$

Ross [12] highlighted some of the problems that involved differential equations as follows: the problem arising from determining the projectile motion, satellite, rocket or planet; the problem of how to determine the charge or current in an electric circuit; the study of chemical reactions and the study of decomposition rate of radioactive substance or population growth rate.

Equation (1) is conventionally solved by a reduction to a system of first order ordinary differential equations (ODEs) and then suitable numerical method for first order would be used to solve the system ([1], [3], [4], [6], and [11]). This method computes the numerical solution at one point at a time. However, the major setbacks for this method are computational burden which affects the accuracy of theSch method in terms of error. In addition, the computer program to examine the accuracy of the method is always found to be complicated ([2], [7]).

In order to overcome the challenges in reduction method, scholars such as Kuboye and Omar [7], Mohammed [8], Olabode [9] and Omar [10] developed block methods for direct solution of higher order ODEs where the accuracy of the methods is better than when it is reduced to system of first order ODEs.

This paper brings improvement on numerical methods by developing new block method with step-length of eight for solving equation (1) without going through the process of reduction.

II. Derivation of the Method

We assume power series of the form

$$y(x) = \sum_{j=0}^{k+3} a_j x^j \qquad (2)$$

as an approximate solution to the general third order of the form (1) where $k = 8$ is the step-length. The first, second and third derivatives of (2) are as follows

$$y'(x) = \sum_{j=1}^{k+3} j a_j x^{j-1} \qquad (3)$$

$$y''(x) = \sum_{j=2}^{k+3} j(j-1) a_j x^{j-2} \qquad (4)$$

$$y'''(x) = \sum_{j=3}^{k+3} j(j-1)(j-2) a_j x^{j-3} = f(x, y, y', y'') \qquad (5)$$

Equation (2) is interpolated at $x = x_{n+i}, i = 4(1)6$ and (5) is collocated at $x = x_{n+i}, i = 0(1)8$.

As a result, we get

$$AX = B \qquad (6)$$

where

$$X = \left(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}\right)^T,$$

$$B = \begin{pmatrix} y_{n+4}, y_{n+5}, y_{n+6}, f_n, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, \\ f_{n+5}, f_{n+6}, f_{n+7}, f_{n+8} \end{pmatrix}^T$$ and A is shown in APPENDIX 1.

In order to find the values of *a's* in (6), Gaussian elimination method is employed and thereafter substituted into (2) to produce a continuous implicit scheme of the form

$$y(x) = \sum_{j=4}^{k-2} \alpha_j(x) y_{n+j} + h^3 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \tag{7}$$

where $x = zh + x_n + 7h$ (8)

Substituting (8) into (7) and simplifying, we have

$$\alpha_4(z) = 1 + \frac{z^2}{2} + \frac{3z}{2}$$

$$\alpha_5(z) = -3 - 4z - z^2$$

$$\alpha_6(z) = 3 + \frac{5z}{2} + \frac{z^2}{2}$$

$$\beta_0(z) = \frac{1}{79833600}(218460 + 3551z + 53669z^2 - 59400z^4 - 34452z^5 + 2310z^6 + 8382z^7 + 3300z^8 + 605z^9 + 55z^{10} + 2z^{11}).$$

$$\beta_1(z) = \frac{1}{39916800}(107844 - 56816z - 250041z^2 + 277200z^4 + 158136z^5 - 12474z^6 - 38808z^7 - 14850z^8 - 26405z^9 - 231z^{10} - 8z^{11}).$$

$$\beta_2(z) = \frac{1}{39916800}(-489984 - 8036z + 1046606z^2 - 1164240z^4 - 648648z^5 + 61908z^6 + 160908z^7 + 59235z^8 + 10120z^9 + 847z^{10} + 28z^{11}).$$

$$\beta_3(z) = \frac{1}{39916800}(1370028 + 206020z - 2599817z^2 + 2910600z^4 + 1563408z^5 - 188034z^6 - 393096z^7 - 136950z^8 - 22220z^9 - 1771z^{10} - 56z^{11}). \tag{9}$$

$$\beta_4(z) = \frac{1}{39916800}(-2343990 - 358285z + 4450215z^2 - 4851000z^4 - 2443980z^5 + 395010z^6 + 624690z^7 + 200475z^8 + 30525z^9 + 2310z^{10} + 70z^{11}).$$

$$\beta_5(z) = \frac{1}{39916800}(22270908 + 29694952z + 4271333z^2 + 5821200z^4 + 2544696z^5 - 621390z^6 - 662376z^7 - 189750z^8 - 26840z^9 - 1925z^{10} - 56z^{11}).$$

$$\beta_6(z) = \frac{1}{39916800}(18775416 + 40666312z + 26010886z^2 - 5821200z^4 - 1380456z^5 + 675444z^6 + 453948z^7 + 113025z^8 + 14740z^9 + 1001z^{10} + 28z^{11}).$$

$$\beta_7(z) = \frac{1}{39916800}(226644 + 3056996z + 7115229z^2 + 6652800z^3 + 2649240z^4 + 8448z^5 - 419958z^6 - 181368z^7 - 38610z^8 - 4620z^9 - 297z^{10} - 8z^{11}).$$

$$\beta_8(z) = \frac{1}{79833600}(21714 - 139405z - 308891z^2 + 415800z^4 + 431244z^5 + 216678z^6 + 63822z^7 + 11550z^8 + 1265z^9 + 77z^{10} - 2z^{11}).$$

The first and second derivatives of (9) are given below

$$\alpha_4'(z) = z + \frac{3}{2}$$

$$\alpha_5'(z) = -4 - 2z$$

$$\alpha_6'(z) = z + \frac{5}{2}$$

$$\beta_0'(z) = \frac{1}{79833600}(3551 + 107338z - 237600z^3 - 172260z^4 + 13860z^5 + 58674z^6 + 26400z^7 + 5445z^8 + 550z^9 + 22z^{10}).$$

$$\beta_1'(z) = \frac{1}{39916800}(-56816 - 500082z + 1108800z^3 + 790680z^4 - 74844z^5 - 27165z^6 - 118800z^7 - 23760z^8 - 2310z^9 - 88z^{10}).$$

$$\beta_2'(z) = \frac{1}{39916800}(-8036 + 2093212z - 4656960z^3 - 3243240z^4 + 371448z^5 + 1126356z^6 + 473880z^7 + 91080z^8 + 8470z^9 + 308z^{10}).$$

$$\beta_3'(z) = \frac{1}{39916800}(206020 - 5199634z + 11642400z^3 + 7817040z^4 - 1128204z^5 - 2751672z^6 - 1095600z^7 - 199980z^8 - 17710z^9 - 616z^{10}). \tag{10}$$

$$\beta_4'(z) = \frac{1}{39916800}(-358285 + 8900430z - 19404000z^3 - 12219900z^4 + 2370060z^5 + 4372830z^6 + 1603800z^7 + 274725z^8 + 23100z^9 + 770z^{10}).$$

$$\beta_5'(z) = \frac{1}{39916800}(29694952 + 8542666z + 23284800z^3 + 12723480z^4 - 3728340z^5 - 4636632z^6 - 1518000z^7 - 241560z^8 - 19250z^9 - 616z^{10}).$$

$$\beta_6'(z) = \frac{1}{39916800}(40666312 + 52021772z - 23284800z^3 - 6902280z^4 + 4052664z^5 + 3177636z^6 + 904200z^7 + 132660z^8 + 10010z^9 + 308z^{10}).$$

$$\beta_7'(z) = \frac{1}{39916800}(3056996 + 14230458z + 19958400z^2 + 10596960z^3 + 42240z^4 - 2519748z^5 - 1269576z^6 - 308880z^7 - 41580z^8 - 2970z^9 - 88z^{10}).$$

$$\beta_8'(z) = \frac{1}{79833600}(-139405 - 617782z + 1663200z^3 + 2156220z^4 + 1300068z^5 + 446754z^6 + 92400z^7 + 11385z^8 + 770z^9 - 22z^{10})$$

$$\alpha_4''(z) = 1$$
$$\alpha_5''(z) = -2$$
$$\alpha_6''(z) = 1$$

$$\beta_0''(z) = \frac{1}{79833600}(107338 - 712800z^2 - 689040z^3 + 69300z^4 + 352044z^5 + 184800z^6 + 43560z^7 + 4950z^8 + 220z^9).$$

$$\beta_1''(z) = \frac{1}{39916800}(-500082 + 3326400z^2 + 3162720z^3 - 374220z^4 - 1629936z^5 - 831600z^6 - 190080z^7 - 20790z^8 - 880z^9).$$

$$\beta_3''(z) = \frac{1}{39916800}(-5199634 + 34927200z^2 + 31268160z^3 - 5641020z^4 - 16510032z^5 - 7669200z^6 - 1599840z^7 - 159390z^8 - 6160z^9).$$

$$\beta_4''(z) = \frac{1}{39916800}(8900430 - 58212000z^2 - 48879600z^3 + 11850300z^4 + 26236980z^5 + 11226600z^6 + 2197800z^7 + 207900z^8 + 7700z^9). \qquad (11)$$

$$\beta_5''(z) = \frac{1}{39916800}(8542666 + 69854400z^2 + 50893920z^3 - 18641700z^4 - 27819792z^5 - 10626000z^6 - 1932480z^7 - 173250z^8 - 6160z^9).$$

$$\beta_6''(z) = \frac{1}{39916800}(52021772 - 69854400z^2 - 27609120z^3 + 20263320z^4 + 19065816z^5 + 6329400z^6 + 1061280z^7 + 90090z^8 + 3080z^9).$$

$$\beta_7''(z) = \frac{1}{39916800}(14230458 + 39916800z + 31790880z^2 + 168960z^3 - 12598740z^4 - 7617456z^5 - 2162160z^6 - 332640z^7 - 26730z^8 - 880z^9).$$

$$\beta_8''(z) = \frac{1}{79833600}(-617782 + 4989600z^2 + 8624880z^3 + 6500340z^4 + 2680524z^5 + 646800z^6 + 91080z^7 + 6930z^8 - 220z^9).$$

Equation (9) is evaluated at the non-interpolation points, i.e. $z = -7, -6, -5, -4, 0$ and 1 to produce the discrete schemes while equations (10) and (11) are evaluated at all the grid points, i.e. $z = -7, -6, -5, -4, -3, -2, -1, 0$ and 1 to give the derivatives of the discrete schemes. The schemes and its derivatives are combined in a matrix form and matrix inversion is applied. This yields the block of the form

$$A'Y_{N+1} = B'Y_N + hC'Y_N' + h^2 D'Y_N'' + h^3(E'F_{N+1} + E''F_N) \qquad (12)$$

where

$$Y_{N+1} = [y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+6}, y_{n+7}, y_{n+8}]^T,$$
$$Y_N = [y_{n-7}, y_{n-6}, y_{n-5}, y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}, y_n]^T$$
$$Y_N' = [y_{n-7}', y_{n-6}', y_{n-5}', y_{n-4}', y_{n-3}', y_{n-2}', y_{n-1}', y_n']^T,$$
$$Y_N'' = [y_{n-7}'', y_{n-6}'', y_{n-5}'', y_{n-4}'', y_{n-3}'', y_{n-2}'', y_{n-1}'', y_n'']^T$$
$$F_{N+1} = [f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}, f_{n+7}, f_{n+8}]^T,$$
$$F_N = [f_{n-7}, f_{n-6}, f_{n-5}, f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n]^T$$

$$A' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{pmatrix},$$

$$D' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{49}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 \end{pmatrix}$$

$$E' = \begin{pmatrix} \frac{409}{2408} & \frac{-483}{2060} & \frac{674}{2285} & \frac{-298}{1107} & \frac{353}{2081} & \frac{-106}{1515} & \frac{338}{19867} & \frac{-165}{89141} \\ \frac{393}{269} & \frac{-4423}{2977} & \frac{2812}{1495} & \frac{-488}{285} & \frac{1051}{976} & \frac{-1249}{2816} & \frac{621}{5764} & \frac{-179}{15283} \\ \frac{2479}{572} & \frac{-752}{237} & \frac{3550}{761} & \frac{-899}{211} & \frac{857}{319} & \frac{-553}{499} & \frac{1077}{3997} & \frac{-173}{5902} \\ \frac{7827}{890} & \frac{-1192}{247} & \frac{6471}{703} & \frac{-2368}{297} & \frac{3357}{667} & \frac{-1987}{957} & \frac{1299}{2573} & \frac{-256}{4661} \\ \frac{7571}{510} & \frac{-1193}{185} & \frac{2687}{168} & \frac{-4499}{365} & \frac{2727}{337} & \frac{-1091}{327} & \frac{943}{1162} & \frac{-460}{5209} \\ \frac{6183}{275} & \frac{-6183}{770} & \frac{3001}{120} & \frac{-1730}{103} & \frac{3324}{269} & \frac{-2691}{550} & \frac{459}{385} & \frac{-999}{7700} \\ \frac{5740}{181} & \frac{-4858}{507} & \frac{8236}{227} & \frac{-2959}{138} & \frac{4339}{236} & \frac{-2077}{331} & \frac{1289}{780} & \frac{-2402}{13423} \\ \frac{15098}{355} & \frac{-843}{76} & \frac{5426}{109} & \frac{-17315}{661} & \frac{10721}{411} & \frac{-4728}{685} & \frac{2425}{914} & \frac{-313}{1374} \end{pmatrix}$$

$$E'' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{37}{408} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{617}{1341} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{259}{232} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4410}{2141} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{760}{231} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7663}{1594} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5481}{829} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1523}{175} \end{pmatrix}$$

The first and second derivatives of (12) yields

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{7257600}(-40187 f_{n+8} + 369744 f_{n+7} - 1522673 f_{n+6} + 3698922 f_{n+5} - 64888311 f_{n+4} + 6488191 f_{n+3} - 5225623 f_{n+2} + 4124231 f_{n+1}. + 1624505 f_n).$$

$$y'_{n+2} = y'_n + 2hy''_n + \frac{h^2}{113400}(-1563 f_{n+8} + 14368 f_{n+7} - 59092 f_{n+6} + 143232 f_{n+5} - 227030 f_{n+4} + 247328 f_{n+3} - 183708 f_{n+2} + 235072 f_{n+1}. + 58193 f_n).$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{h^2}{89600}(-1935 f_{n+8} + 17784 f_{n+7} - 73128 f_{n+6} + 177264 f_{n+5} - 281430 f_{n+4} + 315000 f_{n+3} - 150624 f_{n+2} + 328608 f_{n+1} + 71661 f_n).$$

$$y'_{n+4} = y'_n + 4hy''_n + \frac{h^2}{28350}(-836 f_{n+8} + 7680 f_{n+7} - 31552 f_{n+6} + 76288 f_{n+5} - 118440 f_{n+4} + 160256 f_{n+3} - 46400 f_{n+2} + 148992 f_{n+1}. + 30812 f_n).$$

$$y'_{n+5} = y'_n + 5hy''_n + \frac{h^2}{290304}(-10875 f_{n+8} + 100000 f_{n+7} - 412000 f_{n+6} + 1020600 f_{n+5} - 1283750 f_{n+4} + 2294000 f_{n+3} - 465000 f_{n+2} + 1987000 f_{n+1}. + 398825 f_n).$$

$$y'_{n+6} = y'_n + 6hy''_n + \frac{h^2}{1400}(-63 f_{n+8} + 576 f_{n+7} - 2268 f_{n+6} + 7200 f_{n+5} - 6390 f_{n+4} + 14208 f_{n+3} - 2196 f_{n+2} + 11808 f_{n+1} + 2325 f_n).$$

$$y'_{n+7} = y'_n + 7hy''_n + \frac{7h^2}{1036800}(-8183 f_{n+8} + 84168 f_{n+7} - 145432 f_{n+6} + 1009792 f_{n+5} - 689430 f_{n+4} + 1830248 f_{n+3} - 225008 f_{n+2} + 1484112 f_{n+1} + 288533 f_n).$$

$$y'_{n+8} = y'_n + 8hy''_n + \frac{h^2}{28350}(47104 f_{n+7} - 14848 f_{n+6} + 251904 f_{n+5} - 145280 f_{n+4} + 419840 f_{n+3} - 44544 f_{n+2} + 329728 f_{n+1} + 63296 f_n).$$

$$y''_{n+1} = y''_n + \frac{h}{1069200}(-10004 f_{n+8} + 92186 f_{n+7} - 380447 f_{n+6} + 927046 f_{n+5} - 1482974 f_{n+4} + 1648632 f_{n+3} - 1356711 f_{n+2} + 1316197 f_{n+1} + 315273 f_n).$$

$$y''_{n+2} = y''_n + \frac{h}{113400}(-833 f_{n+8} + 7624 f_{n+7} - 31154 f_{n+6} + 74728 f_{n+5} - 116120 f_{n+4} + 120088 f_{n+3} - 42494 f_{n+2} + 182584 f_{n+1} + 32377 f_n).$$

$$y''_{n+3} = y''_n + \frac{h}{44800}(-369 f_{n+8} + 3402 f_{n+7} - 14062 f_{n+6} + 34434 f_{n+5} - 56160 f_{n+4} + 79934 f_{n+3} + 3438 f_{n+2} + 70902 f_{n+1} + 12881 f_n).$$

$$y''_{n+4} = y''_n + \frac{h}{28350}(-214 f_{n+8} + 1952 f_{n+7} - 7912 f_{n+6} + 18464 f_{n+5} - 18160 f_{n+4} + 65504 f_{n+3} + 488 f_{n+2} + 45152 f_{n+1} + 8126 f_n).$$

$$y''_{n+5} = y''_n + \frac{h}{145152}(-1225 f_{n+8} + 11450 f_{n+7} - 49150 f_{n+6} + 170930 f_{n+5} - 4000 f_{n+4} + 318350 f_{n+3} + 7550 f_{n+2} + 230150 f_{n+1} + 41705 f_n).$$

$$y''_{n+6} = y''_n + \frac{h}{1400}(-9f_{n+8} + 72f_{n+7} + 158f_{n+6} + 2664f_{n+5} - 360f_{n+4} + 3224f_{n+3} + 18f_{n+2} + 2232f_{n+1} + 401f_n).$$

$$y''_{n+7} = y''_n + \frac{h}{518400}(-8183f_{n+8} + 223174f_{n+7} + 522046f_{n+6} + 736078f_{n+5} + 54880f_{n+4} + 1085937f_{n+3} + 48706f_{n+2} + 816634f_{n+1} + 149527f_n).$$

$$y''_{n+8} = y''_n + \frac{h}{28350}(7912f_{n+8} + 47104f_{n+7} - 7424f_{n+6} + 83968f_{n+5} - 36320f_{n+4} + 83968f_{n+3} - 7424f_{n+2} + 47104f_{n+1} + 7912f_n).$$

III. ESTABLISHING THE PROPERTIES OF THE BLOCK METHOD

A. *Order of the Block Method*

In finding the order of the block (12), *y* and *f*–functions are expanded about *x* using Taylor series (refer to *J* in APPENDIX 1).

Hence, comparing the coefficients of h^m and y_n^m, the block has order $(9,9,9,9,9,9,9,9)^T$ with error constants

$$\left(\frac{1}{630939}, \frac{113}{11313}, \frac{155}{6179}, \frac{80}{1701}, \frac{316}{4176}, \frac{403}{3623}, \frac{854}{5555}, \frac{374}{1847}\right)^T.$$

B. *Zero-stability of the Block Method*

In order to find the zero-stability of the block (12), we only put into consideration the coefficients of *y*–function according to Lambert [6]. That is,

$$\rho(r) = \det[rA' - B'] = 0$$

where A' and B' are the coefficients of $y_{n+i}, i = 1(1)8$ and y_n in (12). This gives

$$\det[rA^{(0)} - A^{(1)}] = \left| r\begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix} - \begin{pmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \end{pmatrix} \right| = 0$$

which implies $r = 0,0,0,0,0,0,0,1$. Hence, the method is zero stable.

C. *Consistency of the Block Method*

The block method (12) is consistent because is having an order greater than one. Hence, it is convergent since it is zero-stable and consistent ([5]).

D. *Region of Absolute Stability of the Block Method*

The method known as boundary locus method proposed by Lambert [6] and Henrici [5] is adopted in finding the region of absolute stability of the block (12). This is given as

$\bar{h}(r) = \frac{\ell(r)}{\sigma(r)}$ where $\ell(r)$ and $\sigma(r)$ are first and second characteristics polynomials respectively. The test problem of the form $y''' = \lambda^3 y$ is substituted into the block (12). Substituting $r = \cos\theta + i\sin\theta$, simplifying and equating the imaginary part to zero. This gives the interval of absolute stability to be (0, 531.18). This is shown in the diagram below

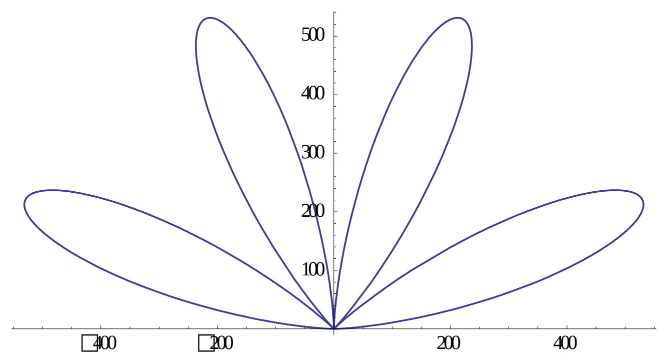

Figure1. Region of absolute stability of the method (12) for third order ODEs

IV. TEST PROBLEMS FOR THIRD ORDER ODES

The accuracy of the new block method is examined by solving third order ODEs displayed below. The same differential problems the existing methods solved are also considered for the purpose of comparison in terms of error. The results are shown in APPENDIX 2.

Problem 1: $y''' = 2y'' - 4,\ y(0) = 1, y'(0) = 2, y''(0) = 6,\ 0 \le x \le 1$

Exact Solution: $y(x) = x^2 + e^{2x}$

Problem 2: $y''' = 8y' - 3y - 4e^x,\ y(0) = 2, y'(0) = -2, y''(0) = 10,\ 0 \le x \le 1$

Exact Solution: $y(x) = e^x + xe^{-3x}$

Problem 3: $y''' = e^x \quad y(0) = 3, y'(0) = 1, y''(0) = 5, h = 0.1\ \ 0 \le x \le 1$

Exact solution: $y(x) = 2 + 2x^2 + e^x$

The following notations are used in Tables 1 and 2.

S2PEB Sequential implementation of the 2-point explicit block method
P2PEB Parallel implementation of the 2-point explicit block method
S3PEB Sequential implementation of the 3-point explicit block method
P3PEB Parallel implementation of the 3-point explicit block method

V. CONCLUSION

This paper contains the development of block methods with step-length $k = 8$ using interpolation and collocation technique for the solution of third order initial value problems of ODEs. The properties of the methods are analyzed. The numerical results generated when the method was applied to third order problems are compared with the numerical methods of step-lengths $k = 8$ and $k = 7$. The accuracy of the new method in terms of error is better than the existing methods (refer to Tables 1 – 3 in APPENDIX 2)).

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APPENDIX 1

$$A=\begin{pmatrix}
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 & x_{n+4}^{10} & x_{n+4}^{11} \\
1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8 & x_{n+5}^9 & x_{n+5}^{10} & x_{n+5}^{11} \\
1 & x_{n+6} & x_{n+6}^2 & x_{n+6}^3 & x_{n+6}^4 & x_{n+6}^5 & x_{n+6}^6 & x_{n+6}^7 & x_{n+6}^8 & x_{n+6}^9 & x_{n+6}^{10} & x_{n+6}^{11} \\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2\mathrm{A} & 120x_n^3 & 210x_n^4 & 336x_n^5 & 504x_n^6 & 720x_n^7 & 990x_n^8 \\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 & 504x_{n+1}^6 & 720x_{n+1}^7 & 990x_{n+1}^8 \\
0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 & 504x_{n+2}^6 & 720x_{n+2}^7 & 990x_{n+2}^8 \\
0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & 336x_{n+3}^5 & 504x_{n+3}^6 & 720x_{n+3}^7 & 990x_{n+3}^8 \\
0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & 336x_{n+4}^5 & 504x_{n+4}^6 & 720x_{n+4}^7 & 990x_{n+4}^8 \\
0 & 0 & 0 & 6 & 24x_{n+5} & 60x_{n+5}^2 & 120x_{n+5}^3 & 210x_{n+5}^4 & 336x_{n+5}^5 & 504x_{n+5}^6 & 720x_{n+5}^7 & 990x_{n+5}^8 \\
0 & 0 & 0 & 6 & 24x_{n+6} & 60x_{n+6}^2 & 120x_{n+6}^3 & 210x_{n+6}^4 & 336x_{n+6}^5 & 504x_{n+6}^6 & 720x_{n+6}^7 & 990x_{n+6}^8 \\
0 & 0 & 0 & 6 & 24x_{n+7} & 60x_{n+7}^2 & 120x_{n+7}^3 & 210x_{n+7}^4 & 336x_{n+7}^5 & 504x_{n+7}^6 & 720x_{n+7}^7 & 990x_{n+7}^8 \\
0 & 0 & 0 & 6 & 24x_{n+8} & 60x_{n+8}^2 & 120x_{n+8}^3 & 210x_{n+8}^4 & 336x_{n+8}^5 & 504x_{n+8}^6 & 720x_{n+8}^7 & 990x_{n+8}^8
\end{pmatrix}$$

$$J=\begin{pmatrix}
\sum_{m=0}^{\infty}\frac{h^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{h^m}{m!}y_n^{(m)}-\frac{3619903}{39916800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(39916800)(m!)}y_n^{(3+m)}\begin{pmatrix}6779886(1)^m-9359135(2)^m+11774146(3)^m\\-10745445(4)^m+6771082(5)^m-2792861(6)^m\\+679110(7)^m-73886(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(2h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(2h)^m}{m!}y_n^{(m)}-\frac{286967}{623700}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(623700)(m!)}y_n^{(3+m)}\begin{pmatrix}911204(1)^m-926646(2)^m+1173140(3)^m\\-1067950(4)^m+671628(5)^m-276634(6)^m\\+67196(7)^m-7305(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(3h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(3h)^m}{m!}y_n^{(m)}-\frac{183384}{492800}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(492800)(m!)}y_n^{(3+m)}\begin{pmatrix}711918(1)^m-521217(2)^m+766290(3)^m\\-699885(4)^m+441306(5)^m-182043(6)^m\\+44262(7)^m-4815(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(4h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(4h)^m}{m!}y_n^{(m)}-\frac{321172}{155925}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(155925)(m!)}y_n^{(3+m)}\begin{pmatrix}137126(1)^m-752480(2)^m+1435264(3)^m\\-1243200(4)^m+784768(5)^m-323744(6)^m\\+78720(7)^m-8564(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(5h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(5h)^m}{m!}y_n^{(m)}-\frac{5253125}{1596672}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(1596672)(m!)}y_n^{(3+m)}\begin{pmatrix}23702750(1)^m-10296375(2)^m+25537250(3)^m\\-19680625(4)^m+12920250(5)^m-5327125(6)^m\\+1295750(7)^m-141000(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(6h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(6h)^m}{m!}y_n^{(m)}-\frac{74034}{15400}h^2y_n''-\sum_{m=0}^{\infty}\frac{h^{3+m}}{(15400)(m!)}y_n^{(3+m)}\begin{pmatrix}346248(1)^m-123660(2)^m+385128(3)^m\\-258660(4)^m+190296(5)^m-75348(6)^m\\+18360(7)^m-1998(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(7h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(7h)^m}{m!}y_n^{(m)}-\frac{37701874}{5702400}h^2y_n''-\sum_{m=0}^{\infty}\frac{7h^{3+m}}{(5702400)(m!)}y_n^{(3+m)}\begin{pmatrix}180838518(1)^m-54639557(2)^m+206894170(3)^m\\-122270925(4)^m+104842066(5)^m-35782103(6)^m\\+9423582(7)^m-1020425(8)^m\end{pmatrix}\\
\sum_{m=0}^{\infty}\frac{(8h)^m}{m!}y_n^m-\sum_{m=0}^{2}\frac{(8h)^m}{m!}y_n^{(m)}-\frac{8(169624)}{155925}h^2y_n''-\sum_{m=0}^{\infty}\frac{8h^{3+m}}{(155925)(m!)}y_n^{(3+m)}\begin{pmatrix}828928(1)^m-216192(2)^m+970240(3)^m\\-510560(4)^m+508416(5)^m-134528(6)^m\\+51712(7)^m-4440(8)^m\end{pmatrix}
\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

APPENDIX 2

TABLE 1: THE COMPARISON OF THE NUMERICAL RESULTS GENERATED FROM THE NEW BLOCK METHOD FOR SOLVING PROBLEM 1 WITH OMAR [10] BLOCK METHOD K=8 IN WHICH MAXIMUM ERRORS WERE CONSIDERED.

h-values	New Method	Omar [10]	Number of Steps	Error in new Method, K=8	Error in Omar [10] K=8
10^{-2}	8-Step Method	S2PEB	54	1.267345E-02	1.64491E+08
		P2PEB	54	1.267345E-02	1.64491E+08
		S3PEB	39	8.063762E-04	7.14468E+11
		P3PEB	39	8.063762E-04	7.14468E+11
10^{-3}	8-Step Method	S2PEB	504	3.161755E-06	5.58385E-03
		P2PEB	504	3.161755E-06	5.58385E-03
		S3PEB	339	1.736923E-07	9.98913E-02
		P3PEB	339	1.736923E-07	9.98913E-02
10^{-4}	8-Step Method	S2PEB	5004	2.526122E-10	3.48576E-05
		P2PEB	5004	2.526122E-10	3.48576E-05
		S3PEB	3339	1.172396E-11	1.53942E-03
		P3PEB	3339	1.172396E-11	1.53942E-03
10^{-5}	8-Step Method	S2PEB	50004	5.236188E-09	3.92325E-05
		P2PEB	50004	5.236188E-09	3.92325E-05
		S3PEB	33339	2.625740E-10	2.22810E-05
		P3PEB	33339	2.625740E-10	2.22810E-05

TABLE 2: COMPARISON OF THE NUMERICAL RESULTS GENERATED FROM THE NEW BLOCK METHOD K=8 FOR SOLVING PROBLEM 2 WITH OMAR [10] BLOCK METHOD K=8 IN WHICH MAXIMUM ERRORS WERE CONSIDERED.

h-values	New Method	Omar [10]	Number of Steps	Error in new Method, K=8	Error in Omar [10] K=8
10^{-2}	8-Step Method	S2PEB	54	5.058909E-02	2.70995E-01
		P2PEB	54	5.058909E-02	2.70995E-01
		S3PEB	39	3.866111E-03	8.01861E-01
		P3PEB	39	3.866111E-03	8.01861E-01
10^{-3}	8-Step Method	S2PEB	504	3.565494E-07	1.19154E-02
		P2PEB	504	3.565494E-07	1.19154E-02
		S3PEB	339	1.674995E-08	1.25684E-02
		P3PEB	339	1.674995E-08	1.25684E-02
10^{-4}	8-Step Method	S2PEB	5004	1.723066E-11	1.17030E-03
		P2PEB	5004	1.723066E-11	1.17030E-03
		S3PEB	3339	2.646772E-13	1.17117E-03
		P3PEB	3339	2.646772E-13	1.17117E-03
10^{-5}	8-Step Method	S2PEB	50004	2.149037E-10	1.17019E-04
		P2PEB	50004	2.149037E-10	1.17019E-04
		S3PEB	33339	1.489830E-11	1.17020E-04
		P3PEB	33339	1.489830E-11	1.17020E-04

TABLE 3: COMPARISON OF THE NUMERICAL RESULTS GENERATED FROM THE NEW BLOCK METHOD K=8 FOR SOLVING PROBLEM 3 WITH KUBOYE AND OMAR [7] BLOCK METHOD K=7.

X	Exact Solution	Computed Solution	Error in New Method	Error in Kuboye and Omar [7]
0.1	3.125170918075647700	3.125170918075645500	2.220446E-15	2.531308E-14
0.2	3.301402758160169700	3.301402758160155500	1.421085E-14	1.612044E-13
0.3	3.529858807576003300	3.529858807575966900	3.641532E-14	4.023448E-13
0.4	3.811824697641270600	3.811824697641202200	6.838974E-14	7.536194E-13
0.5	4.148721270700128200	4.148721270700018900	1.092459E-13	1.212364E-12
0.6	4.542118800390509700	4.542118800390349000	1.607603E-13	1.780798E-12
0.7	4.993752707470477500	4.993752707470256300	2.211564E-13	2.456702E-12
0.8	5.505540928492468600	5.505540928492176400	2.922107E-13	2.212097E-11
0.9	6.079603111156950000	6.079603111156580500	3.694822E-13	5.231993E-11
1.0	6.718281828459045500	6.718281828458578400	4.671818E-13	8.860113E-11
1.1	7.424166023946433800	7.424166023945847600	5.861978E-13	1.314815E-10
1.2	8.200116922736548000	8.200116922735819700	7.283063E-13	1.815206E-10